Solutions 1

1. (a) The joint range of $X$ and $Y$ is given by

$$ R_{(X,Y)} = \{(x, y) : 0 < x < 1, \ 0 < y < 1\}. $$

$R_X$ is given by the projection of $R_{(X,Y)}$ onto the “$x$-axis”.
Hence $R_X = \{x : 0 < x < 1\} = (0, 1)$.

(b) For $x \in R_X$,

$$ f_X(x) = \int_{R_Y} f_{(X,Y)}(x, y) dy = \int_0^1 (x + y) dy = \left[ xy + \frac{y^2}{2} \right]_{y=0}^{1} = x + \frac{1}{2}. $$

2. (a)

$$ R_{(X,Y)} = \{(m, n) : m = 0, 1, 2, \ldots; y = 0, 1, 2, \ldots\} = \{(m, n) : m \in \mathbb{N}; y \in \mathbb{N}\}; $$

$$ R_X = \{0, 1, 2, \ldots\} = \mathbb{N}; \quad R_Y = \{0, 1, 2, \ldots\} = \mathbb{N}. $$

(b) We can see that for $(m, n) \in R_{(X,Y)}$ (which, in this case, is equal to the Cartesian product $R_X \times R_Y$),

$$ p_{(X,Y)}(m, n) = \frac{e^{-\lambda} \lambda^m}{m!} \times \frac{e^{-\mu} \mu^n}{n!}. $$

For $m \in R_X$, setting $p_X(m) = \frac{e^{-\lambda} \lambda^m}{m!}$, yields that $X$ corresponds to a random variable drawn from the Poisson$(\lambda)$ distribution.

For $n \in R_Y$, setting $p_Y(n) = \frac{e^{-\mu} \mu^n}{n!}$, yields that $Y$ corresponds to a random variable drawn from the Poisson$(\mu)$ distribution.

Since the joint p.m.f. of $(X, Y)$ can be written as the product of the marginal p.m.f.’s of $X$ and $Y$, then it follows that $X$ and $Y$ are independent.
3. (a) 

\[ p_X(0) = \sum_{y=0,1} p_{(X,Y)}(0, y) = p_{(X,Y)}(0, 0) + p_{(X,Y)}(0, 1) = 0.4 + 0.2 = 0.6. \]

\[ p_X(1) = \sum_{y=0,1} p_{(X,Y)}(1, y) = p_{(X,Y)}(1, 0) + p_{(X,Y)}(1, 1) = 0.1 + 0.3 = 0.4. \]

\[ p_Y(0) = \sum_{x=0,1} p_{(X,Y)}(x, 0) = p_{(X,Y)}(0, 0) + p_{(X,Y)}(1, 0) = 0.4 + 0.1 = 0.5. \]

\[ p_Y(0) = \sum_{x=0,1} p_{(X,Y)}(x, 1) = p_{(X,Y)}(0, 1) + p_{(X,Y)}(1, 1) = 0.2 + 0.3 = 0.5. \]

(b) We need to check that for \((x, y) \in R_X \times R_Y, p_{(X,Y)}(x, y) = p_X(x)p_Y(y)\).

\[ p_{(X,Y)}(0, 0) = 0.4 \neq 0.3 = 0.6 \times 0.5 = p_X(0)p_Y(0) \]

\[ p_{(X,Y)}(1, 0) = 0.1 \neq 0.2 = 0.4 \times 0.5 = p_X(1)p_Y(0) \]

\[ p_{(X,Y)}(0, 1) = 0.2 \neq 0.3 = 0.6 \times 0.5 = p_X(0)p_Y(1) \]

\[ p_{(X,Y)}(1, 1) = 0.3 \neq 0.2 = 0.4 \times 0.5 = p_X(1)p_Y(1) \]

Condition for independence is not met in any of the above four cases. Therefore \(X\) and \(Y\) are NOT independent.

(c)

\[ p_{Y|X}(0|0) = \frac{p_{(X,Y)}(0, 0)}{p_X(0)} = \frac{0.4}{0.6} = \frac{2}{3}. \]

\[ p_{Y|X}(1|0) = \frac{p_{(X,Y)}(0, 1)}{p_X(0)} = \frac{0.2}{0.6} = \frac{1}{3}. \]

(d)

\[ E[Y|X = 0] = \sum_{y=0,1} yp_{Y|X}(y|0) = 0 \times p_{Y|X}(0|0) + 1 \times p_{Y|X}(1|0) = 0 \times \frac{2}{3} + 1 \times \frac{1}{3} = \frac{1}{3}. \]