1. (a) 

\[ M_X(t) = \int_0^\infty e^{tx} 108x^2 e^{-6x} dx = \int_0^\infty 108x^2 e^{-(6-t)x} dx. \]

Assuming that \( t < 6 \), there are at least 2 ways to proceed with the computation of this integral. **Integration by Parts:**

\[ M_X(t) = \left[ 108 x^2 e^{-(6-t)x} \right]_0^\infty + \int_0^\infty 108 x e^{-(6-t)x} \cdot x e^{-(6-t)x} dx = (0 - 0) + 216 \int_0^\infty x e^{-(6-t)x} dx. \]

**Slick method:**

\[ M_X(t) = 108 \int_0^\infty x^{3-1} e^{-(6-t)x} dx = 108 \frac{\Gamma(3)}{(6-t)^3} \int_0^\infty x^{3-1} e^{-(6-t)x} dx. \]

However, the integrand is the p.d.f. of a r.v. from the Gamma\((3, 6-t)\) distribution. Also, from the handout on the Gamma function, we know that \( \Gamma(n) = (n-1)! \) for \( n \in \mathbb{Z}^+ \). Hence

\[ M_X(t) = 108 \frac{2!}{(6-t)^3} = 216 \frac{1}{(6-t)^3}. \]

(b)

\[ M'_X(t) = \frac{216 \times -3 \times -1}{(6-t)^4} = \frac{648}{(6-t)^4}, \]
\[ M''_X(t) = \frac{2592}{(6-t)^5}, \]
\[ M^{(3)}_X(t) = \frac{12,960}{(6-t)^6}. \]

Hence

\[ E[X^3] = M_X^{(3)}(0) = \frac{5}{18} = 0.278 \text{ (to 3 s.f.)} \]

2. To save time, for \( n, m \in \mathbb{Z}^+ \), can compute

\[ E[X^n Y^m] = \int_0^1 \int_0^1 x^n y^m (x + y) dx dy = \int_0^1 \left\{ \int_0^1 (x^{n+1} y^m + x^n y^{m+1}) dx \right\} dy \]
\[ = \int_0^1 \left[ \frac{x^{n+2}}{n+2} y^m + \frac{x^{n+1}}{n+1} y^{m+1} \right]_0^1 dy = \int_0^1 \left( \frac{y^m}{n+2} + \frac{y^{m+1}}{n+1} \right) dy \]
\[ = \left[ \frac{y^{m+1}}{(m+1)(n+2)} + \frac{y^{m+2}}{(m+2)(n+1)} \right]_0^1 = \frac{1}{(m+1)(n+2)} + \frac{1}{(m+2)(n+1)}. \]
Plugging in $n = 1, m = 0$ yields:

$$E[X] = \frac{1}{3} + \frac{1}{2 \times 2} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

$m = 1, n = 0$:

$$E[Y] = \frac{7}{12}.$$

$m = 1, n = 1$

$$E[XY] = \frac{1}{3}.$$

$m = 2, n = 0$ and $m = 0, n = 2$:

$$E[Y^2] = E[X^2] = \frac{5}{12}.$$

(a)

$$\text{cov}(X, Y) = E[XY] - E[X]E[Y] = \frac{1}{3} - \frac{7}{12} \times \frac{7}{12} = -\frac{1}{144}.$$

(b)

$$\text{var}(X) = \text{var}(Y) = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}.$$

Therefore

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} = -\frac{1}{11}.$$

(c)

$$\text{var}(2X - Y + 4) = \text{var}(2X - Y) = \text{var}(2X) + \text{var}(Y) + 2(1)(-1)\text{cov}(2X, Y)$$

$$= 2^2\text{var}(X) + \text{var}(Y) - 2 \times 2\text{cov}(X, Y) = 4\text{var}(X) + \text{var}(Y) - 4\text{cov}(X, Y) = \frac{59}{144}.$$

3.

(a)

$$f_X(x) = \int_x^2 2 \, dy = \left[\frac{y^2}{2}\right]_x^2 = \frac{2-x}{2} \quad x \in [0, 2].$$

$$f_Y(y) = \int_0^y \frac{1}{2} \, dx = \left[\frac{x}{2}\right]_0^y = \frac{y}{2} \quad y \in [0, 2].$$

(b) One proof for the fact that $X$ and $Y$ are not independent is that the region $R_{(X,Y)}$ is not 'rectangular'. Another proof is that the joint p.d.f. of $X$ and $Y$ is not equal to the product of their marginal p.d.f.'s, i.e.

$$f_X(x)f_Y(y) = \frac{2-x}{2} \times \frac{y}{2} \neq \frac{1}{2} = f_{(X,Y)}(x, y)$$

for $(x, y) \in R_{(X,Y)}$. 

2