2. Lecture II: Pricing European Derivatives

2.1. The fundamental pricing formula for European derivatives. We continue working within the Black and Scholes model introduced in Lecture I. Recall from definition 1.4 that a general European pay-off with exercise date \( T \) is an \( \mathcal{F}_T^W \)-measurable random variable \( X \). The way to think about \( X \) is as a future cash-flow to be delivered at time \( T \), whose (future) value of \( X \) depends only on what will have happened with the stock price \( S_t \) in between now \( (t = 0) \) and \( t = T \). Since there is a one-to-one relationship between \( S_t \) and \( W_t \) (namely \( W_t = \log S_t / \sigma - \mu t \)), this is equivalent to saying that \( X \) only depends on the Brownian \( W_t \) trajectory between 0 and \( T \), which is the same as saying that \( X \) is \( \mathcal{F}_T^W \)-measurable.

\textbf{Step 1: constructing a self-financing replicating portfolio-strategy.} Our first aim is to find a self-financing portfolio strategy \((\varphi_t, \psi_t)\) for \((S_t, B_t)\) which replicates the claim \( X \), in the sense that the portfolio’s value at \( T \) coincides with \( X \):

\[
X = V_T(\varphi, \psi) = \varphi_T S_T + \psi_T B_T.
\]

Since the strategy is to be self-financing, this is equivalent to asking for \((\phi, \psi)\) such that

\[
X = V_0(\varphi, \psi) + \int_0^T \varphi_t dS_t + \psi_t dB_t,
\]

or, after discounting (remembering that \( B_0 = 1 \)),

\[
\tilde{X} = V_0(\varphi, \psi) + \int_0^T \varphi_t d\tilde{S}_t;
\]

cf. corollary 1.3 and formula (9). Such a strategy can be obtained from the martingale representation theorem, as follows.

First, using proposition 1.2, discount everything to present value by dividing by \( B_t \). This doesn’t affect the self-financing property, and it therefore suffices to find a self-financing strategy \((\varphi_t, \psi_t)\) for \((\tilde{S}_t, 1)\), \( \tilde{S}_t =: S_t / B_t \), which replicates the discounted claim,

\[
\tilde{X} := X / B_T.
\]

We now first change Brownian motion and probability measure, using Girsanov’s theorem. If we choose \( \gamma = (\mu - r) / \sigma \) in theorem 1.9, then we find an equivalent measure \( \mathbb{Q} \) such that \( \tilde{W}_t = \gamma t + W_t \) is a \( \mathbb{Q} \)-Brownian motion, and \( \tilde{S}_t \) evolves according to the SDE

\[
d\tilde{S}_t = \sigma \tilde{S}_t d\tilde{W}_t,
\]

see section 1.7. Now by the martingale representation theorem, theorem 1.5, there exist:

- a constant \( X_0 \), and
an adapted process \((h_t)_{0 \leq t \leq T}\), such that:

\[
\tilde{X} = X_0 + \int_0^T h_t d\tilde{W}_t.
\]

We can rewrite this as follows, using (33):

\[
\tilde{X} = X_0 + \int_0^T h_t d\tilde{W}_t = X_0 + \int_0^T \frac{h_t}{\sigma_t} \tilde{S}_t d\tilde{W}_t
\]

where we have put

\[
\phi_t := \frac{h_t}{\sigma_t}.
\]

The financial interpretation of (34) is that \(\phi_t\) is exactly the number of (discounted) stock which you must own at time \(t\), to obtain the (discounted) claim at time \(T\), given an initial investment of \(X_0\). Comparing with (31), we see that (35) is the \(\phi\)-component of the self-financing strategy we are looking for. The corresponding \(\psi\)-component is found by solving

\[
\phi_t \tilde{S}_t + \psi_t = \tilde{V}_t(\varphi, \psi)
\]

\[
= X_0 + \int_0^t \phi_u d\tilde{S}_u,
\]

where we used the self-financing property again, or

\[
\psi_t = \left( X_0 + \int_0^t \phi_u d\tilde{S}_u \right) - \phi_t \tilde{S}_t.
\]

In particular, \(\psi_0 = X_0 - \phi_0 S_0\).

**Step 2: Computing the price as an expectation.** By the law of one price, \(X_0\) should then be the fair value of the claim at time 0. How do we compute \(X_0\)? The essential point is that the Ito-integral in (34) is a \(Q\)-martingale:

\[
\mathbb{E}_Q \left( \int_0^T h_u d\tilde{W}_u | \mathcal{F}_t \right) = \int_0^t h_u d\tilde{W}_u.
\]

(See Math Methods I). In particular, if \(t = 0\), we get 0. Hence, taking expectations in (33), we find that

\[X_0 = \mathbb{E}_Q(\tilde{X}).\]
Let us write $\pi_t(X)$ for the price at $t$ of the European claim $X$ at $T$. Then, remembering that $\tilde{X} = X/B_T = e^{-rT}X$, we have shown that

\begin{equation}
\pi_0(X) = e^{-rT}E_Q(X).
\end{equation}

This can be generalized to any $t$ between 0 and $T$: if $(\varphi, \psi)$ is our replicating strategy, then $\pi_t(X)$ should be equal to the portfolio’s value $V_t(\varphi, \psi)$ at $t$, by absence of arbitrage. Hence, after discounting,

\begin{equation}
\frac{\pi_t(X)}{B_t} = \tilde{V}_t(\varphi, \psi).
\end{equation}

Now, since the portfolio is self-financing, we find that

\begin{align*}
\tilde{X} &= \tilde{V}_t(\varphi, \psi) + \int_t^T d\tilde{V}_u(\varphi, \psi) \\
&= \tilde{V}_t(\varphi, \psi) + \int_t^T \varphi_u d\tilde{S}_u \\
&= \tilde{V}_t(\varphi, \psi) + \int_t^T h_u d\tilde{W}_u.
\end{align*}

Taking conditional expectations, and using the martingale property of Ito integrals\(^3\), we find that

\begin{equation}
E_Q(\tilde{X}|\mathcal{F}_t^W) = \tilde{V}_t(\varphi, \psi) = \frac{\pi_t(X)}{B_t}.
\end{equation}

Hence, remembering that $B_t = e^{-rt}$,

\begin{equation}
\pi_t(X) = B_tE_Q(B_T^{-1}X) = e^{-r(T-t)}E_Q(X).
\end{equation}

We summarize the discussion up till now in the following theorem:

**Theorem 2.1. (European Option Pricing Formula):** The value at time 0 of a European claim $X$ at time $T$ is, in the Black and Scholes model, given by:

\begin{equation}
\pi_0(X) = E_Q \left( \frac{X}{B_T} \right) = e^{-rT}E_Q(X_T).
\end{equation}

More generally, its price at $t$, $0 < t < T$, will be given by:

\begin{equation}
B_tE_Q \left( \frac{X}{B_T} | \mathcal{F}_t \right) = e^{-r(T-t)}E_Q(X_T | \mathcal{F}_t^W),
\end{equation}

where $\mathcal{F}_t^W$ is the Brownian filtration.

\(^3\)Explicitly, $E_Q \left( \int_0^T h_u d\tilde{W}_u | \mathcal{F}_t \right) = E_Q \left( \int_0^T h_u d\tilde{W}_u | \mathcal{F}_t \right) - E_Q \left( \int_0^t h_u d\tilde{W}_u | \mathcal{F}_t \right) = \int_0^t h_u d\tilde{W}_u - \int_0^t h_u d\tilde{W}_u = 0.$
Remark 2.2. Formulas (39), (40) are the simplest examples of the "Risk-Neutral Pricing Principle", which can be stated as:

\[ \text{Price of an asset at } t = \mathbb{E}_Q \left( \sum \text{discounted future cash-flows} | \mathcal{F}_t \right), \]

\( \mathbb{Q} \) being a risk-neutral probability measure, that is, one with respect to which the discounted asset prices are martingales. Observe that we have put the discounting under the expectation-symbol, since in general the discounting factor may be stochastic: you will be asked to work out an example of this in exercise 2.11 below.

We pause with the general discussion to consider the classical example.

Example 2.3. If the claim \( X \) is of the form:

\[ X = g(S_T), \]

for some given function \( g \), then the claims value at 0 is:

\[ \pi_0(X) = e^{-rT} \mathbb{E}_Q \left( g(S_T) \right) = \mathbb{E}_Q \left( g(S_0 e^{(r-\sigma^2/2)T+\sigma \tilde{W}_T}) \right), \]

since \( S_t, \text{ with respect to the } \mathbb{Q} \text{-measure,} \) follows the SDE

\[ dS_t = rS_t dt + \sigma S_t d\tilde{W}_t, \]

whose solution is \( S_t = S_0 e^{(r-\sigma^2/2)(T-t)+\sigma \tilde{W}_t} \) (cf. Math. Methods I). Remembering the pdf of \( \tilde{W}_T, \) and the formula for computing expectations of functions of random variables with known pdf, we find that (42) equals:

\[ \pi_0(X) = \int_{-\infty}^{\infty} g \left( S_0 e^{(r-\sigma^2/2)T+\sigma \tilde{W}} \right) e^{-w^2/2T} \frac{dw}{\sqrt{2\pi T}}. \]

More generally, since

\[ S_T = S_0 e^{(r-\sigma^2/2)(T-t)+\sigma (W_T-W_t)}, \]

we find that

\[ \pi_t(X) = \mathbb{E}_Q \left( g \left( S_t e^{(r-\sigma^2/2)(T-t)+\sigma (W_T-W_t)} \right) | \mathcal{F}_t^W \right) \]

\[ = \mathbb{E}_Q \left( g \left( S_t e^{(r-\sigma^2/2)(T-t)+\sigma (W_T-W_t)} \right) \right) \]

\[ = \int_{\mathbb{R}} g \left( S_t e^{(r-\sigma^2/2)(T-t)+\sigma w} \right) e^{-w^2/2(T-t)} \frac{dw}{\sqrt{2\pi(T-t)}}, \]

where we used that \( W_T - W_t \) is independent of \( \mathcal{F}_t^W \) and \( N(0, T-t) \)-distributed. Observe that in this case (of an \( X \) of the form \( g(S_T) \)), the price is of the form

\[ \pi_t(X) = f(S_t, t), \]
with \( f \) given by

\[
(44) \quad f(S, t) = \int_{\mathbb{R}} g\left(Se^{(r-\sigma^2/2)(T-t)+\sigma w}\right) e^{-w^2/2(T-t)} \frac{dw}{\sqrt{2\pi(T-t)}}.
\]

In the special case of a European call with strike \( E \),

\[ g(S_T) = \max(S_T - E, 0), \]

this integral can be explicitly evaluated, which gives the famous Black and Scholes formula. Writing the call’s price as \( C(S_t, t) \), one computes that

\[
(45) \quad C(S_t, t) = S\Phi(d_+) - e^{-r(T-t)}\Phi(d_-),
\]

where

\[
d_\pm = \frac{1}{\sigma\sqrt{T-t}} \left( \log(S_t/E) + (r \pm \frac{\sigma^2}{2})(T-t) \right).
\]

Here is a simple way to memorize \( d_\pm \): since \( \sigma\sqrt{T-t} \) is the total volatility of the stock-return over the remaining life-time \([t, T]\) of the option, we have:

\[
d_\pm = \log \left( \frac{\text{stock price at } t}{\text{discounted exercise price}} \right) \pm \left( \frac{\text{vol over remaining life-time}}{2} \right)^2.
\]

**Step 3: Replication and hedging.** For financial practice, we not only need to know the price, but also how to set up the replicating portfolio (for this is what the writer of the option should immediately do, after having sold the option, in order to be able to meet his obligation of providing the pay-off at \( T \) to the buyer of the option). We will discuss this for pay-offs of the form \( X = g(S_T) \). In this case we have seen that the option’s price can be written as

\[
\pi_t(X) = f(S_t, t),
\]

for some suitable function \( f(S, t) \). From what you already know, from Pricing I, about option pricing using the PDE-method, you may guess that the amount of stock we should hold is

\[
\varphi_t = \frac{\partial f}{\partial S}(S_t, t),
\]

the options \( \Delta \), but how can we see this using the present martingale formalism? The trick is to use self-financing property of our replicating portfolio \( (\varphi, \psi) \) (which we know exists on abstract grounds, but which we do not know how to construct yet). In fact,

\[
f(S_t, t) = \pi_t(X) = V_t(\varphi, \psi).
\]

We also know, since \((\varphi, \psi)\) is self-financing, that

\[
dV_t = \varphi_t dS_t + \psi_t dB_t = \varphi_t dS_t + r\psi_t B_t dt.
\]

---

4Here, and elsewhere, log stands for the natural logarithm with base \( e \): \( \log = \ln \).
Therefore we should have that
\[ df(S_t, t) = \varphi_t dS_t + r \psi_t B_t dt. \] (46)

Now the left hand side can be evaluated using Ito’s lemma\(^5\). This gives
\[ df(S_t, t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS_t)^2 \] (47)
all derivatives evaluated in \((S_t, t)\). Comparing coefficients of \(dS_t\) with (46), we see that
\[ \varphi_t = \frac{\partial f}{\partial S}(S_t, t), \] (48)
as expected. The number of savings bonds \(\psi_t\) to hold then simply follows from
\[ \varphi_t S_t + \psi_t B_t = V_t(\varphi, \psi) = f(S_t, t) \Rightarrow \psi_t = \frac{f(S_t, t) - \varphi_t S_t}{B_t}, \]
which simply amounts to putting the balance \(f(S_t, t) - \varphi_t S_t\) into a bank account. We therefore know exactly how many stock and how much money in savings to hold, at any (future) time \(t\), as a function of the price \(S_t\) of the underlying.

What about comparing the coefficients of \(dt\) in (46), (47)? It turns out that this does not really give any new information, but allows us to re-encounter an old acquaintance: in exercise 2.12 you will be asked to show that this leads to the Black and Scholes PDE for \(f(S, t)\), thus making the connection with the PDE-pricing method from Pricing I. However, see also exercise 2.13.

2.2. Pricing exotics. One of the big advantages of the present pricing methodology over the one using PDE’s is, that it is rather straightforward to price ”exotic”\(^6\) claims, like asian options, look-back options, barrier options like knock-in’s and knock-outs’s, etc. One just has to write the pay-off as a function of the \(S_t\)’s, \(0 \leq t \leq T\) and plug it in in the risk-neutral pricing formula (39), while recalling that we have an explicit formula for \(S_t\) in terms of the risk-neutral \(Q\)-Brownian motion \(\hat{W}_t\):
\[ S_t = S_0 \exp((r - \sigma^2/2)t + \sigma \hat{W}_t). \]
This will give us the price as an expectation, which in general cannot be evaluated in closed form, but which can be treated numerically by Monte Carlo and simulation of the trajectories of \(\hat{W}_t\) (which is simply a

\(^5\)The function \(f(S, t)\) given by (44) can be shown to be arbitrarily many times differentiable with bounded derivatives, for ”nice” functions \(g = g(x)\), for example those not growing faster than a power of \(x\) as \(x \to \infty\).

\(^6\)basically, everything which is neither a call or a put; the latter are often called ”vanillas”
Brownian motion). We’ll take a brief look at some of the most common exotics.

**Example 2.4.** *Asian options* The pay-off of for example an Asian call depends on the mean of the underlying’s price over the entire life-time of the option, instead of just its value at time $T$. For example, the pay-off of a fixed-strike Asian call is

$$A_T = \max \left( \frac{1}{T} \int_0^T S_t \, dt - E, 0 \right), \tag{49}$$

while that of a floating strike Asian call is

$$A_T = \max \left( \frac{1}{T} \int_0^T S_t \, dt - S_T, 0 \right), \tag{50}$$

allowing to buy at the mean-price over $[0, T]$, instead of the spot-price at $T$. Asian puts are defined similarly. In practice one of course replaces the continuous mean by a discrete one, by discretizing the time-interval:

$$\max \left( \left( \frac{1}{N} \sum_{j=1}^N S_{jT/N} \right) - E, 0 \right).$$

For example, one can take the mean over daily closing prices; $N$ then would be the number of days till expiry $T$. The price of for example the latter option is given by:

$$E_Q \left( \frac{1}{N} \sum_{j=1}^N S_0 \exp \left( (r - \sigma^2/2)T + \sigma \hat{W}_{jT/N} \right) - E, 0 \right).$$

There is little hope of being able to evaluate this explicitly, but it is pretty straightforward to evaluate using Monte-Carlo. However, for the option with the integral pay-off (49), mathematicians have made substantial efforts to obtain a theoretical understanding of the price, and there are “semi-explicit” pricing formulas available, the most famous being the one due to Geman and Yor, which gives an explicit expression for what is basically the Laplace transform (with respect to time $t$) of the Asian option’s price at $\pi_t(A_T)$: see [BK] or [MR] for further precisions and references. To use this semi-explicit formula for pricing and hedging purposes one has to numerically invert the Laplace transform, which can be delicate.

**Example 2.5.** *(Barrier options)* Single barrier options become worthless whenever the stock-price crosses a certain level $H$, where it also matters whether $H$ is crossed from above or from below. For example, a *down-and-out call* with strike $E$ and barrier $H$ will have the same pay-off as an ordinary call, provided $S_t$ stays above $H$ for the entire life-time $[0, T]$ of the option. If it falls below $H$ for even one moment, the option becomes worthless. It’s pay-off can be written as:

$$\max (S_T - E, 0) \mathbb{1}_{\{ \min_{0 \leq t \leq T} S_t \geq H \}},$$
where $I_A$ is the indicator-function of the event $A$. Similarly, the pay-off for a \textit{down and in call}, which only becomes ”alive” once the underlying’s price dips below $H$, is:

$$\max(S_T - E, 0) \, I_{\{\min_{0 \leq t \leq T} S_t \leq H\}}.$$  

Note, that the sum of a ”down and out” and a ”down and in” has the pay-off of a ordinary (”vanilla”) call, $\max(S_T - E, 0)$.

”Up and out” and an ”up and in” calls have pay-offs:

$$\max(S_T - E, 0) \, I_{\{\max_{0 \leq t \leq T} S_t \leq H\}}, \quad \max(S_T - E, 0) \, I_{\{\max_{0 \leq t \leq T} S_t \geq H\}},$$

respectively. Explicit prices for these products can be derived by using the (explicitly known) joint probability distribution functions of $(W_T, \min_{0 \leq t \leq T} W_t)$ and $(W_T, \max_{0 \leq t \leq T} W_t)$: see for example [BK] or [MR].

\textbf{Example 2.6. (Look-back options)} Look-back options allow you to, \textit{a posteriori}, buy an asset $S_t$ at its low, and sell it at its high. For example, a look-back call has pay-off:

$$S_T - \min_{0 \leq t \leq T} S_t,$$

while a \textit{look-back put} pays

$$\max_{0 \leq t \leq T} S_t - S_T,$$

at maturity. Their price at $t = 0$ is, according to the risk-neutral pricing formula (39):

$$e^{-rT} \mathbb{E}_Q \left(S_T - \min_{0 \leq t \leq T} S_t\right),$$

respectively

$$e^{-rT} \mathbb{E}_Q \left(\max_{0 \leq t \leq T} S_t - S_T\right).$$

Plugging in $S_t = \exp((r - \sigma^2/2)t + \sigma W_t)$ these can be evaluated using Monte-Carlo, but one can also give explicit formulas: cf. [BK] and [MR].

2.3. \textbf{Exercises.}

\textbf{Exercise 2.7. (Digital options)} A \textit{European binary call with strike $E$} has a pay-off of 1 if $S_T \geq E$, and 0 otherwise. It’s pay-off function can be written as:

$$I_{\{S_T \geq E\}}.$$  

Similarly, a \textit{binary put} has pay-off

$$I_{\{S_T \leq E\}}.$$  

Find an explicit expression for the price of these instruments, in terms of the cumulative normal distribution function $\Phi$. 
Exercise 2.8. From the risk-neutral option pricing formula (39) derive the Put-Call Parity relation:

$$C(S_0, E, T) - P(S_0, S, E) = S_0 - E e^{-rT}.$$ 

Exercise 2.9. a) Consider a European derivative with pay-off $g(S_T)$. Its price at time $t$ is a function of $f(S_t, t)$ of $S_t$ and $t$ (cf. example 2.3). Show that

$$\Delta_t = \frac{\partial f(S_t)}{\partial S_t} = e^{-r(T-t)} \int_{-\infty}^{\infty} g' \left( S_t e^{(r-\sigma^2/2)(T-t)+\sigma w} \right) e^{(r-\sigma^2/2)(T-t)+\sigma w} e^{-w^2/2(T-t)} \frac{dw}{\sqrt{2\pi(T-t)}} = e^{-r(T-t)} E_Q \left( g'(S_T) \left( \frac{S_T}{S_t} \right) | \mathcal{F}_t \right)$$

b) $\Delta_t$ is the number of stock we need to hold at time $t$ in our replicating portfolio (see next lecture), and is therefore an important quantity to evaluate. Part a) suggests how to evaluate this quantity using Monte-Carlo. Another way of evaluating $\Delta$ would be to approximate it by a finite difference:

$$\frac{\partial f(S)}{\partial S} \approx \frac{f(S+h, t) - f(S, h)}{h},$$

for some small $h$. Which of the two methods is more attractive from a numerical point of view (i.e. is likely to lead to the smallest approximation and round-off errors)?

Exercise 2.10. Show that the Delta of a European call at time 0 is given by:

$$\Delta_0 = \Phi(d_+).$$

(Hint: do not start differentiating the Black and Scholes formula unless you like manipulating complicated formulas; rather, use part a) of the previous exercise).

Exercise 2.11. We now extend the Black and Scholes world by making the bond $B_t$ stochastic also:

$$dB_t = r B_t dt + \rho B_t dW_t.$$ 

that is, we suppose that the interest rate is no longer fixed ($rdt$), but stochastic also ($rdt + \rho dW_t$).

a) Show that

$$d\tilde{S}_t = (\mu - r + \rho^2 - \rho \sigma)\tilde{S}_t dt + (\sigma - \rho)\tilde{S}_t dW_t.$$ 

b) Show that there exists an equivalent probability-measure, $Q$, and a new Brownian motion, $\tilde{W}_t$ with respect to $Q$, such that:

$$d\tilde{S}_t = (\sigma - \rho)\tilde{S}_t d\tilde{W}_t.$$
c) Let $X_T$ be an arbitrary European claim at $T$. Prove that its value at time 0 will be given by:

$$
E_Q \left( \frac{X_T}{B_T} \right).
$$

Generalize to arbitrary $t$.

d) Find the value of a digital call in this model.

e) Find the value of an ordinary call.

**Exercise 2.12.** Show that by equating the coefficients of $dt$ of (46) and (47), we obtain the Black and Scholes PDE for $f(S_t, t)$:

$$
\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} + r \frac{\partial f}{\partial S} = rf.
$$

*(Hint: You will need to use that $\psi_t B_t = f(S_t, t) - S_t \frac{\partial f}{\partial S}(S_t, t)$.)*

**Exercise 2.13.** In the PDE-approach to derivative pricing, one sets up a risk-free hedging portfolio $V_t$ consisting of 1 option long, and $-\frac{\partial f}{\partial S}(S_t, t)$ underlying short, where $f(S, t)$ is the option price at $t$ when the price of the underlying is $S$. The time-$t$ value of this portfolio is clearly

$$
V_t = f(S_t, t) - S_t \frac{\partial f}{\partial S}(S_t, t),
$$

and one derives the Black and Scholes PDE starting off from the relation

(51)

$$
dV_t = df(S_t, t) - \frac{\partial f}{\partial S}(S_t, t) dS_t,
$$

and using Ito’s lemma and absence of arbitrage: cf. Pricing I. Now (51) is equivalent to saying that the edging strategy is self-financing. When one checks this a posteriori, this turns out to be false, as this exercise will show! So there seems to be a fundamental problem with the ”Risk-Free Portfolio Method”, which there isn’t with the ”Replicating Portfolio Method”. See however the discussion at the end of this exercise.

To simplify the formulas, we will denote partial differentiation by subscripts:

$$
\frac{\partial f}{\partial S} = f_S, \quad \frac{\partial f}{\partial t} = f_t, \quad \frac{\partial^2 f}{\partial S^2} = f_{SS}, \quad \text{etc.}
$$

a) Use Ito’s lemma to show that if $dS_t = \mu S_t dt + \sigma S_t dW_t$, then

$$
d(f_S) = (f_{St} + \mu S_t f_{SS} + \frac{\sigma^2}{2} S_t f_{SSS}) dt + \sigma S_t f_{SS} dW_t,
$$

all partial derivatives evaluated in $(S_t, t)$, as usual.
b) Show, that if \( V_t = f(S_t, t) - f_S(S_t, t)S_t \), then the hedging strategy is self-financing iff

\[
S_t df(S_t, t) + dS df(S_t, t) = 0,
\]

that is, iff

\[
(52) \quad S_t \left( f_{st} + \mu S_t f_{s2} + \frac{\sigma^2}{2} S_t f_{ss2} + \sigma^2 S_t f_{s2} \right) dt + \sigma S_t f_{ss} dW_t = 0.
\]

(Hint: \( dV_t = dF - f_s dS_t - S_t d(f_s) - dS_t d(f_s) \).)

c) Use the fact that \( f(S, t) \) satisfies the Black and Scholes equation to show that the left and side of (52) is equal to

\[
\sigma S_t f_{ss} \left( \frac{\mu - r}{\sigma} dt + d\tilde{W}_t \right),
\]

which, using the definition of our risk-neutral Brownian motion, can also be written as:

\[
\sigma S_t f_{ss} d\tilde{W}_t.
\]

Conclude that the hedging portfolio is not self-financing, unless \( f_{ss} \) is identically 0 (What would this mean for the option?).

Discussion: On first sight, this is a nasty surprise, which sheds doubt on the Risk-Free Portfolio Method. However, note that the total additional capital which we would have to inject between \( t \) and \( T \) (and which we normally should have added to the price),

\[
\int_t^T \sigma S_u f_{ss}(S_u, u) d\tilde{W}_u,
\]

has risk-neutral expectation 0, conditional to \( \mathcal{F}_W^t \), by the martingale property of Ito integrals once more. This explains why the Risk-Free Method still leads to the correct answer.