Solutions to Revision Lecture

5. (a) \( T = \{2, 3\}, \ C_1 = \{1\}, \ C_2 = \{4, 5, 6\} \).

(b) For \( C_1 \) with the absorbing state 1 we have simply \( \pi_1 = 1 \).

For \( C_2 \) the stationary distribution satisfies the equations

\[
\begin{align*}
\pi_4 &= \frac{1}{2}\pi_4 + \frac{1}{2}\pi_5 + \frac{1}{2}\pi_6, \\
\pi_5 &= \frac{1}{4}\pi_4 + \frac{1}{2}\pi_5, \\
\pi_6 &= \frac{1}{4}\pi_4 + \frac{1}{2}\pi_5.
\end{align*}
\]

From the 2nd and 3rd equations we have that

\[
\pi_5 = \frac{1}{4}\pi_4 + \frac{1}{8}\pi_4 + \frac{1}{4}\pi_6
\]

which implies that

\[
\frac{3}{4}\pi_5 = \frac{3}{8}\pi_4 \quad \text{i.e.} \quad \pi_6 = \frac{1}{2}\pi_4.
\]

Plugging this solution into the 3rd of the first three equations yields

\[
\pi_6 = \frac{1}{4}\pi_4 + \frac{1}{2} \times \frac{1}{2}\pi_4 = \frac{1}{2}\pi_4.
\]

Hence

\[(\pi_4, \pi_5, \pi_6) = k \times (1, 1/2, 1/2) . \]

But since \( \pi_4 + \pi_5 + \pi_6 = 1 \) then \( k = 1/(1 + \frac{1}{2} + \frac{1}{2}) = 1/2 \). Therefore

\[(\pi_4, \pi_5, \pi_6) = (1/2, 1/4, 1/4) . \]

(c) Let

\[ \tilde{f}_i = \mathbb{P}(X_n \in C_1 \text{ for all large enough } n | X_0 = i) . \]

This quantity solves the following equations:

\[
\begin{align*}
\tilde{f}_2 &= \frac{1}{4}\tilde{f}_2 + \frac{1}{4}\tilde{f}_3, \\
\tilde{f}_3 &= \frac{1}{4}\tilde{f}_2 + \frac{1}{4}\tilde{f}_3 .
\end{align*}
\]

They have the solution, \( \tilde{f}_2 = \frac{1}{8}, \tilde{f}_3 = \frac{3}{8} \).

Clearly, if the chain does not eventually hit \( C_1 \) first, starting from \( i \in T \), then it must eventually hit \( C_2 \) (with probability one).

Thus

\[
\begin{align*}
\tilde{g}_2 &= \mathbb{P}(X_n \in C_2 \text{ for all large enough } n | X_0 = 2) = 1 - \tilde{f}_2 = 1 - \frac{1}{8} = \frac{7}{8} , \\
\tilde{g}_3 &= \mathbb{P}(X_n \in C_2 \text{ for all large enough } n | X_0 = 3) = 1 - \tilde{f}_3 = 1 - \frac{3}{8} = \frac{5}{8} .
\end{align*}
\]
(d) For the transient states \( j = 2, 3, \)
\[
\lim_{n \to \infty} p_{ij}(n) = 0. \\
\text{More on limiting dist.}
\]
For \( i = 2, 3, \)
\[
\lim_{n \to \infty} p_{ii}(n) = \tilde{f}_i \pi_1. \\
\text{Nothing very specific in lecture notes.}
\]
For \( i = 2, 3, j = 4, 5, 6, \)
\[
\lim_{n \to \infty} p_{ij}(n) = \tilde{g}_i \pi_j. \\
\text{Some lateral thinking reqd.}
\]
All the limiting values of \( p_{ij}(n) \) are given in the table below.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>( j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 2 )</td>
<td>( \frac{1}{8} )</td>
<td>0</td>
<td>0</td>
<td>( \frac{7}{16} )</td>
<td>( \frac{7}{32} )</td>
<td>( \frac{7}{32} )</td>
<td></td>
</tr>
<tr>
<td>( i = 3 )</td>
<td>( \frac{3}{8} )</td>
<td>0</td>
<td>0</td>
<td>( \frac{5}{16} )</td>
<td>( \frac{5}{32} )</td>
<td>( \frac{5}{32} )</td>
<td></td>
</tr>
</tbody>
</table>
6. (a) Data have been read into S+ as a regular time series object, with the time points indexed at monthly intervals, starting from Jan. 1974. The data have been plotted. A plot of differenced data of period 12 has also been produced.

There is periodicity of period 12 months (so we look at a plot of the differenced data with that period). There appears to be no trend in the original data, and so the seasonal differencing could hopefully be enough. Indeed, the plot of the differenced data could plausibly be stationary.

(b)

\[ p = q = d = 0, \quad P = 2, \quad D = 1, \quad Q = 0, \quad s = 12. \]

\[ \Phi(L^s)(1 - L^s)Y_t = \epsilon_t \]

where \( \Phi(z^s) = 1 - \Phi_1 z^s - \Phi_2 z^{2s} \).

(c) Let \( W_t = (1 - L^{12})Y_t \). Then

\[ W_t = \Phi_1 W_{t-12} + \Phi_2 W_{t-24} + \epsilon_t \]

\[ \Rightarrow Y_t = (1 + \Phi_1)Y_{t-12} + (\Phi_2 - \Phi_1)Y_{t-24} - \Phi_2 Y_{t-36} + \epsilon_t. \]

Hence, from the output, we deduce that the estimated equation is

\[ Y_t = (1 - 0.6892666)Y_{t-12} + (-0.2290979 + 0.6892666)Y_{t-24} + 0.2290979Y_{t-36} + \epsilon_t \]

i.e.

\[ Y_t = 0.31073Y_{t-12} + 0.46017Y_{t-24} + 0.22910Y_{t-36} + \epsilon_t. \]

It is not being suggested that this can be used for prediction: this is just explaining the behaviour within our current data set.

(d) The numbers are the squares of the standard errors (or estimated variances) of the estimates of \( \Phi_1 \) and \( \Phi_2 \) (which are the autoregressive parameters for the differenced data).

None of the p-values indicate that any of the Portmanteau statistics is significant. Also, nothing untoward in respect of the standardized residuals and the a.c.f. of the residuals. Hence, we can endorse the model as providing an adequate fit.

(e) Say something like: Deaths peak about once every 12 months at around January of each year. Pattern of deaths seem to follow a cycle which repeats itself once every 12 months. No long term trend upward or downward over time. Pattern of deaths for each month seems to depend on the number of deaths at the same point in the year for the last 3 years, as well as some noise corrupting factor of constant variability which is, in a sense, unrelated to what happens at other time instants.
7. Model equation:

\[ Y_t = \frac{2}{35} Y_{t-1} + \frac{1}{35} Y_{t-2} + \varepsilon_t \]

\textbf{Spring Term}

Ch. 5

Sec. 5.1

\( \phi(z) = 1 - \frac{2}{35} z - \frac{1}{35} z^2 = 0 \Rightarrow z^2 + 2z - 35 = 0 \)

\[ \Rightarrow z = \frac{-2 \pm \sqrt{4 + 4 \times 1 \times 35}}{2} \]

\[ = \frac{-2 \pm \sqrt{144}}{2} = \frac{-2 \pm 12}{2} = -7, 5. \]

Both roots are bigger than 1 in modulus, hence process is stationary.

(b) Multiplying model equation by \( Y_{t-\tau} \), for \( \tau > 0 \), and taking expectations term by term, yields

\textbf{Spring Term}

Ch. 5

Sec. 5.4

\[ E[Y_{t-\tau} Y_t] = \frac{2}{35} E[Y_{t-\tau} Y_{t-1}] + \frac{1}{35} E[Y_{t-\tau} Y_{t-2}] + E[Y_{t-\tau} \varepsilon_t] \]

\[ \gamma_\tau = \frac{2}{35} \gamma_{\tau-1} + \frac{1}{35} \gamma_{\tau-2}. \]

for an equation linking the auto-covariances

Dividing through by \( \gamma_0 \), yields

\[ \rho_\tau = \frac{2}{35} \rho_{\tau-1} + \frac{1}{35} \rho_{\tau-2}, \quad \tau \geq 1. \]

(c) Conditions to be invoked are \( \rho_0 = 1 \) and \( \rho_1 = \rho_{-1} \). Hence solving Y-W equations for \( \tau = 1 \) and \( \tau = 2 \) yields

\[ \rho_1 = \frac{2}{35} \rho_0 + \frac{1}{35} \rho_{-1} = \frac{1}{35} (2\rho_0 + \rho_1) \]

\[ \Rightarrow 35 \rho_1 = 2\rho_0 + \rho_1 \Rightarrow 34 \rho_1 = 2\rho_0 = 2 \Rightarrow \rho_1 = \frac{1}{17}. \]

\[ \rho_2 = \frac{2}{35} \rho_1 + \frac{1}{35} \rho_0 = \frac{2}{35} \times \frac{1}{17} + \frac{1}{35} \times 1 = \frac{1}{35} \left( \frac{2 + 17}{17} \right) = \frac{19}{595} = 0.031933. \]

(d) General solution of the form:

\[ \rho_\tau = A_1 \left( \frac{1}{5} \right)^\tau + A_2 \left( \frac{-1}{7} \right)^\tau. \]

\[ \tau = 0 \text{ under eq.(23)} \]

\[ A_1 + A_2 = 1 \]
\[ \tau = 1 \]

\[ \frac{A_1}{5} - \frac{A_2}{7} = \frac{1}{17} \]

Therefore

\[ \frac{A_1}{5} \left( 1 - A_1 \right) = 7A_1 - 5 + 5A_1 \]
\[ = \frac{12A_1}{35} \cdot \frac{1}{7} = \frac{12A_1}{35} \Rightarrow \frac{12A_1}{35} = \frac{24}{119} \Rightarrow A_1 = \frac{35 \times 24}{119} = 70 \]

Hence

\[ A_2 = 1 - A_1 = \frac{49}{119} \]

8. Model equation

\[ Y_t = Y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}, \quad t \in \mathbb{Z} \]

(a) 

\[ \hat{y}_T(h) = E[Y_{T+h}|\mathcal{H}_T] \]

(b) Noting that \( \{\epsilon_t\} \) terms for \( t \leq T \) are (implicitly) known, and that those for \( t > T \) are independent of \( \mathcal{H}_T \) and have mean 0, we have:

\[ h = 1 \]

\[ Y_{T+1} = Y_T + \epsilon_{T+1} + \theta_1 \epsilon_T + \theta_2 \epsilon_{T-1} \]

\[ \hat{y}_T(1) = E[Y_{T+1}|\mathcal{H}_T] = y_T + \theta_1 \epsilon_T + \theta_2 \epsilon_{T-1} \]

\[ \text{Ch. 8 sec. 8.7} \]

\[ \text{eq. (8)} \]

\[ \hat{y}_T(2) = E[Y_{T+2}|\mathcal{H}_T] = \hat{y}_T(1) + \theta_2 \epsilon_T \]

\[ \text{Ch. 8 sec. 8.7} \]

\[ \text{eq. (8)} \]

\[ \text{[Notes on \( E[\epsilon_{T+h}|\mathcal{H}_T] \) just below it.] \]

\[ h \geq 3 \]

\[ Y_{T+h} = Y_{T+h-1} + \epsilon_{T+h} + \theta_1 \epsilon_{T+h-1} + \theta_2 \epsilon_{T+h-2} \]

\[ \hat{y}_T(h) = E[Y_{T+h}|\mathcal{H}_T] = \hat{y}_T(h-1) \]

To be useful, need to express the noise terms into (a truncated form of) the infinite autoregressive format.
(c) i. The candidate needs to deduce that

\[ Y_{T+3} = y_T + \epsilon_{T+3} + (1 + \theta_1)\epsilon_{T+2} + (1 + \theta_1 + \theta_2)\epsilon_{T+1} + (\theta_1 + \theta_2)\epsilon_T + \theta_2\epsilon_{T-1}. \]

It does involve a bit of working out. Appropriate credit will be given for the substantive nature of the progress made on this.

\[ \hat{y}_T(3) = y_T + (\theta_1 + \theta_2)\epsilon_T + \theta_2\epsilon_{T-1}. \]

\[ \epsilon_T(3) = Y_{T+3} - \hat{y}_T(3) = \epsilon_{T+3} + (1 + \theta_1)\epsilon_{T+2} + (1 + \theta_1 + \theta_2)\epsilon_{T+1} \]

ii. Spring Term Ch. 8: parameterize eqn. q sec. 8.6

\[ V(3) = \text{var}(\epsilon_T(3)) = [1 + (1 + \theta_1)^2 + (1 + \theta_1 + \theta_2)^2]s^2. \]

(d) The model studied in parts (a)-(c), is exactly this one with \( \theta_1 = -\frac{5}{6} \) and \( \theta_2 = \frac{1}{6}. \)

Hence

\[ V(3) = \left[ 1 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2 \right] \times 1.739 = 1.138889 \times 1.739 = 1.98053 \]

and so

\[ \sqrt{V(3)} = 1.40731. \]

A 95% prediction interval for \( Y_{T+3} \) is given by

\[ \hat{y}_T(3) \pm z_{0.025}\sqrt{V(3)} = 4.9 \pm 1.96 \times 1.40731 = 4.9 \pm 2.75832 = (2.14, 7.66) \]

to 3 s.f.

It is assumed that the estimates of the parameters can be taken to be the true values.

c.f. Spring Term Ch. 9. sec. 8.6 Final eqn.