A Portfolio Approach to Investment and Annuitization During Retirement*

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Abstract

We apply Merton(1969) to the investment allocation decision of individuals in retirement who can invest in both equities and annuities. We derive optimal switching rules between equities and annuities during retirement. Our model predicts a critical age at which an individual ceases to hold any equity. This age does not depend on the volatility of markets nor on the risk aversion of individuals, only on the difference between the *equity premium* and the *annuity premium*, the excess return to annuities because of cohort mortality. While the cross-over age does not depend on relative risk aversion, the speed of approach to the cross-over age does depend on relative risk aversion. The bequest motive influences both the level of consumption and the share of equity. Our results suggest a maximal annuitization age should be flexible as mortality improves. The UK's maximum annuitization age of 75 years appears at least five years too low given current mortality experience. We also calculate welfare losses from mandatory annuitization at earlier ages and find that consumer detriment can be substantial. For moderate levels of risk aversion, a typical individual annuitizing at age 60 can experience declines in risk-adjusted monetary value of up to 20% relative to an optimal portfolio.

1 Introduction

As individuals live longer, the issue of how they invest during their retirement becomes more pertinent. In the early 1990s, a 60 year old female in the UK who retires would expect to live over 22 years [8]; longevity for higher social classes is yet longer [3]. In the early 1950s, the average 60 year old female would only expect to live an additional 18 years. Such increases in life expectancy and the period workers spend in retirement are mirrored across other developed countries.

As workers expect to live longer in retirement and have more private pension money available, the question arises how they should invest this money. This paper adapts the neoclassical approach in [7] [10] [9] to the investment decision of retired individuals. As in [12], longevity is uncertain but here rational individuals do not annuitize fully. Instead, individuals have a choice between annuities and equity products and choose

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consumption and portfolios to maximize their welfare as well as to provide for their estates.

The US Mutual Fund Vanguard [2] had suggested that a typical retiree just below 60 should have 60% of assets in stocks. On the other hand, an average worker wishing to retire early in the UK will typically have to annuitize at retirement and the few equity-linked annuity options available mean that this worker will probably end up investing mainly in gilt-linked fixed annuities. If the worker's optimal strategy is to invest in equities but is constrained by regulations or product designs to invest largely in gilts, the value of saving to the worker will be diminished; this has feedback effects to the incentives of workers to build up capital before retirement.

The magnitude of the equity premium over investment in bonds is considerable. [6] estimates it at 6% for the U.S. Historical data for the U.K. indicate that it is of similar if not greater magnitude [11]. At the same time, annuities yield a rate of return which can be considerably higher than that on bonds because of *mortality drag*, the phenomenon whereby returns of capital on those who die is allocated to the living members of a cohort. As mortality rates increase, mortality drag increases and returns for the living cohort rise. At some point, the returns from holding an annuity exceed the mean return on equity and are much less risky so that all investors without a bequest motive switch eventually into annuities.

In Fig. (1), we show the top rates on UK level annuities compared to gilt returns. The figure shows the annuity premium of roughly 3% for a 65 year old man which comes from purchasing an annuity. Since women have longer life expectancy than men, the annuity premium for women will be lower (around 2% for a 65 year old woman).

The annuity premium increases rapidly with age as shown in Fig. (2) which plots level annuity yields as a function of age for open market option annuities on May 9, 1999. We note that the empirical annuity premium in Fig. (1) may be significantly lower than an annuity premium calculated using population mortality. Differences are due to adverse selection, socioeconomic selection and some cost/profit loadings. Because they are effectively commoditized and have low servicing costs, gilt-backed annuities tend to have low commissions and the better quoted annuity rates have relatively low implied profit margins. Nevertheless, rational individuals will make decisions based on marginal yields from annuities and any selection or cost loadings on annuity serve to delay the annuity purchase decisions of a rational investor.

The paper is structured as follows. The next section lays out a simple model with plain vanilla gilt-linked annuities and equities only. Investors are assumed not to want to leave bequests. In the absence of a bequest motive, no investor will hold bonds when there are gilt-linked annuities. The assumption of no bequests is made because we wanted to isolate the effects of annuitization and look at only the assets which are managed for the purpose of providing retirement income. In Section 3, we develop a method for calculating the risk-adjusted monetary cost to investors of not following the optimal strategy; we apply this method to look at the costs of early annuitization and to investment of pension money entirely in equities. A final section concludes.

2 Model

We let x be the age of a retired investor with no income. An investor aged x is assumed to die with some probability $\delta(x)$. Investors can invest in equities or annuities. Equities yield a mean return of μ but this return is random with standard deviation σ. The

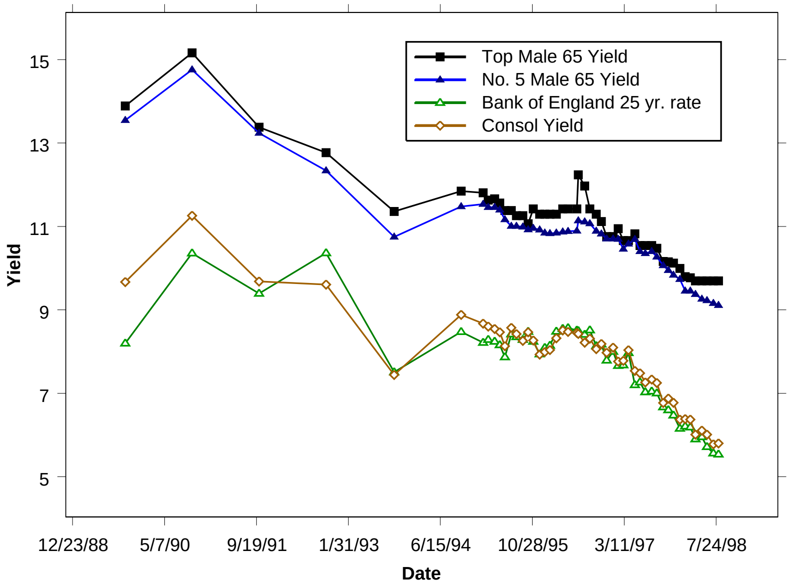

Figure 1: Male annuity yields from 1989 to 1998.

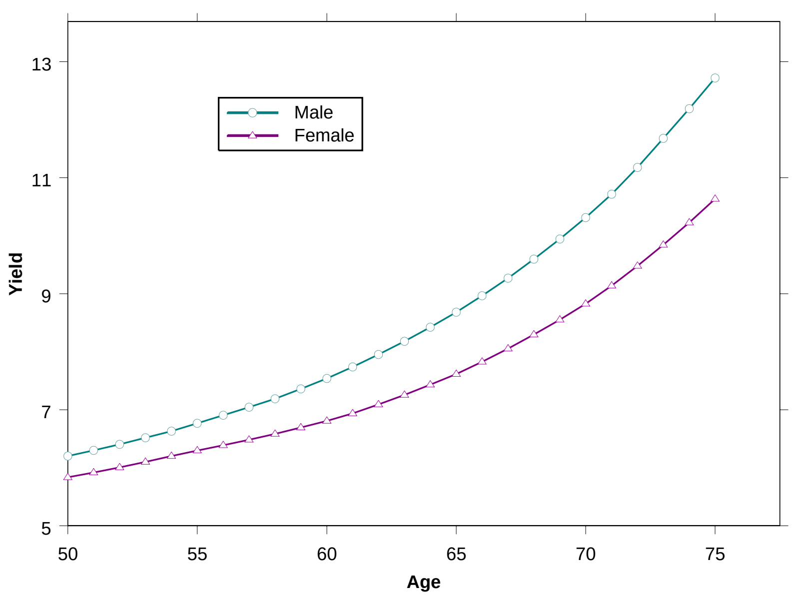

Figure 2: Age structure of annuity yields for men and women on May 9, 1999

annuities are tontines which produce a yield to those alive of $r + \delta(x)$. With perfect markets, the payment structure of the annuity is not relevant as individuals who desire different payment structures can borrow money each period at interest rate r while paying for life assurance to secure the principal at rate $\delta(x)$.[1]

The stochastic differential equation which describes the evolution of assets $A(x)$ is:

$$dA = \left(\left[w(x)\mu + (1 - w(x))(r + \delta(x))\right] A - c\right) dx + \sigma A w(x) dW_x \tag{1}$$

where $w(x)$ is the share of assets invested in risky securities at age x, $c(x)$ is the consumption at age x and dW_x is a Brownian motion increment.

Investors receive value from consumption so that the expected discounted utility of an investor aged x is:

$$V(A, x) = \sup_{c, 0 \le w \le 1} E\left[\int_x^\infty e^{-\int_y^\infty (\rho + \delta(z))dz} \left[u\left(c(y)\right)\right] dy\right] \tag{2}$$

where u is an instantaneous utility function, ρ is a discount rate and the supremum in Eq. (2) is taken over all consumption and portfolio strategies. The theory of dynamic programming [5] allows us to express the present value in Eq. (2) in terms of the following Hamilton-Bellman-Jacobi Equation:

$$\begin{aligned}(\rho + \delta(y))V(A, y) \quad &= \sup_{c, 0 \le w \le 1} [u(c(y)) + \frac{1}{2}\sigma^2 w(x)^2 A^2 V_{AA} \\ &\quad V_A \left[w(x)\mu + (1 - w(x))(r + \delta(x))A - c\right] + V_x] \end{aligned} \tag{3}$$

Taking the supremum in Eq. (3) with respect to the portfolio strategy $w(x)$ in the absence of bequests, we obtain the interior portfolio solutionfor equity share of:

$$w(x) = \frac{-V_A}{V_{AA}A} \frac{\mu - r - \delta(x)}{\sigma^2} \tag{4}$$

The portfolio shares in Eq. (4) depend on the risk aversion of the individual (the left-hand side term) and the risk-adjusted equity premium (the right-hand side term). The risk-adjusted equity premium declines with age because the return to annuitization rises with the death of other members of the cohort.

For consumption the individual equates the marginal utility of consumption with the marginal value associated with savings:

$$u_c = V_A \tag{5}$$

Eq. (4) and Eq. (5) are substituted back into Eq. (3) and this nonlinear partial differential equation is then solved for $V(A, y)$ given boundary conditions which determine the value of being bankrupt as well as the marginal value of an additional unit of wealth for someone infinitely rich.

While closed-form solutions of the partial differential equation Eq. (3) are rare, Appendix A shows that the particular solution in [7] can be extended to our problem where the utility function is constant relative risk aversion:

[1] In practice, one impact of imperfect markets may be more upfront payment of annuities and this is one potential most of the UK market involves flat annuities. Given that providers are able to structure their annuity payments in this way, imperfect markets would be relevant if the elderly wanted to go into debt early in retirement based on future income and were unable to do so. We do not think this is likely as the elderly do not ordinarily have wage income.

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}. \quad (6)$$

for $\gamma > 0$ where γ is relative risk aversion .

Appendix A shows the following interior solution for $w(x)$:

$$w(x) = \frac{1}{\gamma} \frac{\mu - r - \delta(x)}{\sigma^2} \quad (7)$$

The model predicts a critical age x at which an individual ceases to hold any equity. This age does not depend on the volatility of markets or risk aversion; the swtiching age only depends on the difference between the *equity premium* and the *annuity premium*, the excess return to annuities because of cohort mortality.[2] We show below that the cross-over age does not depend on relative risk aversion but the speed of approach to the cross-over age does depend on relative risk aversion.

In Fig. (3), we show the optimal asset allocation for insured men and women using the latest pensioner mortality tables assuming relative risk aversion of 2, an equity premium of 7% and a standard deviation of equity returns of 0.25. A typical 80 year old life office pensioner with moderate risk aversion would still be holding over 10% of her assets in equity.[3]

The population English Life Tables have heavier mortality and hence predict a faster switch into annuities. Indeed, the 1980-82 English Life Tables predict an optimal switching age for men of 72-73 which was below the legal maximum at the time. However, because of improvements in mortality and potentially also a change in the mean equity premium, the formula Eq. (7) calls for an increase in the mandatory annuitization age.

3 Welfare Analysis

We now consider two polar cases. The first is an individual who invests only in annuities and the second is the individual who invests only in equities and compare these with the optimal solution above. To capture the costs of these polar cases, we calculate what level of assets an individual would need to have to be equally well off as an individual following an optimal strategy. In Appendix A, we show that the solution to the problem is:

$$V(A, x) = h(x) \frac{A^{1-\gamma}}{1-\gamma} \quad (8)$$

where $h(x)$ is determined through solution of an ordinary differential equation. Similarly, an individual who invests only in annuities has value:

$$V_a(A_a, x) = h_a(x) \frac{A_a^{1-\gamma}}{1-\gamma} \quad (9)$$

where $h_a(x)$ solves a nonlinear ordinary differential equation defined in Appendix A. An individual who invests only in annuities would need more assets to be equally well

[2]The relevant mortality rate is that charged by the insurance company issuing the annuity. Private information about individual mortality will influence consumption behaviour but not asset allocation with constant relative risk aversion utility.

[3]Portfolio share here includes the reserves held by the insurance company. Bequest motives may lead to higher equity shares, depending on risk aversion about the size of the bequest.

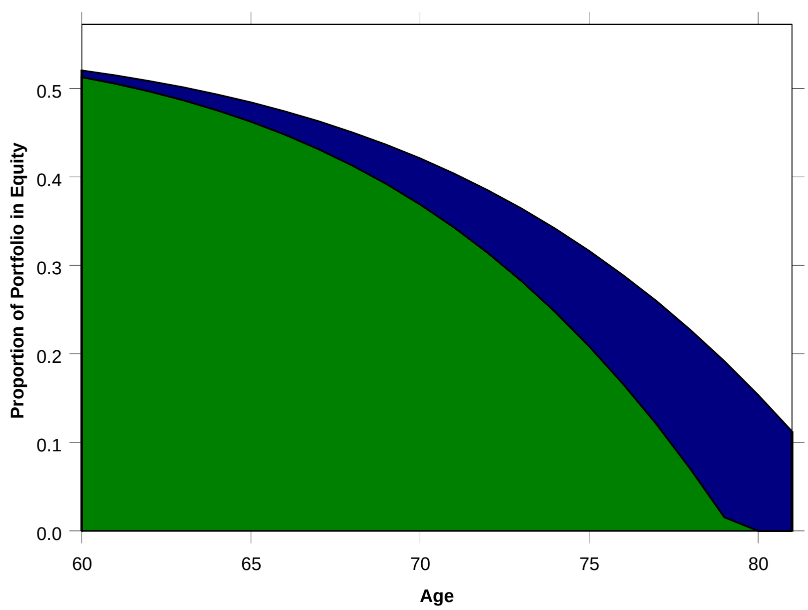

Figure 3: Portfolio allocation during retirement for typical life office pensioner mortality.

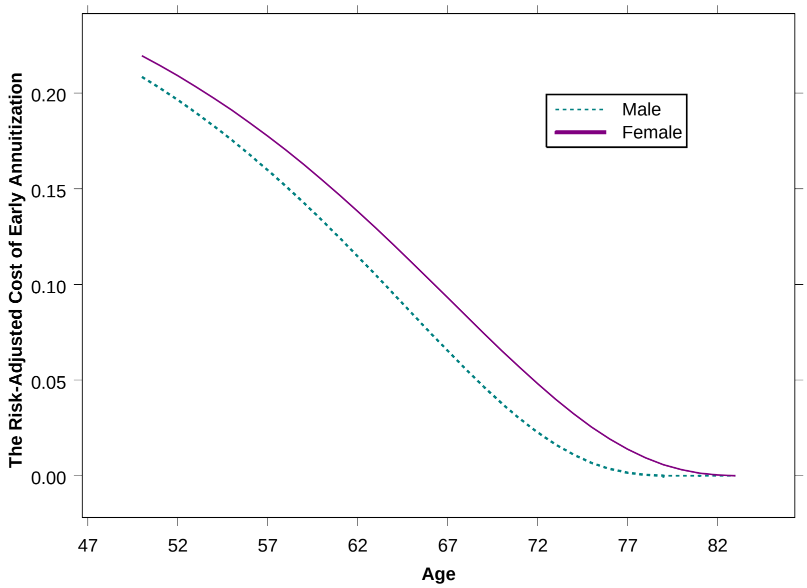

Figure 4: The costs of early annuitization as expressed in fraction of annuitant assets.

off. Specifically, the person would be equally well off with a level of assets A_a which satisfies:

$$\left[\frac{h_a(x)}{h(x)}\right]^{\frac{1}{1-\gamma}} A_a = A. \tag{10}$$

The ratio $\left[\frac{h_a(x)}{h(x)}\right]^{\frac{1}{1-\gamma}}$ hence has a monetary interpretation. It explicitly measures the cost of market features such as product design or regulatory constraints which keep portfolio allocations away from their optimal level. The closer $h_a(x)$ is to $h(x)$, the smaller the percentage cost of these constraints in terms of retirement assets. Similarly, Appendix A details an equation for h_e which applies to equity only retirement investment and the ratio $\frac{h_e(x)}{h(x)}$ measures how efficient investing an entire portfolio in equities. We note that these measures correct for both risk and the optimal consumption decisions of individuals and hence present a simple measure of the costs associated with regulatory or other constraints in the market.

As an example, we have used this method to compute costs associated with early annuitization where $\mu = 0.14$, $\gamma = 1.25$, $r = 0.07$, $\sigma^2 = 0.25$ using the new CMI insured mortality tables[4]. The results are shown in Fig. (4)

The cost to a woman aged 60 of early annuitization and moderate risk aversion is 15.5% of her assets. The CMI tables are base tables and do not take into account mortality improvements. If we build in a $(0.85)^{\frac{1}{15}}$ per year improvement factor, this figure rises to about 16.3%. Such costs are significantly higher than UK profit loadings on annuities. However, individuals who invest in equities and forego annuities would

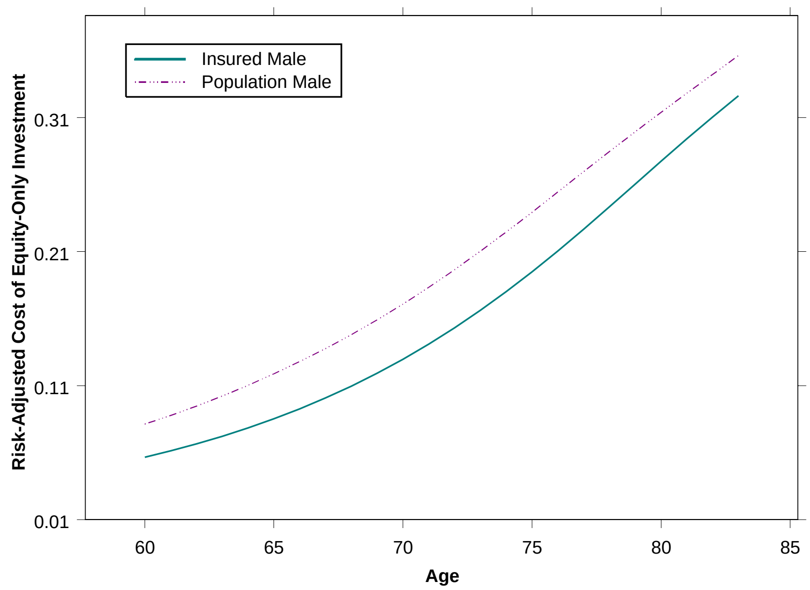

Figure 5: The costs of equity-only investment as expressed in fraction of annuitant assets.

also suffer high costs. These costs are shown in Fig. (5) for a typical insured pensioner male and for a typical member of the population with English Life Tables 15 mortality. Unlike in Fig. (4) where the costs of annuitization are decreasing with age, the costs of equity investment rise with age as one is unable to make advantage of the rising annuity premium with age.

4 Conclusion

In this paper, we have derived the optimal investment decisions of an individual who retires with a given level of assets and decides to invest in annuities and equities in order to provide income in retirement. We have found that the optimal portfolio decision depends on risk aversion but that all individuals switch into annuities as they get older. This date depends only on average bond and equity returns and cohort mortality. We find that the risk-adjusted losses from early annuitization can be significant and for comparison purposes are higher than current estimates of cost loadings on annuities and other aspects of costs. On the other hand, there are also significant costs associated with pure equity investment and not annuitizing.

Appendix A: Solution of the Model without Bequests

We let $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$. Eq. (3) in the absence of bequests is:

$$
\begin{aligned}
(\rho+\delta(x))V(A,x) &= \frac{(\gamma)\,V_A^{\frac{\gamma-1}{\gamma}}}{1-\gamma}+ &(11)\\
&+(r+\delta(x))V_A A &(12)\\
&-\frac{1}{2}\frac{(\nu(x)-r-\delta(x))^2}{\sigma^2}\frac{V_A^2}{V_{AA}}+V_x &(13)
\end{aligned}
$$

where $\nu(x)=\mu+\max(r+\delta-\mu,0.0)$. In this case, in a manner analogous to [7], we wish to show a solution has the form:

$$
V(A,x)=h(x)\frac{A^{1-\gamma}}{1-\gamma} \tag{14}
$$

We note that, using Eq. (5), optimal consumption is:

$$
c(x)=V_A^{-\frac{1}{\gamma}}=\left[h(x)^{-\frac{1}{\gamma}}\right]A(x) \tag{15}
$$

Substitution of Eq. (14) into Eq. (11) yields:

$$
\begin{aligned}
(\rho+\delta(x))h(x)\frac{A^{1-\gamma}}{1-\gamma} &= \frac{(1-\gamma)\,A^{1-\gamma}h(x)^{\frac{\gamma}{\gamma-1}}}{1-\gamma} &(16)\\
&+(r+\delta(x))A^{1-\gamma}h(x) &(17)\\
&-\frac{1}{2}\frac{(\nu(x)-r-\delta(x))^2}{\sigma^2(-\gamma)}h(x)A^{1-\gamma}+h'(x)\frac{A^{1-\gamma}}{1-\gamma} &(18)
\end{aligned}
$$

which implies that:

$$
h'(x)=\tau(x)h(x)-\gamma h(x)^{\frac{\gamma-1}{\gamma}} \tag{19}
$$

where $\tau(x)=(\delta(x)+\rho)-(1-\gamma)\left(r+\delta(x)+\frac{(\nu(x)-r-\delta(x))^2}{2\sigma^2(\gamma)}\right)$.

Now consider the case where the investor chooses to invest entirely in annuities. Eq. (3) is:

$$
\begin{aligned}
(\rho+\delta(x))h(x)\frac{A^{1-\gamma}}{1-\gamma} &= \frac{(1-\gamma)\,A^{1-\gamma}h(x)^{\frac{\gamma}{\gamma-1}}}{1-\gamma} &(20)\\
&+(r+\delta(x))A^{1-\gamma}h(x) &(21)\\
&+h'(x)\frac{A^{1-\gamma}}{1-\gamma} &(22)
\end{aligned}
$$

The candidate solution:

$$
V_a(A,x)=h_a(x)\frac{A^{1-\gamma}}{1-\gamma} \tag{23}
$$

solves Eq. (20) where $h_a(x)$ satisfies the nonlinear differential equation:

$$
h_a'(x)=\tau_a(x)h_a(x)-\gamma h_a(x)^{\frac{\gamma-1}{\gamma}} \tag{24}
$$

where $\tau_a(x)=\rho+\delta(x)-(1-\gamma)\,(r+\delta(x))$.

If the individual chooses to invest entirely in equities, similar steps lead to the differential equation:

$$h'_e(x) = \tau_e(x)h_e(x) - \gamma h_e(x)^{\frac{-\gamma}{1-\gamma}} \tag{25}$$

where $\tau_a(x) = \rho + \delta(x) - (1-\gamma)(\mu - \frac{\sigma^2}{2}\gamma)$

Eqs. (19), (24) (25) are nonlinear ordinary differential equations; because they are Bernoulli equations, they can be solved explicitly through the transformation $f(x) = h(x)^{\frac{1}{\gamma}}$ which leads to a linear differential equation in the transformed variable $f(x)$ ([1], p. 20). Using this transformation, we obtain:

$$f'(x) = \frac{1}{\gamma}\tau(x)f(x) - 1 \tag{26}$$

so that:

$$f(x) = e^{\frac{-\int_x^\infty \tau(p)dp}{\gamma}}[\bar{c} + \int_x^\infty e^{\frac{\int_z^\infty \tau(z)dz}{\gamma}}dy] \tag{27}$$

If the boundary conditions are such that at an infinitely old age, there is no value (no bequest value), $\lim_{x\to\infty} f(x) = 0$ so $\bar{c} = 0$.Then:

$$f(x) = \int_x^\infty e^{\frac{-\int_x^z \tau(z)dz}{\gamma}}dz] \tag{28}$$

Similarly:

$$f_a(x) = \int_x^\infty e^{\frac{-\int_x^z \tau_a(z)dz}{\gamma}}dz] \tag{29}$$

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