MAS programmes: Stochastic Models and Forecasting

Solutions 9

1. (i) Squaring both sides of the model equation and taking expectations, and using the fact that \( \epsilon_{t-1} \) and \( Y_{t-1} \) are uncorrelated with \( \epsilon_t \), we obtain

\[
\gamma_0 = \phi^2 \gamma_0 + (1 + \theta^2) \sigma^2 + 2 \theta \phi E[Y_{t-1}\epsilon_{t-1}].
\]

Multiplying the model equation by \( \epsilon_t \) and taking expectations,

\[
E[Y_t \epsilon_t] = \sigma^2.
\]

Hence also, by the stationarity, \( E[Y_{t-1}\epsilon_{t-1}] = \sigma^2 \), and substituting into the above equation we obtain

\[
\gamma_0 = \frac{1 + \theta^2 + 2 \theta \phi}{1 - \phi^2} \sigma^2.
\]

(ii) The autocovariances \( \gamma_t^{12} \) are given by \( \gamma_t^{12} = E[Y_t \epsilon_{t-\tau}] \). Because \( Y_t \) is uncorrelated with \( \epsilon_s \) for \( s > t \), it follows that \( \gamma_t^{12} = 0, \tau \leq -1 \). From the solution to part (i), \( \gamma_0^{12} = E[Y_t \epsilon_t] = \sigma^2 \). Multiplying through by \( \epsilon_{t-1} \) in the model equation, and taking expectations,

\[
\gamma_t^{12} = E[Y_{t-1}\epsilon_{t-1}] = \phi E[(Y_{t-1}\epsilon_{t-1}) + \theta E[\epsilon_{t-1}^2] = (\phi + \theta) \sigma^2.
\]

Multiplying through by \( \epsilon_{t-\tau} \) in the model equation, where \( \tau \geq 2 \), and taking expectations,

\[
\gamma_t^{12} = E[Y_{t-1}\epsilon_{t-\tau}] = \phi E[Y_{t-1}\epsilon_{t-\tau}] = \phi \gamma_{t-1}^{12}.
\]

Iterating this relationship we find that

\[
\gamma_t^{12} = \phi^{\tau-1}(\phi + \theta)\sigma^2 \quad \tau \geq 1.
\]

The autocorrelations \( \rho_t^{12} \) are found from the autocovariances \( \gamma_t^{12} \) by dividing through by

\[
\sqrt{\text{var}(Y_t)\text{var}(\epsilon_t)} = \sigma \sqrt{\frac{1 + \theta^2 + 2 \theta \phi}{1 - \phi^2}}.
\]
The following SAS program may be used for this exercise.

```sas
proc arima data=DJAO;
  identify var=lnDJ(1);
  identify var=lnAO(1);
  identify var=lnDJ(1) crosscor=lnAO(1);
proc varmax data=DJAO;
  model lnDJ lnAO / p=1 dif=(lnDJ(1) lnAO(1))
       output lead=5;
proc varmax data=DJAO;
  model lnDJ lnAO / p=1 dif=(lnDJ(1) lnAO(1)) noint;
       output lead=5;
proc arima data=DJAO;
  identify var=lnDJ(1);
  estimate;
  forecast lead=5;
  identify var=lnAO(1) crosscor=lnDJ(1);
  estimate q=1 input=(1 lnDJ);
  forecast lead=5;
proc arima data=DJAO;
  identify var=lnDJ(1);
  estimate noint;
  forecast lead=5;
  identify var=lnAO(1) crosscor=lnDJ(1);
  estimate q=1 input=(1 lnDJ) noint;
  forecast lead=5;
run;
```

(i) The cross-correlation between the first differences of \( \text{lnDJ} \) and \( \text{lnAO} \) that stands out as significant is the one at lag -1, that is, the correlation between the first difference of \( \text{lnAO} \) and the first difference of \( \text{lnDJ} \) on the previous day. So the two series appear not to be independent of each other: the value of the return for the AO on a given day is positively correlated with the value of the return for the DJ on the previous day.

(ii) If we write \( X_t \) and \( Y_t \) for the return on Day \( t \) for the DJ and AO, respectively, then the equations of the fitted model, including constant terms, are

\[
X_t = 0.00026 - 0.01366X_{t-1} + 0.03603Y_{t-1} + \epsilon_t \\
Y_t = 0.00010 + 0.67337X_{t-1} + 0.09875Y_{t-1} + \zeta_t.
\]

The equations of the fitted model, not including constant terms, are

\[
X_t = -0.01212X_{t-1} + 0.03689Y_{t-1} + \epsilon_t \\
Y_t = 0.67397X_{t-1} + 0.09908Y_{t-1} + \zeta_t.
\]

In both cases \((\epsilon_t, \zeta_t)\) is a bivariate white noise process with estimated covariance matrix, to 5 d.p.,

\[
\begin{pmatrix}
0.00004 & 0.00000 \\
0.00000 & 0.00006
\end{pmatrix}.
\]
The equation of the fitted transfer function model, including constant terms, is
\[ Y_t = 0.0001171 + 0.67168X_{t-1} + \epsilon_t + 0.16361\epsilon_{t-1}, \]
where \( \{\epsilon_t\} \) is a white noise process.

The equation of the fitted transfer function model, not including constant terms, is
\[ Y_t = 0.67218X_{t-1} + \epsilon_t + 0.16388\epsilon_{t-1}. \]

In both cases, the \( p \)-values of the portmanteau statistics are not significant. Thus our models provide a satisfactory fit to the data.

Using S+, the following commands produce the required ccf and fit the VAR(1) model. (The VAR model of best fit according to the AIC criterion is a VAR(1) model.)

```r
attach(DJAO)
lnDJ <- log(DJ)
lnAO <- log(AO)
DLDJ <- diff(lnDJ)
DLAO <- diff(lnAO)
DLDJA0 <- data.frame(DLDJ,DLAO)
DJAO.rts <- rts(DLDJA0)
DJAO.acf <- acf(DJAO.rts)
DJAO.acf
DJAO.ar <- ar(DJAO.rts)
DJAO.ar$order
DJAO.ar$ar
mean(DLDJ)
mean(DLAO)
DJAO.ar$var.pred
```

3