Integrating exponentials over a simplex:

a surprising link with exponential Brownian motion

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For any nonsingular matrix

$$V = (v_1 \quad \cdots \quad v_n) \in \mathbb{R}^{n \times n},$$

let

$$K(V) = \mathrm{conv}\{0, v_1, \ldots, v_n\}.$$

We want to integrate exponentials over the simplex $K(V)$:

$$\frac{1}{|\det V|} \int_{K(V)} \exp(a^T y)\, dy$$

For the exponential function,

$$\exp[a_0, \ldots, a_n] = \int_{S_n} \exp(\sum_{k=0}^{n} t_k a_k)\, dt_1 \cdots dt_n,$$

where the domain of integration is the normalized simplex:

$$S_n = \{t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}_+^n \colon \sum_{k=1}^{n} t_k \le 1\}$$

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Then

$$\frac{1}{|\det V|} \int_{K(V)} \exp(a^T y)\, dy$$

is equal to

$$\exp[0, (V^T a)_1, \ldots, (V^T a)_n].$$

[$(V^T a)_j$ is jth component of $V^T a$.]

Hermite–Genocchi: Let $f \in C^{(n)}(\mathbb{R})$ and let $a_0, a_1, \ldots, a_n$ be (not necessarily distinct) real numbers. Then

$$f[a_0, a_1, \ldots, a_n]$$
$$= \int_{S_n} f^{(n)}(t_0 a_0 + t_1 a_1 + \cdots + t_n a_n)\, dt_1 \cdots dt_n,$$
$$= \int_0^1 dt_1 \cdots \int_0^{1-\sum_{k=1}^{n-1} t_k} dt_n \;\; f^{(n)}(\sum_{k=0}^{n} t_k a_k)$$

integrating over the simplex

$$S_n = \{t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}_+^n\colon\; \sum_{k=1}^{n} t_k \leq 1\}$$

and

$$t_0 = 1 - \sum_{k=1}^{n} t_k.$$

If

$$V = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

then

$$\int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} dx_1 \exp\left(\sum_{k=1}^n a_k x_k\right)$$
$$= \exp[0, a_n, a_n + a_{n-1}, \ldots, a_n + a_{n-1} + \cdots + a_1].$$

Exponential [Geometric] Brownian Motion:

$$S(t) = e^{(r-\sigma^2/2)t+\sigma W_t}, \qquad t \geq 0,$$

where W_t Brownian motion. This is the *risk-neutral* form, i.e. written in this way to ensure $\mathbb{E}S(t) = e^{rt}$.

Fixing t,

$$S(t) = e^{(r-\sigma^2/2)t+\sigma\sqrt{t}Z_t}, \qquad t \geq 0,$$

where Z_t is $N(0,1)$ [Gaussian random variable].

Time average is

$$A(T) = \frac{1}{T}\int_0^T S(t)\,dt, \qquad T > 0.$$

Asian Options are insurance policies that enable an asset to be bought for its time-averaged price – a smoothing functional.

Empirical discovery: $S(T)$ and $A(T)$ typically highly correlated – coefficient $\approx$ 0.85. Hence we calculate

$$R = \frac{\mathbb{E}S(T)A(T) - \mathbb{E}S(T)\mathbb{E}A(T)}{\sqrt{\mathbb{V}S(T)\mathbb{V}A(T)}}.$$

Others have tried parts of this calculation. Even with Maple, it's a **mess** without divided differences.

First link with divided differences:

$$\begin{aligned}\mathbb{E}A(T) &= \frac{1}{T}\int_0^T \mathbb{E}S(t)\,dt \\ &= \frac{e^{rT}-1}{rT} \\ &= \exp[0, rT].\end{aligned}$$

Coincidence? Let's try another. We need a simple Lemma:

$$\mathbb{E}S(a)S(b) = e^{a(r+\sigma^2)}e^{br}, \qquad \text{for } 0 \le a \le b.$$

Key point here is independent Gaussian increments property for Brownian motion, i.e. if we write

$$W_b = W_a + (W_b - W_a)$$

then $W_a \sim N(0, a)$, $W_b - W_a \sim N(0, b - a)$ and they're independent. Also need $\mathbb{E}\exp(\lambda Z) = \exp(\lambda^2/2)$ for $Z \sim N(0, 1)$.

Then

$$\mathbb{E}S(a)S(b)$$

$$\begin{aligned}
&= \mathbb{E}e^{(r-\sigma^2/2)a+(r-\sigma^2/2)b+2\sigma W_a+\sigma(W_b-W_a)} \\
&= e^{(r-\sigma^2/2)a+(r-\sigma^2/2)b}\left(\mathbb{E}e^{2\sqrt{a}Z_1}\right)\left(e^{\sigma\sqrt{b-a}Z_2}\right) \\
&= e^{(r-\sigma^2/2)a+(r-\sigma^2/2)b+2\sigma^2 a+\sigma^2(b-a)/2} \\
&= e^{a(r+\sigma^2)}e^{br},
\end{aligned}$$

where Z_1, Z_2 are independent $N(0,1)$.

Then

$$
\begin{aligned}
\mathbb{E}S(T)A(T) &= T^{-1}\int_0^T \mathbb{E}S(t)S(T)\,dt \\
&= T^{-1}\int_0^T e^{(r+\sigma^2)t}e^{rT}\,dt \\
&= \frac{e^{(2r+\sigma^2)T} - e^{rT}}{(r+\sigma^2)T} \\
&= \mathrm{exp}[rT, (2r+\sigma^2)T].
\end{aligned}
$$

Now it's known that

$$\mathbb{E}\left(A(T)^2\right)$$

is given by

$$\frac{2e^{(2r+\sigma^2)T}}{(r+\sigma^2)(2r+\sigma^2)T} + \frac{2}{rT^2}\left(\frac{1}{2r+\sigma^2} - \frac{e^{rT}}{r+\sigma^2}\right).$$

This *is* a divided difference:

$$\mathbb{E}\left(A(T)^2\right) = 2\,\mathrm{exp}[0, rT, (2r+\sigma^2)T].$$

Why? $\mathbb{E}(A(T))^2$

$$
\begin{aligned}
&= T^{-2}\int_0^T\Big(\int_0^T \mathbb{E}S(t_1)S(t_2)\,dt_2\Big)\,dt_1 \\
&= 2T^{-2}\int_0^T\Big(\int_0^{t_1} \mathbb{E}S(t_1)S(t_2)\,dt_2\Big)\,dt_1 \\
&= 2T^{-2}\int_0^T\Big(\int_0^{t_1} e^{r(t_1+t_2)}e^{\sigma^2 t_2}\,dt_2\Big)\,dt_1 \\
&= 2T^{-2}\int_0^T e^{rt_1}\Big(\frac{e^{(r+\sigma^2)t_1}-1}{r+\sigma^2}\Big)\,dt_1 \\
&= \frac{2}{(r+\sigma^2)T}\Big[\mathrm{exp}[0,(2r+\sigma^2)T]-\mathrm{exp}[0,rT]\Big] \\
&= 2\,\mathrm{exp}[0,rT,(2r+\sigma^2)T],
\end{aligned}
$$

Now we *expect* to see divided differences:

$$
\begin{aligned}
&\mathbb{E}S(T)A(T)-\mathbb{E}S(T)\mathbb{E}A(T) \\
&\quad= \mathrm{exp}[rT,(2r+\sigma^2)T]-e^{rT}(e^{rT}-1)/(rT) \\
&\quad= \mathrm{exp}[rT,(2r+\sigma^2)T]-\mathrm{exp}[rT,2rT] \\
&\quad= \sigma^2T\,\mathrm{exp}[rT,2rT,(2r+\sigma^2)T],
\end{aligned}
$$

and for the variance

$$
\begin{aligned}
&\mathbb{V}S(T) \\
&\quad = \mathbb{E}(S(T)^2) - (\mathbb{E}S(T))^2 \\
&\quad = e^{(2r+\sigma^2)T} - e^{2rT} \\
&\quad = \sigma^2 T \,\mathsf{exp}[2rT, (2r+\sigma^2)T].
\end{aligned}
$$

Finally, the correlation coefficient R is given by

$$\frac{\exp[rT, 2rT, (2r+\sigma^2)T]}{\sqrt{2\exp[2rT, (2r+\sigma^2)T]\exp[0, rT, 2rT, (2r+\sigma^2)T]}}.$$

But *why* do these iterated integrals lead to divided differences? The answer is the *Hermite–Genocchi* formula:

Hermite–Genocchi: Let $f \in C^{(n)}(\mathbb{R})$ and let $a_0, a_1, \ldots, a_n$ be (not necessarily distinct) real numbers. Then

$$\begin{aligned} & f[a_0, a_1, \ldots, a_n] \\ &= \int_{S_n} f^{(n)}(t_0 a_0 + t_1 a_1 + \cdots + t_n a_n)\, dt_1 \cdots dt_n, \\ &= \int_0^1 dt_1 \cdots \int_0^{1-\sum_{k=1}^{n-1} t_k} dt_n \; f^{(n)}(\sum_{k=0}^{n} t_k a_k) \end{aligned}$$

integrating over the simplex

$$S_n = \{t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}_+^n : \sum_{k=1}^{n} t_k \le 1\}$$

and

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$$\int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} dx_1 \exp\left(\sum_{k=1}^{n} a_k x_k\right)$$
$$= \exp[0, a_n, a_n + a_{n-1}, \ldots, a_n + a_{n-1} + \cdots + a_1].$$

Now we can compute higher moments of $A(T)$. Yor and Oshanin *et al* showed that, for the easier case $r = \sigma^2/2$, the mth moment $\mathbb{E}\,(A(T))^m$ is the sum

$$\frac{\Gamma(m)}{(rT)^m \Gamma(2m)} \left(c_m + \sum_{\ell=0}^{m} \binom{2m}{\ell} (-1)^\ell e^{rT(m-\ell)^2} \right),$$

where $c_m = -\frac{1}{2}(-1)^m \binom{2m}{m}$. We find the (equivalent) nice formula

$$m! \exp[0, rR, 4rT, 9rT, \ldots, m^2 rT].$$

For us, the assumption $r = \sigma^2/2$ is no longer needed to simplify the algebra – divided difference theory achieves that. We obtain

$$m! \exp[b_0 T, b_1 T, \ldots, b_m T],$$

where

$$b_k = rk + \sigma^2 k(k-1)/2, \quad k \geq 0.$$

Theorem The correlation coefficient

$$\frac{\exp[rT, 2rT, (2r+\sigma^2)T]}{\sqrt{2\exp[2rT, (2r+\sigma^2)T]\exp[0, rT, 2rT, (2r+\sigma^2)T]}}.$$

is **always** $\geq 1/\sqrt{2}$.

Proof: Pólya frequency functions.