Solutions to Trial Examination Paper

1. (a) For e.g.
\[ p_{(X,Y)}(0,0) = \frac{1}{4} \neq \frac{1}{4} \times \frac{1}{2} = p_X(0)p_Y(0). \]

Hence \( X \) and \( Y \) are not independent.

(b)
\[ E[X] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1. \]
\[ E[Y] = -1 \times \frac{1}{4} + 0 \times \frac{1}{2} + 1 \times \frac{1}{4} = 0. \]
\[ E[XY] = \sum_{x,y} xy p_{(X,Y)}(x,y) = 0 \times \frac{1}{4} + 1 \times \frac{1}{4} - 1 \times \frac{1}{4} + 0 \times \frac{1}{4} = 0. \]

Therefore
\[ \text{cov}(X,Y) = E[XY] - E[X]E[Y] = 0. \]

Hence \( X \) and \( Y \) are uncorrelated.

2. Rather brief - however, there is much that the student needs to decode from the supplied information to get to this stage! Let \( Z \) be a random variable drawn from the \( N(0,1) \) distribution. Then
\[ P(Y > 202) = P\left( \frac{Y - \mu}{\sigma/\sqrt{n}} > \frac{202 - 200}{15/6} \right) \approx P\left( Z > \frac{2}{15/6} \right) \]
\[ = P(Z > 0.8) = 1 - \Phi(0.8) = 1 - 0.7881 = 0.2119 \]
where the approximation arises by invoking the Central Limit Theorem.

3. (a)
\[ E\left[ \sum_{i=1}^{3} X_i \right] = \sum_{i=1}^{3} E[X_i] = 3 \times \frac{1}{\lambda} = 3 \times \frac{1}{3} = 1. \]

The expected time until 3 births occur is 1 day.

(b)
\[ P(X_{11} > 2) = P(X > 2) = e^{-2\lambda} = e^{-2 \times 3} = e^{-6} = 0.00248. \]

4. (a) i.
\[ c_T = 1 \frac{1}{T} \sum_{t=t+1}^{T} (y_t - \bar{y})(y_{t-\tau} - \bar{y}). \]
\[ r_\tau = \frac{c_\tau}{c_0} = \frac{\sum_{t=\tau+1}^{T} (y_t - \bar{y})(y_{t-\tau} - \bar{y})}{\sum_{t=1}^{T} (y_t - \bar{y})^2} \]

(b) Set \( n = T - \tau \), \( g_t = y_t - \bar{y} \), and \( h_t = y_{t+\tau} - \bar{y} \). With this setting

\[
\left( \sum_{t=\tau+1}^{T} (y_t - \bar{y})(y_{t-\tau} - \bar{y}) \right)^2 = \left( \sum_{t=1}^{T-\tau} (y_{t+\tau} - \bar{y})(y_t - \bar{y}) \right)^2 \\
\leq \sum_{t=1}^{T-\tau} (y_{t+\tau} - \bar{y})^2 \sum_{t=1}^{T-\tau} (y_t - \bar{y})^2 \leq \sum_{t=1}^{T} (y_{t+\tau} - \bar{y})^2 \sum_{t=1}^{T} (y_t - \bar{y})^2.
\]

Replacing \( y_{t+\tau} \) by \( y_t \) due to stationarity, establishes that \( r_\tau^2 \leq 1 \).

5. (a)

\[ \theta(z) = 1 + 2 - \frac{1}{z} - \frac{1}{z^2}. \]

(b) Need to find and classify the roots of \( \theta(z) = 0 \).

\[ \theta(z) = 0 \Rightarrow z^2 - 2z - 5 = 0 \Rightarrow z = \frac{2 \pm \sqrt{4 - 4 \times -5}}{2} = 1 \pm \sqrt{6} \]

Hence the roots are \( z_1 = 1 - \sqrt{6} \) and \( z_2 = 1 + \sqrt{6} \). Both roots lie strictly outside the unit circle of the complex plane. Hence this process is invertible.

6. The sample a.c.f. for \( \{y_t\} \) appears to cut off after lag 2. To justify this, we calculate the approximate 95% probability limits:

\[ \pm 2 \sqrt{\frac{1 + 2(0.4939^2 + 0.5150^2)}{100}} = \pm 0.2841. \]

Clearly, all of the presented sample autocorrelations are within these limits beyond lag 2 (and outside for earlier lags). Also, the sample partial autocorrelations shows oscillating decrease, quite possibly geometric, towards 0. This is evidence to conclude that \( \{y_t\} \) is consistent with an MA(2) process. This corresponds to a stationary ARMA(\( p, q \)) process, with \( p = 0 \) and \( q = 2 \).

7. (a)

\[ \phi(z) = 1 - \frac{1}{4}z - \frac{3}{4}z^2 = 0. \]
\[
\frac{3}{4}z^2 + \frac{1}{4}z - 1 = 0 \Rightarrow z^2 + \frac{1}{3}z - \frac{4}{3} = 0
\]
\[
\Rightarrow \left(z + \frac{4}{3}\right)(z - 1) = 0.
\]
So the roots are at \(z_1 = -\frac{4}{3}\) and \(z_2 = 1\). Since both roots are not strictly larger than 1 in modulus (\(z_2\) causing the problem), then we conclude that the process is not stationary.

8. (a) The model equation can be written as
\[
(1 - \phi L)(1 - L)^2V_t = (1 + \theta L)\epsilon_t
\]
(b) The above can be written as
\[
(1 - \phi L)(1 - 2L + L^2)V_t = (1 + \theta L)\epsilon_t
\]
i.e.
\[
(1 - 2L + L^2 - \phi L + 2\phi L^2 - \phi L^3)V_t = (1 + \theta L)\epsilon_t
\]
i.e.
\[
\{1 - (2 + \phi)L + (1 + 2\phi)L^2 - \phi L^3\}V_t = (1 + \theta L)\epsilon_t
\]
i.e.
\[
V_t = (2 + \phi)V_{t-1} - (1 + 2\phi)V_{t-2} + \phi V_{t-3} + \epsilon_t + \theta \epsilon_{t-1}, \quad t \in \mathbb{Z}.
\]
9. (a) $T = \{2, 3\}$, $C_1 = \{1\}$, $C_2 = \{4, 5, 6\}$.

(b) For $C_1$ with the absorbing state 1 we have simply $\pi_1 = 1$.

For $C_2$ the stationary distribution satisfies the equations

$$
\begin{align*}
\pi_4 &= (1/2)\pi_4 + (1/2)\pi_5 + (1/2)\pi_6 \\
\pi_5 &= (1/4)\pi_4 + (1/2)\pi_6 \\
\pi_6 &= (1/4)\pi_4 + (1/2)\pi_5
\end{align*}
$$

From the 2nd and 3rd equations we have that

$$\pi_5 = \frac{1}{4}\pi_4 + \frac{1}{8}\pi_4 + \frac{1}{4}\pi_5$$

which implies that

$$\frac{3}{4}\pi_5 = \frac{3}{8}\pi_4 \text{ i.e. } \pi_5 = \frac{1}{2}\pi_4.$$

Plugging this solution into the 3rd of the first three equations yields

$$\pi_6 = \frac{1}{4}\pi_4 + \frac{1}{2} \times \frac{1}{2}\pi_4 = \frac{1}{2}\pi_4.$$

Hence

$$(\pi_4, \pi_5, \pi_6) = k \times (1, 1/2, 1/2).$$

But since $\pi_4 + \pi_5 + \pi_6 = 1$ then $k = 1/(1 + \frac{1}{2} + \frac{1}{2}) = 1/2$. Therefore

$$(\pi_4, \pi_5, \pi_6) = (1/2, 1/4, 1/4).$$

(c) Let

$$\tilde{f}_i = \mathbb{P}(X_n \in C_1 \text{ for all large enough } n | X_0 = i) .$$

This quantity solves the following equations:

$$
\begin{align*}
\tilde{f}_2 &= \frac{1}{4}\tilde{f}_2 + \frac{1}{4}\tilde{f}_3, \\
\tilde{f}_3 &= \frac{1}{4} + \frac{1}{4}\tilde{f}_2 + \frac{1}{4}\tilde{f}_3.
\end{align*}
$$

They have the solution, $\tilde{f}_2 = \frac{1}{8}$, $\tilde{f}_3 = \frac{3}{8}$.

Clearly, if the chain does not eventually hit $C_1$ first, starting from $i \in T$, then it must eventually hit $C_2$ (with probability one).

Thus

$$
\begin{align*}
\tilde{g}_2 &= \mathbb{P}(X_n \in C_2 \text{ for all large enough } n | X_0 = 2) = 1 - \tilde{f}_2 = 1 - \frac{1}{8} = \frac{7}{8} , \\
\tilde{g}_3 &= \mathbb{P}(X_n \in C_2 \text{ for all large enough } n | X_0 = 3) = 1 - \tilde{f}_3 = 1 - \frac{3}{8} = \frac{5}{8} .
\end{align*}
$$
(d) For the transient states $j = 2, 3,$

$$\lim_{n \to \infty} p_{ij}(n) = 0.$$ 

For $i = 2, 3,$

$$\lim_{n \to \infty} p_{i1}(n) = \tilde{f}_i \pi_1.$$ 

For $i = 2, 3, j = 4, 5, 6,$

$$\lim_{n \to \infty} p_{ij}(n) = \tilde{g}_i \pi_j.$$ 

All the limiting values of $p_{ij}(n)$ are given in the table below.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 2$</td>
<td>$\frac{1}{8}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{7}{16}$</td>
<td>$\frac{7}{32}$</td>
<td>$\frac{7}{32}$</td>
</tr>
<tr>
<td>$i = 3$</td>
<td>$\frac{3}{8}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{5}{16}$</td>
<td>$\frac{5}{32}$</td>
<td>$\frac{5}{32}$</td>
</tr>
</tbody>
</table>
10. (a) Data have been read into S+ as a regular time series object, with the time
time points indexed at monthly intervals, starting from Jan. 1974. The data have been
plotted. A plot of differenced data of period 12 has also been produced.

There is periodicity of period 12 months (so we look at a plot of the differenced
data with that period). There appears to be no trend in the original data, and
so the seasonal differencing could hopefully be enough. Indeed, the plot of the
differenced data could plausibly be stationary.

(b) \[ p = q = d = 0, \quad P = 2, \quad D = 1, \quad Q = 0, \quad s = 12. \]

\[ \Phi(L^s)(1 - L^s)Y_t = \epsilon_t \]
where \( \Phi(z^s) = 1 - \Phi_1 z^s - \Phi_2 z^{2s}. \)

(c) Let \( W_t = (1 - L^{12})Y_t. \) Then

\[ W_t = \Phi_1 W_{t-12} + \Phi_2 W_{t-24} + \epsilon_t \]
\[ \Rightarrow Y_t = (1 + \Phi_1)Y_{t-12} + (\Phi_2 - \Phi_1)Y_{t-24} - \Phi_2 Y_{t-36} + \epsilon_t. \]

Hence, from the output, we deduce that the estimated equation is

\[ Y_t = (1 - 0.6892666)Y_{t-12} + (-0.2290979 + 0.6892666)Y_{t-24} + 0.2290979Y_{t-36} + \epsilon_t \]

i.e.

\[ Y_t = 0.31073Y_{t-12} + 0.46017Y_{t-24} + 0.22910Y_{t-36} + \epsilon_t. \]

It is not being suggested that this can be used for prediction: this is just explaining
the behaviour within our current data set.

(d) The numbers are the squares of the standard errors (or estimated variances) of the
estimates of \( \Phi_1 \) and \( \Phi_2 \) (which are the autoregressive parameters for the differenced
data).

None of the \( p \)-values indicate that any of the Portmanteau statistics is significant.
Also, nothing untoward in respect of the standardized residuals and the a.c.f. of the
residuals. Hence, we can endorse the model as providing an adequate fit.

(e) Say something like: Deaths peak about once every 12 months at around January of
each year. Pattern of deaths seem to follow a cycle which repeats itself once every 12
months. No long term trend upward or downward over time. Pattern of deaths for
each month seems to depend on the number of deaths at the same point in the year
for the last 3 years, as well as some noise corrupting factor of constant variability
which is, in a sense, unrelated to what happens at other time instants.
11. Model equation:

\[ Y_t = \frac{2}{35} Y_{t-1} + \frac{1}{35} Y_{t-2} + \epsilon_t \]

(a) \[
\phi(z) = 1 - \frac{2}{35} z - \frac{1}{35} z^2 = 0 \Rightarrow z^2 + 2z - 35 = 0
\]

\[
\Rightarrow z = \frac{-2 \pm \sqrt{4 + 4 \times 1 \times 35}}{2} = \frac{-2 \pm \sqrt{144}}{2} = \frac{-2 \pm 12}{2} = -7, 5.
\]

Both roots are bigger than 1 in modulus, hence process is stationary.

(b) Multiplying model equation by \( Y_{t-\tau} \), for \( \tau > 0 \), and taking expectations term by term, yields

\[
E[Y_{t-\tau} Y_t] = \frac{2}{35} E[Y_{t-\tau} Y_{t-1}] + \frac{1}{35} E[Y_{t-\tau} Y_{t-2}] + E[Y_{t-\tau} \epsilon_t]
\]

i.e.

\[
\gamma_\tau = \frac{2}{35} \gamma_{\tau-1} + \frac{1}{35} \gamma_{\tau-2}.
\]

for an equation linking the auto-covariances

Dividing through by \( \gamma_0 \), yields

\[
\rho_\tau = \frac{2}{35} \rho_{\tau-1} + \frac{1}{35} \rho_{\tau-2}, \quad \tau \geq 1.
\]

(c) Conditions to be invoked are \( \rho_0 = 1 \) and \( \rho_1 = \rho_{-1} \). Hence solving Y-W equations for \( \tau = 1 \) and \( \tau = 2 \) yields

\[
\rho_1 = \frac{2}{35} \rho_0 + \frac{1}{35} \rho_{-1} = \frac{1}{35} (2 \rho_0 + \rho_1)
\]

\[
\Rightarrow 35 \rho_1 = 2 \rho_0 + \rho_1 \Rightarrow 34 \rho_1 = 2 \rho_0 \Rightarrow \rho_1 = \frac{1}{17}.
\]

\[
\rho_2 = \frac{2}{35} \rho_1 + \frac{1}{35} \rho_0 = \frac{2}{35} \times \frac{1}{17} + \frac{1}{35} \times 1 = \frac{1}{35} \left( \frac{2 + 17}{17} \right) = \frac{19}{595} = 0.031933.
\]

(d) General solution of the form:

\[
\rho_\tau = A_1 \left( \frac{1}{5} \right)^\tau + A_2 \left( -\frac{1}{7} \right)^\tau.
\]

\[
\tau = 0 \Rightarrow A_1 + A_2 = 1
\]
\( \tau = 1 \)

\( \frac{A_1}{5} - \frac{A_2}{7} = \frac{1}{17} \).

Therefore

\[ \frac{A_1}{5} \frac{(1 - A_1)}{7} = \frac{7A_1 - 5 + 5A_1}{35} = \frac{12A_1}{35} - \frac{1}{7} = \frac{12A_1}{35} = \frac{24}{119} \Rightarrow A_1 = \frac{35}{12} \times \frac{24}{119} = \frac{70}{119}. \]

Hence

\[ A_2 = 1 - A_1 = \frac{49}{119}. \]

12. Model equation

\[ Y_t = Y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}, \quad t \in \mathbb{Z}. \]

(a)

\[ \hat{y}_T(h) = E[Y_{T+h} | \mathcal{H}_T] \]

(b) Noting that \( \{\epsilon_t\} \) terms for \( t \leq T \) are (implicitly) known, and that those for \( t > T \) are independent of \( \mathcal{H}_T \) and have mean 0, we have:

\[ h = 1 \]

\[ Y_{T+1} = Y_T + \epsilon_T + \theta_1 \epsilon_{T-1} + \theta_2 \epsilon_{T-2} \]

\[ \hat{y}_T(1) = E[Y_{T+1} | \mathcal{H}_T] = y_T + \theta_1 \epsilon_T + \theta_2 \epsilon_{T-1} \]

\[ h = 2 \]

\[ Y_{T+2} = Y_{T+1} + \epsilon_{T+2} + \theta_1 \epsilon_{T+1} + \theta_2 \epsilon_T \]

\[ \hat{y}_T(2) = E[Y_{T+2} | \mathcal{H}_T] = \hat{y}_T(1) + \theta_2 \epsilon_T \]

\[ h \geq 3 \]

\[ Y_{T+h} = Y_{T+h-1} + \epsilon_{T+h} + \theta_1 \epsilon_{T+h-1} + \theta_2 \epsilon_{T+h-2} \]

\[ \hat{y}_T(h) = E[Y_{T+h} | \mathcal{H}_T] = \hat{y}_T(h-1). \]

To be useful, need to express the noise terms into (a truncated form of) the infinite autoregressive format.
(c) i. The candidate needs to deduce that

\[ Y_{T+3} = y_T + \epsilon_{T+3} + (1 + \theta_1)\epsilon_{T+2} + (1 + \theta_1 + \theta_2)\epsilon_{T+1} + (\theta_1 + \theta_2)\epsilon_T + \theta_2\epsilon_{T-1}. \]

It does involve a bit of working out. Appropriate credit will be given for the substantive nature of the progress made on this.

\[ \hat{y}_T(3) = y_T + (\theta_1 + \theta_2)\epsilon_T + \theta_2\epsilon_{T-1}. \]

\[ e_T(3) = Y_{T+3} - \hat{y}_T(3) = \epsilon_{T+3} + (1 + \theta_1)\epsilon_{T+2} + (1 + \theta_1 + \theta_2)\epsilon_{T+1} \]

ii.

\[ V(3) = \text{var}(e_T(3)) = [1 + (1 + \theta_1)^2 + (1 + \theta_1 + \theta_2)^2]\sigma^2. \]

(d) The model studied in parts(a)-(c), is exactly this one with \( \theta_1 = -\frac{5}{6} \) and \( \theta_2 = \frac{1}{6} \). Hence

\[ V(3) = \left[ 1 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2 \right] \times 1.739 = 1.138889 \times 1.739 = 1.98053 \]

and so

\[ \sqrt{V(3)} = 1.40731. \]

A 95% prediction interval for \( Y_{T+3} \) is given by

\[ \hat{y}_T(3) \pm z_{\alpha/2}\sqrt{V(3)} = 4.9 \pm 1.96 \times 1.40731 = 4.9 \pm 2.75832 = (2.14, 7.66) \]

to 3 s.f.

It is assumed that the estimates of the parameters can be taken to be the true values.