Measuring correlation risk

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ADDITIONAL INFORMATION IS AVAILABLE UPON REQUEST.
Introduction:

- Correlation as measure of dependence for pricing complex derivatives.
- Limitations to the use of correlation
- Practical considerations often force use of correlation as dependence-measure
- What is the sensitivity of prices to correlation parameters? *Correlation VaR.*
Outline:

- What is correlation?
- What do we need: distribution of correlation matrices
- How to get perturbations of correlations and hence distributions
- Examples
- Discussion
Correlation

- Linear correlation is most used measure of dependence between random variable $X$ and $Y$ with finite variances.

$$\rho(X, Y) = \frac{\text{Cov}[X, Y]}{\sqrt{\sigma^2[X] \sigma^2[Y]}}$$

- The pitfalls and limitations are investigated in detail in by Embrechts, McNeil, Straumann [1999]
Constraints

For \( n \) random variables \( X_1, X_2, \ldots, X_n \) we have correlations \( \rho_{ij} \) between \( X_i, X_j \)

Correlation matrices must satisfy the following properties \((i, j = 1, \ldots, n)\):

- \([−1, 1]$: \(-1 \leq \rho_{ij} \leq 1\)
- \(\rho_{ii} = 1\)
- \(\rho_{ij} = \rho_{ji}\)
- The matrix has to be positive \emph{semidefinite}
Estimating correlations

- Historical data.
- *Implied* correlations
Perturbing the correlation matrix

To get random sample of correlation matrices we apply four methods:

• Bootstrapping

• Element-wise perturbation

• Perturbation with the help of angle parametrization

• Perturbation of the eigen-values.
Bootstrap method

The bootstrap is a Monte Carlo \textit{resampling} method.

1. Generate $N$ uniform i.i.d. random integers $n_1, n_2, \ldots, n_N \in [1, N]$.

2. Create a sample $\hat{X} \equiv \{X_{n_i}, Y_{n_i}\}_{i=1}^N$

3. Calculate $\hat{\rho}$ using the sample $\hat{X}$.

4. Repeat the previous steps $M$ times with $M$ being a large number.
Perturbing individual correlations

Turkey, Epperlein, Christofides [2003] propose to perturb the correlation matrix locally to a desired target matrix with use of

- Re-ordering
- Cholesky decomposition to localize the perturbation to the last entries in the Cholesky matrix

They obtain an analytical solution for the bounds of a single correlation entry.
Angle representation (TAP)

Proposed by Brigo, Mercurio, Rapisarda [2002].

- Parametrize correlation matrix via *unique* lower-triangular matrix of angles in $[0, \pi]$.
- This angle-representation maps to a Cholesky matrix.
- Automatically satisfies the correlation constraints.
- Angles are found via a robust and efficient procedure.
- Through these angles the space of all correlation matrices is covered.
Angle representation (TAP)

Correlation matrix $\rho = LL^T$:

$$\rho_{ij} = \sum_{k=1}^{N} l_{ik} l_{jk}$$

The elements of the lower-triangular matrix $L$ are parameterized in terms of $N(N-1)/2$ angles $\theta_{ij} \in [0, \pi]$ ($j < i$) with

$$l_{ij} = \begin{cases} 
\cos \theta_{ij} \prod_{k=1}^{j-1} \sin \theta_{ik} & j < i \\
\prod_{k=1}^{i-1} \sin \theta_{ik} & j = i 
\end{cases}$$
Angle representation (TAP)

Simple algorithm

1. Generate random angles $\theta_{ij}$ around the base angles $\theta_{ij}^0$ with some distribution $\pi(\theta_{ij} | \theta_{ij}^0)$, which is symmetric and centered around the base-correlation $\theta_{ij}^0$ for every $i, j$.

2. Historical analysis indicates angles distributed around the mean, with a standard deviation in the order of $\sigma = 5\%$. 
Angle representation (TAP)

We propose two perturbation methods:

- Perturb angles using one standard normal variate $z \sim N(0, 1)$

$$\hat{\theta}_{ij} = \arctan(\tan(\theta_{ij} + \frac{\pi}{2})(1 + \sigma z)) + \frac{\pi}{2}$$

- Perturb angles $\theta_{ij}$ using i.i.d. standard normal variates $z_{ij} \sim N(0, 1)$

$$\hat{\theta}_{ij} = \arctan(\tan(\theta_{ij} + \frac{\pi}{2})(1 + \sigma z_{ij})) + \frac{\pi}{2}$$
Perturbing eigenvalues

Generate random correlation matrices around the base correlation matrix through the *perturbation of eigen-values*.

\[ \rho_{ij} = \sum_{k,l=1}^{N} V_{ik} \Lambda_{kl} V_{lj} \]

where \( \Lambda_{kl} \equiv \lambda_k \delta_{kl} \) and \( \lambda_1 \geq \lambda_2 \geq \ldots \lambda_N \geq 0 \).

Note the constraint \( \sum_{k=1}^{N} \lambda_i = N \).
Perturbing eigenvalues

Algorithms to perturb the eigen-values.

1. Generate $N$ i.i.d random standard normal variates $z_i \sim N(0, \sigma_i)$

2. Compute perturbed eigenvalues $\hat{\lambda}_i = \lambda_i e^{\sigma_i z_i}$

3. Normalize the eigen-values such that $\sum_{k=1}^{N} \hat{\lambda}_i = N$. 
Perturbing eigenvalues

- The other algorithms are variations of the following method

1. Generate random index $K \in [1, \ldots, n]$ according to distribution $p_i \geq 0$ with $\sum_{i=1}^{n} p_i = 1$.
2. Generate an i.i.d. standard normal variable $z \sim N(0, 1)$.
3. Compute the perturbed eigen-value $\lambda_K$

$$\tilde{\lambda}_K = \lambda_K e^{\sigma K z}$$
Perturbing eigenvalues

The distribution $p_i$ could be chosen as follows:

1. Perturb the largest eigen-values more: $p_i = \frac{\lambda_i}{n}$

2. Perturb the eigen-values uniformly: $p_i = \frac{1}{n}$

3. Pick one specific eigen-value: $p_i = \delta_{iK}$
Perturbing eigenvalues

Having perturbed eigenvalues, we define the perturbed correlation matrix via

$$\tilde{\rho}_{ij} = \sum_{k,l=1}^{n} V_{ik} \tilde{\Lambda}_{kl} V_{lj}$$

where $\tilde{\Lambda}_{kl} \equiv \tilde{\lambda}_k \delta_{kl}$ is the diagonal matrix with the perturbed eigen-values $\tilde{\lambda}_i$. To ensure that $\sum_{i=1}^{n} \rho_{ii} = n$ we need to renormalize the random correlation matrix $\tilde{\rho}$:

$$\tilde{\rho}_{ij} = \frac{\tilde{\rho}_{ij}}{\sqrt{\tilde{\rho}_{ii} \tilde{\rho}_{jj}}}$$
Correlation VaR

To compute the correlation VaR, we perturb the correlation matrix according to some perturbation scheme, which effectively means that we have some distribution $\pi(\rho|\rho^0)$ and compute the density for the portfolio value as a function of the correlation as follows.

$$
\pi(v) = \int \delta(v - V(\rho)) \pi(\rho|\rho^0) d\rho
$$

From this we can then determine the correlation VaR.
Examples

We consider the distribution for two basket options as we perturb the correlation matrix. The basket consists of nine industrial indices, five commodities indices and prices for nine physical commodities.

- In the first example the payoff is defined by

\[
\left( \sum_{i=1}^{n} w_i \frac{X_i(T)}{X_i(0)} - 1 \right)^+ \]

with \( w_i = \frac{1}{n} = \frac{1}{23} \)
Examples

• In the second example we consider a spread option on two baskets, with the first basket being a basket on the industrial indices and the second basket on everything else, i.e. either commodities indices or commodities prices.

The option payoff in this case is given by

\[ \left( \sum_{i=1}^{m} w_1^i \frac{X_i(T)}{X_i(0)} - \sum_{i=m+1}^{n} w_2^{i-m} \frac{X_i(T)}{X_i(0)} \right)^+ \]

with \( w_1^i = \frac{1}{m} \) and \( w_2^i = \frac{1}{n-m} \)
Discussion and conclusions

• Various methods to perturb the correlation matrix and compute the correlation VaR: bootstrap, local perturbation, angle perturbation, eigenvalues perturbation.

• Numerical results for these methods applied to two basket options.

• Different methods lead to different results for the correlation VaR.
Discussion and conclusions

- Use of some parametrized method, be it angles or eigenvalues, gives us a convenient tool to stress-test complicated portfolios of instruments and to do it in a well-defined manner.

- A consistent application of the correlation VaR measures described in this paper improves our understanding of product sensitivity to correlation and provides us with the useful approach to product comparison.
In the following figure we plot the histograms for the value of the first basket option with all different methods discussed in the article. The sample size is 1000.
Simulation of the biggest eigenvalue

Weighted simulation of eigenvalues

Uniform simulation of eigenvalues

Simulation of angles by parallel shift

Simulation of uncorrelated angles
In the following figure we plot the histograms for the value of the second basket option with all different methods discussed in the article. The sample size is 1000.
Simulation of the biggest eigenvalue

Weighted simulation of eigenvalues

Uniform simulation of eigenvalues

Simulation of angles by parallel shift

Simulation of uncorrelated angles
Comparison of the two examples for the three different cases: bootstrap, largest eigenvalue, parallel shift angles. The left-hand graph shows the results for the first basket option. The right-hand graph shows the results for the second basket option.
<table>
<thead>
<tr>
<th>Method</th>
<th>Largest eigenvalue</th>
<th>Weighted eigenvalues</th>
<th>Uniform eigenvalues</th>
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<td>Mean (%)</td>
<td>6.463</td>
<td>6.463</td>
<td>6.455</td>
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<td>Standard deviation (%)</td>
<td>0.113</td>
<td>0.096</td>
<td>0.094</td>
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<td>95% CorVaR (%)</td>
<td>0.183</td>
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<table>
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<th>Method</th>
<th>Parallel shift angles</th>
<th>Uncorrelated angles</th>
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<td>6.459</td>
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<td>Standard deviation (%)</td>
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<td>95% CorVaR (%)</td>
<td>0.226</td>
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Basket 1 with value 6.469. Sample size is 1000.
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<td>95% CorVaR (%)</td>
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<td>Standard deviation (%)</td>
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<td>95% CorVaR (%)</td>
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Basket 2 with value 11.460. Sample size is 1000.
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Sample of correlation angles for different subsets of the data.