B.Sc./Grad. Dip.: Probability Models and Time Series

Examples 3

1. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n = 40$, with population mean $\mu = 6.53$, and population variance $\sigma^2 = 10$.

   By using the Central Limit Theorem, find the approximate probability that $\sum_{i=1}^{40} X_i$ lies between 245 and 255.

2. Suppose that $\{N(t) : t \geq 0\}$ is a Poisson process with rate $\lambda$ and let $S_n$ be the time of the $n$-th arrival, for $n \in \mathbb{Z}^+$. Let $S = \sum_{n=1}^{\infty} g(S_n)$.

   Assuming that $g(\cdot)$ is sufficiently well-behaved (to permit the switching of the order of the operations of either expectation, or integration, with infinite summation), show that:

   (a) $E[S] = \int_0^\infty g(x) \sum_{n=1}^{\infty} f_{S_n}(x) \, dx$,

   (b) $E[S]$ can be further simplified to yield $E[S] = \lambda \int_0^\infty g(x) \, dx$.

   [Hint: you may use the fact that $S_n \sim \text{Gamma}(n, \lambda)$, i.e.
   
   \[ f_{S_n}(t) = e^{-\lambda t} \lambda^n t^{n-1} \frac{1}{(n-1)!}, \quad t \geq 0. \]

3. Suppose that $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are independent Poisson processes with rates $\lambda_1$ and $\lambda_2$, respectively. Set $Y(t) = N_1(t) + N_2(t)$, $t \geq 0$.

   (a) Prove that, for $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$,

   \[ \mathbb{P}(Y(t) = n) = \sum_{r=0}^{n} \mathbb{P}(N_1(t) = n-r) \mathbb{P}(N_2(t) = r). \]

   [Hint: You could try conditioning on the value of $N_2(t)$].

   (b) Hence show that $Y(t) \sim \text{Poisson}((\lambda_1 + \lambda_2)t)$, i.e. for $n \in \mathbb{N}$,

   \[ \mathbb{P}(Y(t) = n) = e^{-(\lambda_1 + \lambda_2)t} \frac{(\lambda_1 + \lambda_2)t^n}{n!}. \]