6 Regenerative and renewal-reward processes

6.1 Regenerative processes

A stochastic process \( \{X(t) : t \geq 0\} \) is said to be *regenerative* if there exists a random point of time \( S_1 \) such that the behaviour of the process starting at time \( S_1 \) is a probabilistic replicate of its behaviour starting at time zero and is independent of its behaviour before time \( S_1 \). The following is a formal definition of a regenerative process.

A stochastic process \( \{X(t) : t \geq 0\} \) is said to be a *regenerative process* if, with probability 1, there exists a *regeneration epoch*, a positive random variable \( S_1 \) such that:

1. \( \{X(t + S_1) : t \geq 0\} \) is independent of \( \{X(t) : 0 \leq t < S_1\} \),
2. \( \{X(t + S_1) : t \geq 0\} \) has the same distribution as \( \{X(t) : t \geq 0\} \).

Note that we are now using the notation \( \{X(t) : t \geq 0\} \) for a stochastic process. This is because it will turn out to be useful to have the notation \( \{N(t) : t \geq 0\} \) available for another related process.

If, with probability 1, there is one regeneration epoch \( S_1 \) then, since the process in probabilistic terms restarts at a regeneration epoch, it follows that, with probability 1, there will be a sequence of regeneration epochs \( \{S_n : n \geq 1\} \), such that each \( S_n \) has the same properties as \( S_1 \).

- Any of the queueing models with Poisson arrivals that we have described so far may be characterized as a regenerative process — if the values of the parameters are such as to allow an equilibrium distribution to exist. If we suppose that the queueing system is initially empty then the successive points of time at which a customer departs to leave the system empty are regeneration epochs. If we suppose that at time zero a customer has just arrived to find the system empty then the successive points of time at which a customer arrives to find the system empty are regeneration epochs.

In general, for any regenerative process, if we define \( S_0 = 0 \) and

\[
T_n = S_n - S_{n-1}, \quad n \geq 1
\]

then \( \{T_n : n \geq 1\} \) is a sequence of i.i.d. r.v.s. Hence we may define a renewal process \( \{N(t) : t \geq 0\} \) by

\[
N(t) = \sup\{n \geq 0 : S_n \leq t\} \quad t \geq 0.
\]

In the present setting, the renewals are the occurrences of successive regeneration epochs, and the intervals of time between successive regeneration epochs are referred to as *renewal cycles*. Thus \( T_n \) is the length of the \( n \)th renewal cycle, \( \{t : S_{n-1} \leq t < S_n\} \). In particular, \( T_1 = S_1 \).
6.2 Renewal-reward processes

In the following we consider the renewal process associated with a regenerative process and assume that

\[ 0 < E(T_1) < \infty. \]

We shall often need to consider problems where there is a reward structure associated with a regenerative process. Rewards may be accumulated continuously over time or be earned at specific time points when particular transitions take place. In any case, let \( R_n \) denote the reward earned during the \( n \)-th renewal cycle for \( n \geq 1 \). It is assumed that \( \{R_n : n \geq 1\} \) is a sequence of i.i.d. r.v.s. Usually, however, \( R_n \) is correlated with \( T_n \). It may be the case that \( R_n \) can take on both positive and negative values, and we shall assume that \( E(\|R_1\|) < \infty \).

Let \( R(t) \) denote the cumulative reward earned up to time \( t \) for \( t \geq 0 \). The process \( \{R(t) : t \geq 0\} \) is called a renewal-reward process.

It turns out that the long-run average reward per unit time is equal to the expected reward earned during any cycle divided by the expected length of any cycle. This may be put formally as follows.

**Theorem 1 (The Renewal-Reward Theorem)** With probability 1

\[ \lim_{t \to \infty} \frac{R(t)}{t} = \frac{E(R_1)}{E(T_1)}. \]

**Proof.** Recall first from Section 5.2 that we used the Strong Law of Large Numbers to establish that with probability 1

\[ \lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{E(T_1)}. \]  

(1)

For the present assume that all rewards are non-negative, in which case, for any \( t > 0 \),

\[ \sum_{i=1}^{N(t)} R_i \leq R(t) \leq \sum_{i=1}^{N(t)+1} R_i. \]  

(2)

Dividing through by \( t \) in the inequalities (2), it follows that, for any \( t \) such that \( N(t) > 0 \),

\[ \frac{\sum_{i=1}^{N(t)} R_i}{N(t)} \frac{N(t)}{t} \leq \frac{R(t)}{t} \leq \sum_{i=1}^{N(t)+1} R_i \frac{N(t) + 1}{t}. \]  

(3)

According to the Strong Law of Large Numbers, with probability 1, as \( n \to \infty \),

\[ \frac{\sum_{i=1}^{n} R_i}{n} \to E(R_1). \]

Now letting \( t \to \infty \) in the inequalities (3), noting that \( N(t) \to \infty \) as \( t \to \infty \), and using the result of Equation (1), we find that, with probability 1, as \( t \to \infty \),

\[ \frac{R(t)}{t} \to \frac{E(R_1)}{E(T_1)}. \]  

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which gives us the result of the theorem for the case when all rewards are non-negative.

If rewards can be either positive or negative (i.e., there can be gains or losses) then the inequalities (2) are not valid. We have to adjust the argument by defining $R^+(t)$ to be the cumulative positive reward earned up to time $t$ and $R^-(t)$ to be the cumulative negative reward earned (or cost incurred) up to time $t$, so that $R(t) = R^+(t) - R^-(t)$. Similarly let $R_n^+$ denote the positive reward and $R_n^-$ the negative reward earned during the $n$th renewal cycle, so that $R_n = R_n^+ - R_n^-$. The above proof then has to be applied to the $R^+$ terms and the $R^-$ terms separately. Finally, with probability 1, as $t \to \infty$,

\[
\frac{R(t)}{t} = \frac{R^+(t) - R^-(t)}{t} \to \frac{E(R_1^+)}{E(T_1)} = \frac{E(R_1)}{E(T_1)}.
\]

Let $A$ be a subset of the state space $S$ of the regenerative process $\{X(t) : t \geq 0\}$. A useful application of the Renewal-Reward Theorem is in the evaluation of the long-term proportion of time that the process spends in the set of states $A$. Define the indicator function $I_A(t)$ by

\[
I_A(t) = \begin{cases} 
1 & \text{if } X(t) \in A \\
0 & \text{if } X(t) \notin A
\end{cases}
\]

and the cumulative reward $R(t)$ by

\[
R(t) = \int_0^t I_A(u)du,
\]

so that $R(t)$ is the length of time and $R(t)/t$ is the proportion of time up to time $t$ that the process spends in the set of states $A$. Correspondingly, the reward $R_1$ earned during the first cycle is given by

\[
R_1 = \int_0^{S_1} I_A(u)du = T_1^A,
\]

where the r.v. $T_1^A$ represents the length of time in the first cycle that the process spends in the set of states $A$. Thus $E(R_1) = E(T_1^A)$. From the Renewal-Reward Theorem we obtain the following result.

**Theorem 2** With probability 1

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t I_A(u)du = \frac{E(T_1^A)}{E(T_1)}.
\]

Thus the long-term proportion of time that the process spends in the set of states $A$ is given by

\[
\frac{E(T_1^A)}{E(T_1)}.
\]

If the process has an equilibrium distribution $\{\pi_x : x \in S\}$ then $\sum_{x \in A} \pi_x$ is the long-term proportion of time that the process spends in $A$. Hence, if an equilibrium distribution $\{\pi_x : x \in S\}$ exists then we have the result that

\[
\sum_{x \in A} \pi_x = \frac{E(T_1^A)}{E(T_1)}.
\]
6.3 Little’s “Law”

Recall the simple single-server queueing system dealt with in Chapter 3, with customer arrival rate $\lambda$ and service rate $\mu$. The ratio $\rho \equiv \lambda/\mu$ is the traffic intensity of the queueing system. An equilibrium distribution exists if and only if $\rho < 1$, i.e., $\lambda < \mu$, in which case

$$\pi_j = (1 - \rho)\rho^j, \quad j \geq 0.$$  

The equilibrium distribution is a geometric distribution with mean $L$, where

$$L = \frac{\rho}{1 - \rho} = \frac{\lambda}{\mu - \lambda}.$$  

We also found that, in equilibrium, an arriving customer’s sojourn time distribution is exponential with parameter $\mu - \lambda$ and hence with mean $W$, where

$$W = \frac{1}{\mu - \lambda}.$$  

It follows that $L = \lambda W$. It turns out that such a relationship holds in a wide variety of settings.

**Theorem 3 (Little’s Theorem)** *For any regenerative process that models a system with individual arrivals and departures, the equation*

$$L = \lambda W$$

*is valid, where $L$ is the mean number of individuals in the system, $\lambda$ is the mean arrival rate of individuals who join the system, and $W$ is the mean length of time spent by an individual in the system.*

**Proof.** Let $F(\cdot)$ be the distribution function for the length of time spent in the system (sojourn time) and $Q(\cdot)$ the corresponding survivor function. Let $t$ be an arbitrary point in the steady state. The customers that are in the system at time $t$ must: (a) have arrived prior to time $t$; and (b) depart after time $t$. Consider an infinitesimally small interval at a distance $x$ in the past, $(t - x, t - x + \delta x)$. An average of $\lambda \delta x$ customers arrive during that interval, and each one of those customers is still in the system at time $t$ with probability $1 - F(x)$ i.e. $Q(x)$. Hence, the interval $(t - x, t - x + \delta x)$ contributes $\lambda(1 - F(x))\delta x$ to the average $L$. Taking the limit as $\delta x \to 0$ and integrating over all $x$ from 0 (current time) to $\infty$ (back into the infinite past), one obtains:

$$L = \int_0^\infty \lambda(1 - F(x))dx = \int_0^\infty \lambda Q(x)dx = \lambda W.$$

6.4 The busy period distribution for the server in a simple queueing system

In looking at the equilibrium distribution for the simple single-server queueing system we were adopting the viewpoint of an outside observer. In looking at the waiting time and sojourn
time distributions we were adopting the viewpoint of an arriving customer. We now look at
the queue from the viewpoint of the server.

The server is busy as long as the system is non-empty and is idle when the system is empty. Assuming that \( \rho < 1 \), i.e., \( \lambda < \mu \), so that the system size settles down to equilibrium, periods during which the server is busy alternate with periods during which it is idle. An idle period is terminated by the arrival of a customer according to a Poisson process with rate \( \lambda \). Hence the idle periods are i.i.d., having an exponential distribution with parameter \( \lambda \). The busy periods are also i.i.d., but their distribution is much more complicated.

However, we may use the Renewal-Reward Theorem to obtain the expected length of the busy period quite easily. Consider the simple queueing system as a regenerative process, where the regeneration epochs are the points in time where a customer departs to leave the system empty. Each cycle starts with a length of time during which the system is empty and the server idle and then has a length of time during which the system is non-empty and the server is busy. When this busy period terminates then so does the cycle. The expected length of the idle period during which the system is empty is simply \( 1/\lambda \). Denote by the r.v. \( B \) the length of the busy period, so that the expected length of a cycle is

\[
\frac{1}{\lambda} + E(B).
\]

So, according to Theorem 2, the long-term proportion of time that the system is empty is equal to

\[
\frac{1}{\lambda + E(B)} = \frac{1}{1 + \lambda E(B)}.
\]

But the long-term proportion of time that the system is empty is also given by the equilibrium probability,

\[
\pi_0 = 1 - \rho = 1 - \frac{\lambda}{\mu}.
\]

Equating these two expressions we obtain

\[
\frac{1}{1 + \lambda E(B)} = 1 - \frac{\lambda}{\mu},
\]

which gives

\[
E(B) = \frac{1}{\mu - \lambda}.
\]  \hspace{1cm} (4)

We now turn to the more difficult issue of finding an expression for the distribution of the length of the busy period. The length \( B \) of the busy period initiated by an arriving customer is the sum of the first service time \( Y \) and the lengths of the busy periods generated by all the customers who arrive during the first service time.

- Note that the queueing discipline, that is, the rules that determine the order in which customers are served, does not affect the value of \( B \). It is convenient to assume a “last in first out” (LIFO) queueing discipline in the analysis of the busy period distribution.
Let the r.v. $M$ denote the number of customers who arrive during the first service time. Because customers arrive in a Poisson process with rate $\lambda$,

$$P(M = m|Y = y) = e^{-\lambda y} \frac{\lambda^m y^m}{m!} \quad m \geq 0.$$ 

Let $b^*(s)$ denote the Laplace transform of $B$. The Laplace transform of the length of the busy period generated by each customer who arrives during the first service time is also $b^*(s)$. Furthermore, the lengths of the busy periods generated by each of the customers who arrive during the first service time are mutually independent. Thus the Laplace transform of $B$ conditional upon the values of $M$ and $Y$ is given by

$$E(e^{-sB}|M = m, Y = y) = e^{-sy} [b^*(s)]^m.$$ 

Because the service times are exponentially distributed with parameter $\mu$, the r.v. $Y$ has p.d.f.

$$\mu e^{-\mu y}, \quad y \geq 0.$$ 

Hence, unconditionally,

$$b^*(s) = E(e^{-sB}) = \int_0^\infty E(e^{-sB}|Y = y) \mu e^{-\mu y} dy$$

$$= \int_0^\infty \sum_{m=0}^\infty E(e^{-sB}|M = m, Y = y) P(M = m|Y = y) \mu e^{-\mu y} dy$$

$$= \int_0^\infty \sum_{m=0}^\infty e^{-sy} [b^*(s)]^m e^{-\lambda y} \frac{\lambda^m y^m}{m!} \mu e^{-\mu y} dy$$

$$= \int_0^\infty \mu e^{-(\mu + \lambda + s)y} e^{\lambda b^*(s)} dy$$

$$= \frac{\mu}{\mu + \lambda + s - \lambda b^*(s)}.$$ 

Thus $b^*(s)$ satisfies the quadratic equation

$$\lambda b^*(s)^2 - (\mu + \lambda + s) b^*(s) + \mu = 0. \quad (5)$$

Equation (5) has the solution

$$b^*(s) = \frac{\mu + \lambda + s - \sqrt{(\mu + \lambda + s)^2 - 4\lambda \mu}}{2\lambda}, \quad (6)$$

where the minus sign has been taken in front of the square root so that the condition $b^*(0) = 1$ is satisfied. This is not the Laplace transform of a well-known distribution, but we can check that it gives the same simple expression for the mean as we obtained in Equation (4). To obtain the mean, we can differentiate the expression in Equation (6), but we can also return to Equation (5) and differentiate:

$$2\lambda b^*(s) \frac{db^*}{ds} - (\mu + \lambda + s) \frac{db^*}{ds} - b^*(s) = 0. \quad (7)$$
Setting \( s = 0 \) in Equation (7) we obtain

\[-2\lambda E(B) + (\mu + \lambda)E(B) - 1 = 0 ,\]

and hence

\[E(B) = \frac{1}{\mu - \lambda} ,\]

which is the expression of Equation (4).

To obtain the second moment \( E(B^2) \) we differentiate Equation (7):

\[2\lambda \left( \frac{db^*}{ds} \right)^2 + 2\lambda b^*(s)\frac{d^2b^*}{ds^2} - (\mu + \lambda + s)\frac{d^2b^*}{ds^2} - 2\frac{db^*}{ds} = 0 . \tag{8}\]

Setting \( s = 0 \) in Equation (8), and using Equation (4), we obtain

\[\frac{2\lambda}{(\mu - \lambda)^2} - (\mu - \lambda)E(B^2) + \frac{2}{\mu - \lambda} = 0 ,\]

and hence

\[E(B^2) = \frac{2\mu}{(\mu - \lambda)^3} .\]

It follows that

\[
\text{var}(B) = \frac{2\mu}{(\mu - \lambda)^3} - \frac{1}{(\mu - \lambda)^2} = \frac{\mu + \lambda}{(\mu - \lambda)^3} .
\]

The coefficient of variation, \( C_B \), is given by

\[
C_B^2 = \frac{\text{var}(B)}{[E(B)]^2} = \frac{\mu + \lambda}{\mu - \lambda} = \frac{1 + \rho}{1 - \rho} .
\]

We may note that, as \( \rho \uparrow 1 \), \( C_B \uparrow \infty \). The p.d.f. of \( B \) has very long tails when \( \rho \) is close to 1.