The properties of AR(1) and MA processes

4.1 Autocovariances and autocorrelations for an AR(1) process

We derive the autocovariance and autocorrelation functions for an AR(1) process \{Y_t\}, using two alternative methods. The first method is based upon the use of the model equation,

\[ Y_t = \phi Y_{t-1} + \epsilon_t, \quad t \in \mathbb{Z}. \]  

(1)

Recall the infinite moving average expression,

\[ Y_t = \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}, \]  

(2)

derived in Section 3.7. The representation was shown to hold in the sense of mean square convergence for \(|\phi| < 1\). However, as we will see later, the representation also holds in the sense of “almost sure” or “probability one” convergence: with such an interpretation, it is apparent that \(Y_t\) and \(\epsilon_{t+i}\) are uncorrelated for \(i \geq 1\). Therefore, it can be argued that \(Y_s\) and \(\epsilon_t\) are uncorrelated for \(s < t\), so that, since the process mean is zero,

\[ E[Y_s \epsilon_t] = 0 \quad s < t. \]

Squared Equation (1) and taking expectations,

\[ E[Y_t^2] = \phi^2 E[Y_{t-1}^2] + 2\phi E[Y_{t-1} \epsilon_t] + E[\epsilon_t^2], \]

i.e.,

\[ \gamma_0 = \phi^2 \gamma_0 + \sigma^2, \]

where \(\sigma^2 = \text{var}(\epsilon_t)\). Hence,

\[ \gamma_0 = \frac{\sigma^2}{1 - \phi^2}. \]  

(3)

Multiplying Equation (1) by \(Y_{t-\tau}\), where \(\tau \geq 1\), and taking expectations, yields

\[ \gamma_{\tau} = \phi \gamma_{\tau-1}, \quad \tau \geq 1. \]  

(4)

Equation (4) together with the initial condition of Equation (3) has the solution

\[ \gamma_{\tau} = \frac{\sigma^2}{1 - \phi^2} \phi^\tau, \quad \tau \geq 0. \]  

(5)

Recalling that \(\rho_{\tau} = \gamma_{\tau}/\gamma_0\), we obtain

\[ \rho_{\tau} = \phi^\tau, \quad \tau \geq 0. \]  

(6)
Note the geometric decline of the autocorrelation function. If \( \phi < 0 \) then the autocorrelation function oscillates and has negative correlation at lag 1.

A slight variant of this method for obtaining the expression for \( \rho_\tau \) is to divide through in Equation (4) by \( \gamma_0 \) to obtain the recurrence relation

\[
\rho_\tau = \phi \rho_{\tau - 1} \quad \tau \geq 1.
\]

Equation (7) together with the initial condition \( \rho_0 = 1 \) has the solution obtained previously as Equation (6). Using the symmetry property that \( \gamma_{-\tau} = \gamma_\tau \) and \( \rho_{-\tau} = \rho_\tau \), we may, if we wish, extend the range of the values of \( \tau \) in the solutions for the autocovariance function \( \{\gamma_\tau\} \) and the autocorrelation function \( \{\rho_\tau\} \). Thus, for example, we may write

\[
\rho_\tau = \phi^{|\tau|} \quad \tau \in \mathbb{Z}.
\]

An alternative approach to finding the expressions for \( \{\gamma_\tau\} \) and \( \{\rho_\tau\} \) is based upon use of the infinite moving average expression of Equation (2). For \( \tau \geq 0 \),

\[
\begin{align*}
\gamma_\tau &= E[Y_t Y_{t-\tau}] \\
&= E\left[\sum_{i=0}^{\infty} \phi^i \epsilon_{t-i} \sum_{j=0}^{\infty} \phi^j \epsilon_{t-\tau-j}\right] \\
&= E\left[\sum_{i=0}^{\infty} \phi^i \epsilon_{t-i} \sum_{j=0}^{\infty} \phi^j \epsilon_{t-\tau-j}\right] \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \phi^{i+j-\tau} E[\epsilon_{t-i} \epsilon_{t-j}] \\
&= \sum_{j=\tau}^{\infty} \phi^{2j-\tau} \sigma^2 \\
&= \sigma^2 \frac{1 - \phi^2 \phi^\tau}{1 - \phi^2}.
\end{align*}
\]

Thus we have again the result of Equation (5). Note that in the above derivation we have used the properties of the autocovariance function of a white noise process as given in Equations (1) and (2) of Section 3.3.

### 4.2 The lag operator

It is often convenient to use the lag operator \( L \) to characterize models and to carry out mathematical manipulations. (An alternative terminology is backward shift operator with corresponding notation \( B \).) The operator is defined by

\[
LY_t = Y_{t-1} \quad t \in \mathbb{Z}.
\]

Defining \( L^j \) to be the ‘\( j \)-fold’ composition of \( L \), then we note that

\[
L^j Y_t = Y_{t-j}.
\]
Equation (1) for the AR(1) model may be written as
\[(1 - \phi L)Y_t = \epsilon_t.\]
The infinite moving average representation of \(Y_t\) may be more simply derived using the formalism (and associated algebra) of the lag operator.
\[
Y_t = (1 - \phi L)^{-1} \epsilon_t
= \sum_{i=0}^{\infty} \phi^i L^i \epsilon_t
= \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i},
\]
assuming that the sum converges.

### 4.3 Linear processes

**Definition 4.3.1 (Linear Process)**

A process \(\{Y_t\}\) that has the representation
\[
Y_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i},
\]
where \(\\{\epsilon_t\}\) is a white noise process and \(\{\psi_i\}\) is a sequence of coefficients such that \(\sum_{j=0}^{\infty} |\psi_j| < \infty\) is referred to as a linear process.

**Proposition 4.3.2 (Convergence and Stationarity of Linear Process)**

For a linear process \(\{Y_t\}\), with representation \(Y_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}\):

(a) \(Y_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}\) is well defined, in the sense that the right hand side is almost surely bounded (i.e. bounded with probability one);

(b) the sequence of partial sums, \(\sum_{i=0}^{n} \psi_i \epsilon_{t-i}\), converges to \(Y_t\) in mean square as \(n \to \infty\);

(c) \(\{Y_t\}\) is (weakly) stationary, with mean zero and autocovariance function given by
\[
\gamma_\tau = E[Y_t Y_{t-\tau}]
= E \left[ \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i} \sum_{j=0}^{\infty} \psi_j \epsilon_{t-\tau-j} \right]
= E \left[ \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i} \sum_{j=\tau}^{\infty} \psi_j \epsilon_{t-j} \right]
= \sum_{i=0}^{\infty} \sum_{j=\tau}^{\infty} \psi_i \psi_{j-\tau} E[\epsilon_{t-i} \epsilon_{t-j}]
= \sigma^2 \sum_{j=\tau}^{\infty} \psi_j \psi_{j-\tau} \quad \tau \geq 0.
\]
In particular,
\[ \text{var}(Y_t) = \gamma_0 = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2. \]

But since \( \sum_{j=0}^{\infty} |\psi_j| < \infty \), then \( \sum_{j=0}^{\infty} \psi_j^2 < \infty \), and so \( \text{var}(Y_t) \) is finite. Finiteness of \( \gamma_\tau \) for \( \tau \neq 0 \) follows from the calculation used in Examples 3 Qu. 3.

**Remarks 4.3.3**
(i) A linear process \( \{Y_t\} \), as defined above, is sometimes said to be *causal* or a causal function of \( \{\epsilon_t\} \).

(ii) We see from the infinite moving average representation of Equation (8) that the value \( Y_s \) of the process at time \( s \) depends only on \( \{\epsilon_t : t \leq s\} \) and not on \( \{\epsilon_t : t > s\} \).

(iii) As for the special case of the AR(1) process, \( Y_s \) and \( \epsilon_t \) are uncorrelated for \( s < t \), so that
\[ E[Y_s \epsilon_t] = 0, \quad s < t, \quad (10) \]
a fact that we shall make use of later.

(iv) In the special case of \( \{\psi_i = \phi^i : i \in \mathbb{N}\} \) we have the infinite moving average representation (2) of the AR(1) process, and thus \( \sum_{j=0}^{\infty} |\psi_j| < \infty \) is satisfied if and only if \( |\phi| < 1 \). It is readily checked by substitution that the process \( \{Y_t\} \) defined by \( Y_t = \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i} \) satisfies Equation (1). It follows that a viable (i.e. stationary) AR(1) process with autoregressive parameter \( \phi \) exists if \( |\phi| < 1 \).

(v) The square summability of the coefficients i.e. \( \sum_{j=0}^{\infty} \psi_j^2 < \infty \) is actually both necessary and sufficient for mean square convergence of the linear process representation.

### 4.4 The moving average model

Consider a market that every working day receives fresh information which affects the price of a certain commodity. Let \( Y_t \) denote the price change on day \( t \). The immediate effect of the information received on day \( t \) upon \( Y_t \) is represented by \( \epsilon_t \), where \( \{\epsilon_t\} \) is assumed to be a white noise process. But there is also a residual effect, such that \( Y_t \) is affected by the information received on the \( q \) previous days. A simple model represents \( \{Y_t\} \) as a moving average process.

A *moving average process of order* \( q \), an MA(\( q \)) process, with zero mean is a process \( \{Y_t\} \) which satisfies the relation
\[ Y_t = \epsilon_t + \sum_{i=1}^{q} \theta_i \epsilon_{t-i} \quad t \in \mathbb{Z}, \quad (11) \]
where \( \{\epsilon_t\} \) is a white noise process with mean zero and variance \( \sigma^2 \) and where \( \theta_q \neq 0 \).

For convenience, we may define \( \theta_0 = 1 \) and rewrite Equation (11) as
\[ Y_t = \sum_{i=0}^{q} \theta_i \epsilon_{t-i} \quad t \in \mathbb{Z}. \quad (12) \]
Note that we are dealing here with a finite moving average, having a finite number of moving average parameters, as against the infinite moving average representation that we described in the discussion of the AR(1) process.

Historically, “moving averages” were introduced rather differently, with the coefficients $\theta_i$ defined in such a way that $\sum \theta_i = 1$. Each $Y_t$ value is then a weighted average of the $\epsilon_t$ values. The average “moves” as $t$ moves through successive values.

An MA($q$) process is a special case of a linear process as defined in Section 4.3, with

$$\psi_i = \begin{cases} \theta_i & \text{for } 0 \leq i \leq q \\ 0 & \text{for } i > q \end{cases}.$$  

Because all but a finite number of the coefficients $\psi_i$ are zero, an MA($q$) process necessarily satisfies the summability conditions on the coefficients in the definition of a linear process. Thus, for all moving average parameter values, $\theta_1, \theta_2, \ldots, \theta_q$, Equation (11) or (12) defines a stationary process with mean zero.

Applying the result of Equation (9) to the present case, for $\tau \geq 0$,

$$\gamma_\tau = \sigma^2 \sum_{j=\tau}^{q} \theta_j \theta_{j-\tau}.$$  

Thus

$$\gamma_\tau = \begin{cases} \sigma^2 \sum_{i=0}^{q-\tau} \theta_i \theta_{i+\tau} & \text{for } 0 \leq \tau \leq q \\ 0 & \text{for } \tau > q \end{cases}.$$  

In particular,

$$\text{var}(Y_t) = \gamma_0 = \sigma^2 \sum_{i=0}^{q} \theta_i^2.$$  

The autocorrelation function is given by

$$\rho_\tau = \begin{cases} \sum_{i=0}^{q-\tau} \theta_i \theta_{i+\tau} / \sum_{i=0}^{q} \theta_i^2 & \text{for } 0 \leq \tau \leq q \\ 0 & \text{for } \tau > q \end{cases}.$$  

Note that, for an MA($q$) process, there is a cut-off point in the autocorrelation function – all autocorrelations beyond the $q$-th are zero. By comparison, for the AR(1) process and, as we shall see, for more general autoregressive processes, all autocorrelations are generally non-zero but die away geometrically as a function of the lag. This fact may be borne in mind when we are examining the sample autocorrelation function of an observed time series and considering what model to fit to the data.
4.5 The first order moving average process

In the special case of the MA(1) process \( \{Y_t\} \), which satisfies the equation
\[
Y_t = \epsilon_t + \theta \epsilon_{t-1} \quad t \in \mathbb{Z},
\]
(13)
the autocorrelation function is given by
\[
\rho_0 = 1
\]
\[
\rho_1 = \frac{\theta}{1 + \theta^2}
\]
\[
\rho_\tau = 0, \quad \tau \geq 2.
\]
Note that if \( \theta > 0 \) then the MA(1) process is smoother than a white noise process but that if \( \theta < 0 \) then the MA(1) process is more jagged than a white noise process.

Using Equation (13) recursively,
\[
Y_t = \epsilon_t + \theta (Y_{t-1} - \theta \epsilon_{t-2})
\]
\[
= \epsilon_t + \theta Y_{t-1} - \theta^2 (Y_{t-2} - \theta \epsilon_{t-3})
\]
\[
= \ldots
\]
\[
= \epsilon_t - \sum_{k=1}^{n} (-\theta)^k Y_{t-k} - (-\theta)^{n+1} \epsilon_{t-n-1}.
\]
(14)
Here we would like to take the limit as \( n \to \infty \) in some appropriate sense for Equation (14) to obtain the infinite order autoregressive representation of \( \{Y_t\} \),
\[
Y_t = -\sum_{k=1}^{\infty} (-\theta)^k Y_{t-k} + \epsilon_t.
\]
(15)
Let us consider again the concept of convergence in mean square.
\[
E \left[ \left( -\sum_{k=1}^{n} (-\theta)^k Y_{t-k} + \epsilon_t - Y_t \right)^2 \right] = E \left[ \theta^2 (n+1) \epsilon_{t-n-1}^2 \right]
\]
\[
= \theta^2 (n+1) \sigma^2,
\]
which tends to zero as \( n \to \infty \) if and only if \( |\theta| < 1 \).

As a matter of fact, it can be shown that equation (15) is an exact representation (with probability one), in which case (as we will see later) the MA(1) process is an invertible process: this is the case if and only if \( |\theta| < 1 \).

Using the lag operator, we may rewrite Equation (13) as
\[
Y_t = (1 + \theta L) \epsilon_t.
\]
Formally inverting this relationship, we may write

\[ \epsilon_t = (1 + \theta L)^{-1} Y_t = \sum_{k=0}^{\infty} (-\theta)^k L^k Y_t = Y_t + \sum_{k=1}^{\infty} (-\theta)^k Y_{t-k}, \]

which is equivalent to Equation (15). As shown above, this procedure is justifiable if and only if \(|\theta| < 1\).

Note that the MA(1) processes with parameters \(\theta\) and \(\theta^{-1}\), respectively, have the same autocorrelation function, since

\[ \frac{\theta^{-1}}{1 + \theta^{-2}} = \frac{\theta}{1 + \theta^2}. \]

Hence, given the autocorrelation function of an MA(1) process, it is impossible to identify uniquely the parameter value. But if we impose the condition that the process must be invertible then there is a unique parameter value corresponding to the given autocorrelation function.

The MA(1) process with \(|\theta| = 1\) is exceptional in that it is not invertible but may be uniquely identified from the autocorrelation function.

4.6 Invertibility

Using the lag operator, we may write the general MA(\(q\)) model of Equation (11) in the form

\[ Y_t = \theta(L) \epsilon_t, \quad (16) \]

where \(\theta(z)\) is the MA characteristic polynomial, defined by

\[ \theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \ldots + \theta_q z^q. \quad (17) \]

Thus \(\theta(z)\) is the generating function of the moving average coefficients. The corresponding MA characteristic equation is

\[ 1 + \theta_1 z + \theta_2 z^2 + \ldots + \theta_q z^q = 0. \quad (18) \]

Definition 4.6.1 (Invertibility)

A stationary process \(\{Y_t\}\) is said to be invertible if it can be expressed in the form

\[ Y_t = \sum_{k=1}^{\infty} \pi_k Y_{t-k} + \epsilon_t, \quad (19) \]

such that \(\sum_{j=1}^{\infty} |\pi_j| < \infty\).

For an MA(\(q\)) process, the following result can be established.

Theorem 4.6.2 (Invertibility condition for MA(\(q\)))

A necessary and sufficient condition for the moving average parameters \(\theta_1, \theta_2, \ldots, \theta_q\) of the MA(\(q\)) model to specify an invertible moving average process is that all the roots of the characteristic equation, Equation (18), lie strictly outside the unit circle in the complex plane (i.e., all the roots are greater than one in modulus/absolute value).  \(\square\)
We see from the expression (19) that the value at time $t$ of an invertible process can be expressed as a linear combination of the values of the process at previous time points, with the addition of the white noise term at time $t$. This makes it plausible that it should be possible to forecast future values of such a process in a straightforward manner, given knowledge of its history.

Generally, for purposes of estimation and forecasting, the invertibility condition is imposed on stationary processes. Given the autocorrelation function of a MA process, there is generally a multiplicity of parameter values that will yield the given autocorrelations, but there is only one corresponding set of parameters which specifies an invertible MA process.

Assuming that a given MA($q$) process is invertible, we may rewrite (19) as

$$\epsilon_t = \pi(L)Y_t,$$  \hspace{1cm} (20)

where

$$\pi(z) = 1 - \sum_{k=1}^{\infty} \pi_k z^k.$$  

We may invert (16) to obtain

$$\epsilon_t = \theta(L)^{-1}Y_t.$$  \hspace{1cm} (21)

It follows from equations (20) and (21) that we may write

$$\pi(z) = \theta(z)^{-1}$$

or, equivalently,

$$\theta(z)\pi(z) = 1.$$  

In the special case of the MA(1) process, $\theta(z) = 1 + \theta z$, $\pi(z) = (1 + \theta z)^{-1}$, and the invertibility condition reduces to $|\theta| < 1$. 


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