LECTURE 10.1

Default risk in Merton’s model

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1 MERTON’S MODEL

1.1 Introduction

Credit risk is the risk of suffering a financial loss due to the decline in the creditworthiness of a counterparty in a financial transaction. For instance: the risk that the market value of a bond declines due to decline in the credit rating of the issuer, the risk of suffering a loss if a bank’s debtor defaults, loss on CDS if a credit event occurs. The source of credit risk is the default risk, that is the risk that a counterparty will not fulfill her/his contractual obligations. In the area of credit risk the main focus is on a judicious specification of the (random) time of default. A large part of these last three lectures will be devoted to this issue.

There are two main methods of modeling credit risk: I) so-called structural approach and II) intensity-based approach (also known as reduced form approach). Due to the limited time we have, the focus will be on method I) which attempts to model explicitly the event triggering default, that is, the process driving the value of the assets. On the contrary, with the intensity-based approach there is no attempt to model default and a credit event is taken as an unpredictable event, meaning that the date of its occurrence is a totally inaccessible stopping time with respect to an underlying filtration. The modeling of a default time is essentially reduced to the specification of the so-called hazard process with respect to some reference filtration. Under some circumstances, this is equivalent to the modeling of a default time in terms of its intensity process.

1.2 General setting to model default risk

In order to model a defaultable claim the basic ingredients we need are:
- The short-term interest rate process $r$, and thus also a default-free term structure model.
- The firm’s value process $V_t$, (so a model for the total value of the firm’s assets).
Contract specification
- The barrier process $v$, which will be used in the specification of the default time $\tau$.
- The promised contingent claim $L$ representing the firm’s liabilities to be redeemed at maturity date $T$.
- The process $C$, which models the promised coupons, i.e., the liabilities stream that is redeemed continuously or discretely over time to the holder of a defaultable claim.

Recovery
- The recovery claim $L$ representing the recovery payoff received at time $T$, if default occurs prior to or at the claim’s maturity date $T$.
- The recovery process $Z$, which specifies the recovery payoff at time of default, if it occurs prior to or at the maturity date $T$.

Default Time
It is essential to emphasize that the various approaches to valuing and hedging of defaultable securities differ between themselves with regard to the ways in which the default event occurs and thus also the default time $\tau$ and the default barrier $v$ are modeled. In the structural approach, the default time $\tau$ will be typically defined in terms of the value process $V$ and the barrier process $v$. In particular $\tau = \inf\{t > 0 : t \in \Theta$ and $V_t \leq v_t\}$ Depending on the model and the purpose, we may have that $\Theta = \{T\}$ as in the classical Merton model, $\Theta = [0, T]$ or $\Theta = \{T_1, T_2, \ldots, T_n\}$ if default can only happen (in the sense “declared”) at some discrete time instants, such as the coupon payment dates. In most structural models $\Theta = [0, T]$ (or even $\Theta = [0, \infty)$ for perpetual claims). So we can set

$$\tau = \inf\{t > 0 : t \in [0, T] \text{ and } V_t \leq v_t\}.$$

Predictability of Default Time
Since the underlying filtration $\mathcal{F}$ in most structural models is generated by a standard Brownian motion, $\tau$ will be an $\mathcal{F}$-predictable stopping time (as any stopping time with respect to a Brownian filtration). The latter property means that within the framework of the structural approach there exists a sequence of increasing stopping times announcing the default time; in
this sense, the default time can be forecasted with some degree of certainty. In some structural models, the value process \( V \) is assumed to follow a jump diffusion, in which case the default time is not predicable with respect to the reference filtration, in general. Some other structural models are constructed so that the barrier process is not adapted to the reference filtration \( \mathcal{F} \) consequently \( \tau \) cannot be an \( \mathcal{F} \)-predictable stopping time.

**Recovery Rules**

If default does not occur before or at time \( T \), the promised claim \( L \) is paid in full at time \( T \). Otherwise, depending on the market convention, either (1) the amount \( L \) is paid at the maturity date \( T \), or (2) the amount \( Z \) is paid at time \( \tau \) (in reality, the recovery payment may also be distributed over time). However, for the modeling purposes it suffices to consider recovery payment only at default time or at maturity, as other possibilities can be reduced to the above by means of forward or backward discounting. In the case when default occurs at maturity, i.e., on the event \( \tau = T \), we postulate that only the recovery payment \( L \) is paid.

## 2 Merton’s model

The very earliest papers on contingent claims analysis stressed that options theory has important implications for modelling corporate debt. Black and Scholes (1973) and Merton (1974) applied options models to the valuation of default premia on corporate bonds. Classic structural models, like Merton’s model, are based on the assumption that markets are frictionless \(^1\) and the dynamics of a riskless asset is

\[
\text{d}B_t = rB_t \text{d}t
\]

the value of the firm’s assets, \( V_t \), follows a geometric Brownian motion:

\[
\text{d}V_t = \mu V_t \text{d}t + \sigma V_t \text{d}W_t.
\]

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\(^1\)This means that: 1) there are no transaction costs or taxes, 2) assets are perfectly divisible and traded continuously, 3) there are no borrowing and short-selling restrictions.
$W_t$ a Brownian motion. There is a single homogeneous class of debt, with maturity $T$, zero coupons ($C = 0$) and principal repayment (face value) at time $T$ equal to $L$.

At maturity $T$, the assets may be more or less than the contracted repayment, $L$. Thus, assuming that APR (absolute priority rule$^2$ applies, the payoff to the debt-holders, say $D_T$ is:

$$D_T = \min\{L, V_T\} = L + \min\{V_T - L, 0\}$$  \hfill (3)

$$= L - \max\{L - V_T, 0\}.$$ \hfill (4)

The debt-holder’s payoff is thus the sum of a safe claim payoff and a short position in a put option written on the firm’s assets (or in other words the bond can be hedged by buying a put). Thus, the put option represents the loss given default. The equity holders (because residual claimants) receive:

$$E_T = \max\{V_T - L, 0\}.$$ \hfill (5)

that is, the payoff of a call option written on the firm’s assets and strike equal to $L$. Trivially, notice the payoffs structure is such that $E_T + D_T = V_T$ so one could derive one claim value simply by exploiting the additivity of the payoffs.

Pricing these claims is then completely straightforward as the assumptions correspond with those of the Black-Scholes model of call and put option pricing. Therefore, by solving the risk neutral expected discounted payoffs (or the replicating portfolio PDE) we find that debt value is given by

$$D_t = Le^{-r(T-t)} - P_t(V_t, L, T-t)$$

$$= Le^{-r(T-t)} - \left[Le^{-r(T-t)}N(-d_2) - V_tN(-d_1)\right]$$

$$= Le^{-r(T-t)}N(d_2) + V_tN(-d_1)$$ \hfill (6)

$$= \frac{\ln V_t / L + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}$$ \hfill (7)

with $d_1 = \frac{\ln V_t / L + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}$

$^2$Upon bankruptcy/liquidation, senior claims (e.g. debt claims) are repaid before more junior ones (equity).
where \( P_t \) denotes the put price. Given the isomorphic relationship between levered equity and a call option, one can analyze the effect of various parameters by looking at the Greeks of the call option. That is, as \( D_t = V_t - E_t \) and \( E_t \) is a call option, then we have:

\[
\begin{align*}
\frac{\partial D_t}{\partial V_t} &= 1 - \frac{\partial E}{\partial V} = 1 - N(d_1) \geq 0 \\
\frac{\partial D_t}{\partial (T - t)} &= -\frac{\partial E}{\partial (T - t)} < 0 \\
\frac{\partial D_t}{\partial r} &= -\frac{\partial E}{\partial r} < 0 \\
\frac{\partial D_t}{\partial \sigma^2} &= -\frac{\partial E}{\partial \sigma^2} < 0 \\
\frac{\partial D_t}{\partial L} &= -\frac{\partial E}{\partial L} > 0
\end{align*}
\]

so we can summarize the above results by writing the debt value as a function \( D_t = D(V_t^+, L^+, (T - t)^-, \sigma^-, r^-) \) where the superscript\(^\pm\) represents the sign of the above derivatives with respect to the corresponding argument.

### 2.1 Risk structure of Interest Rates

In this context, we can obtain analytical expressions for the yield to maturity, the credit spread and the default probability. The yield to maturity is defined as the solution to:

\[
D_t = Le^{-y(T-t)}
\]

therefore

\[
y_{t,T} = -\frac{\ln(D_t/L)}{T-t}
\]

from which we can derive the credit spread, as the risk premium with respect to the risk free rate:

\[
s_{t,T} = y_{t,T} - r
\]
\[
\frac{1}{T-t} \ln \left( \frac{L e^{-r(T-t)} N (d_2) + V_t N (-d_1)}{L} \right) - r \quad (11)
\]

\[
\frac{1}{T-t} \ln \left( N (d_2) + \frac{V_t}{L e^{-r(T-t)}} N (-d_1) \right) \quad (12)
\]

where the term \( \frac{V_t}{L e^{-r(T-t)}} \) is the inverse of the “quasi-debt ratio” \( d = \frac{L e^{-r(T-t)}}{V_t} \) (a form of leverage ratio). Notice that as \( d_1 \) and \( d_2 \) can be written as functions of \( d \) (and \( \sigma \) and \( T-t \)), therefore one can write the spread as function of \( d, \sigma \) and \( T-t \):

\[
s_{T,t} = f(d, \sigma, T-t)
\]

Denote as \( P \) the argument of the log in equation 12, it is can be proved that

\[
\frac{\partial s_{t,T}}{\partial d} = -\frac{1}{T-t} \frac{\partial P/\partial d}{P} = -\frac{1}{T-t} \frac{N(-d_1)}{P d^2} > 0
\]

(13)

as one would expect, the level of indebtedness relative to assets should make default more likely and thus increase the (default) risk premium. It can also be found, as we will show below that \( N(-d_1)/(P d) \) is equal to the ratio \( \sigma D/\sigma \) where \( \sigma D \) is the instantaneous volatility of the debt value. So the above derivative can be rearranged as

\[
\frac{\partial s_{t,T}}{\partial d} = \frac{\sigma D}{\sigma(T-t)d}
\]

Also one can find the following results\(^3\).

\[
\frac{\partial s_{t,T}}{\partial \sigma^2} = \frac{1}{2\sigma \sqrt{T-t}} \frac{\sigma D N'(d_1)}{\sigma N(d_1)} > 0 \quad (14)
\]

\[
\frac{\partial s_{t,T}}{\partial(T-t)} = \left[ \ln P + \frac{\sigma \sqrt{T-t} \sigma D N'(-d_1)}{2 \sigma} \frac{N(-d_1)}{N(d_1)} \right] / (T-t)^2 \quad (15)
\]

the positive sign of 14 is also intuitive, this is essentially due to the negative effect on the debt value of an increase in the assets volatility (see equ. 8). In turn, this happens because

\(^3I\) just give you these results in case you want to prove them, I don’t derive them as the algebra is elaborate and uninteresting

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the equity have limited downside risk and unlimited upside potential so higher volatility shifts value from debt to equity. Less intuitive is the result in equ. 15. In particular, if \( d \geq 1 \) (the firm is economically insolvent), the sign of the derivative is negative. The reason is that, the firm is currently insolvent so to avoid default will need increased earnings. As \( T - t \) increases there is more time for increased earnings to occur (things cannot get worse, there are chances they can get better). If instead \( d < 1 \), the firm is solvent and the sign of the derivative is unclear. It is positive for shorter maturities and negative for longer maturities. In other words, for \( d < 1 \) the credit spreads is a hump-shaped function of the time to maturity \( T - t \).

One debated issue in Merton’s model is that there is an implicit fall of the leverage over time (the ratio \( d = Le^{-\gamma(T-t)} / V_t \) tends to 0 as the maturity increases). So as \( T - t \) tends to infinity, the spread evanishes (there can never be a default on a perpetual bond).

Merton’s model has been extended in many ways. For most companies it seems more realistic to assume that there is a target leverage ratio which they try to maintain by issuing or retiring debt. Collin-Dufresne and Goldstein (2001) assume constant leverage, which results in much larger spreads and a term structure in which spreads increase with maturity. Leland and Toft (1996) use a slightly different approach, they assume that default on a coupon date is avoided by issuing new equity. This “option to reduce leverage” also results in much larger spreads than in the basic Merton model.

Another risk measure of interest in Merton’s model is the volatility of the debt value, \( \sigma_D \), which can be easily derived by applying Ito’s lemma to \( D(V, t) \), this will yield

\[
\sigma_D = \frac{\partial D}{\partial V} \sigma \quad (16)
\]

\[
= N(-d_1) \frac{V}{D} \sigma \quad (17)
\]

\[
= \frac{N(-d_1)}{N(-d_1) + \frac{Le^{-\gamma(T-t)}}{V_t} N(d_2)} \sigma \quad (18)
\]

which is strictly less than the total assets volatility \( \sigma \). While the spread is the promised risk premium over the remaining life of the bond, the volatility represents the default risk over the next instant. We will not prove this, but it is worth to mention that when the firm is solvent...
(d1) the volatility $\sigma_D$ increases with the maturity unlike the spread which instead tends to zero.

2.2 Implicit recovery and probability of default

It is useful to rewrite the debt price in a way that highlights the recovery rate and the default probability. This can be easily done by writing the debt value as a safe claim ($Le^{-r(T-t)}$) minus a put option (as previously done) and rearranging as follows:

$$D_t = Le^{-r(T-t)} - \left[ Le^{-r(T-t)} N(-d_2) - V_t N(-d_1) \right]$$

if we now go back to our pricing problem we should be able to rewrite

$$D_t = \left[ 1 - Q(V_T < L) \left( 1 - \frac{V_t}{Le^{-r(T-t)} Q(V_T < L)} \right) \right]$$

Last, denote as $\delta_T$ the fractional recovery $V_T/L$ and as $p^* = Q(V_T < L)$ the probability of default, therefore

$$D_t = Le^{-r(T-t)} \left[ 1 - p^* \left( 1 - \frac{E_Q(V_T 1_{\{V_T < L\}} e^{-r(T-t)}}{p^*} \right) \right]$$

where in the last line we have used a property of the conditional expectation of a random variable $X$, $E_P(X \mid B) = E_P(X 1_{\{B\}})/P(B)$.

2.3 Debt issues of different seniority level

Imagine that we keep the same setting as in Merton and add another debt class with seniority provisions and face value $L_s$. Thus, we have, a senior class, a junior one with face value $L_j$
and the equity holders. Let’s assume that all debt classes have the same maturity at time $T$. If APR applies, the pricing of senior debt is not affected by adding more junior classes. Therefore we can focus on the pricing of junior debt. The payoff at maturity of the senior and junior classes, $D_{s,T}$ and $D_{j,T}$ respectively, is:

$$D_{s,T} = \min\{L_s, V_T\}$$

$$D_{j,T} = \min\{L_j, V_T - D_{s,T}\}$$

therefore one can easily derive the t-price of the junior debt by calculating the expected discounted payoff (with expectation under the risk neutral $Q$ and discounting using the risk free asset) of

$$D_{j,T} = F_j \mathbb{1}_{\{V_T \geq L\}} + (V_T - L_s) \mathbb{1}_{\{L_s \leq V_T < L\}}.$$

Another way of tackling the problem of pricing the junior class is by noticing that the junior debt can be written as

$$D_{j,T} = D_T(L) - D_{s,T}$$

$$= \min\{L, V_T\} - \min\{L_s, V_T\}$$

where $D(L)$ is the total debt with face value $L := L_j + L_s$ (therefore it is the sum of the junior and the senior debt). The t-price of the junior debt is simply given by the difference between the price of a debt claim with face value $L$ and a debt claim with face value $L_S$, both these claims can be priced straightforwardly by using Merton’s approach.

Last it is not difficult to prove that the payoffs in equation 2.3 and 22 are the same.
**Exercise**

**Ex. 1** Show that the asset value is invariant to the capital structure of the firm (this is known as the Modigliani-Miller theorem, that is, “the irrelevance of the capital structure in a frictionless market”)

**Ex. 2** Derive the instantaneous volatility of the equity and show that it is greater than $\sigma$.

**Ex. 3** Within the setting of Merton’s model, price a junior debt claim, with face value $L_j$ (total debt face value $L$). In particular, calculate the price at time $t$, by using the junior debt $T$–payoff as defined in equation 2.3.
Bibliography

