Statistics: Theory and Practice

Solutions 2

1. Set $X = -\frac{1}{\lambda} \ln U$. Then

$$F_X(x) = \mathbb{P}(-\frac{1}{\lambda} \ln U \leq x) = \mathbb{P}(\ln U \geq -\lambda x) = \mathbb{P}(U \geq e^{-\lambda x}) = 1 - F_U(e^{-\lambda x})$$

Hence,

$$f_X(x) = \frac{d}{dx} F_X(x) = -f_U(e^{-\lambda x}) \times -\lambda e^{-\lambda x} = \lambda e^{-\lambda x}$$

for $x \in R_X = \{x : x \geq 0\}$. So $X$ has the p.d.f. of $Exp(\lambda)$.

2. (a) Coefficient of $\theta^1$ is equal to $\mathbb{P}(X = 1) = 0.65$.

(b) Coefficient of $\theta^{10}$ is clearly equal to 0, and so $\mathbb{P}(X = 10) = 0$.

(c) Coefficient of $\theta^0$ is the constant term 0.35, i.e. $\mathbb{P}(X = 0) = 0.35$.

(d) $E[X] = G'_X(1) = 0.65$.

3. $M_X(t) = \int_0^\infty e^{tx}f_X(x)dx = \int_0^\infty e^{tx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\beta x}dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta-t)^{\alpha}} \int_0^\infty \frac{(\beta-t)^{\alpha}}{\Gamma(\alpha)} e^{-(\beta-t)x}x^{\alpha-1}dx$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta-t)^{\alpha}} \times 1 = \left(\frac{\beta}{\beta-t}\right)^{\alpha}$$

for $t < \beta$.

$$M'_X(t) = \beta^\alpha \times -\frac{\alpha}{(\beta-t)^{\alpha+1}} \times -1 = \frac{\alpha}{\beta} \left(\frac{\beta}{\beta-t}\right)^{\alpha+1}.$$ 

$$M''_X(t) = \frac{\alpha}{\beta} \beta^{\alpha+1} \times -(\alpha+1) \frac{1}{(\beta-t)^{\alpha+2}} \times -1 = \alpha \beta^{\alpha+1} \left(\frac{\alpha+1}{(\beta-t)^{\alpha+2}} = \frac{\alpha(\alpha+1)}{\beta^2} \left(\frac{\beta}{\beta-t}\right)^{\alpha+2}.$$ 

Hence

$$E[X] = M'_X(0) = \frac{\alpha}{\beta}, \quad M''_X(0) = \frac{\alpha(\alpha+1)}{\beta^2}$$

$$\text{var}(X) = E[X^2] - E[X]^2 = \frac{\alpha(\alpha+1)}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha}{\beta^2}.$$
4.

\[ M'_W(t) = \frac{-20}{(1 - 7t)^{21}} \times -7 = \frac{140}{(1 - 7t)^{21}}. \]

\[ M''_W(t) = \frac{140 \times -21}{(1 - 7t)^{22}} \times -7 = \frac{20,580}{(1 - 7t)^{22}}. \]

Hence

\[ E[W] = M'_W(0) = 140, \quad E[W^2] = M''_W(0) = 20580. \]

\[ \text{var}(W) = E[W^2] - E[W]^2 = 20580 - (140)^2 = 980. \]