Consider a stationary process $\{Y_t\}$ that satisfies the model equation

$$Y_t = \phi Y_{t-1} + \epsilon_t + \theta \epsilon_{t-1} \quad t \in \mathbb{Z},$$

where $\{\epsilon_t\}$ is a white noise process with variance $\sigma^2$ and the parameter $\phi$ satisfies $|\phi| < 1$.

(i) Show that the process variance $\gamma_0$ is given by

$$\gamma_0 = \frac{1 + \theta^2 + 2\theta \phi}{1 - \phi^2} \sigma^2.$$

(ii) Show that the cross-correlation function $\{\rho_{12}^\tau\}$ between $\{Y_t\}$ and $\{\epsilon_t\}$, where $\rho_{12}^\tau \equiv \text{cor}(Y_t, \epsilon_{t-\tau})$, is given by

$$\rho_{12}^\tau = \begin{cases} 
0 & \tau \leq -1 \\
\sqrt{\frac{1 - \phi^2}{1 + \theta^2 + 2\theta \phi}} - \phi^{\tau-1} & \tau \geq 1
\end{cases}$$

Consider the bivariate time series consisting of daily closing values of the Dow-Jones Index of stocks on the New York Stock Exchange and the Australian All-Ordinaries Index of Share Prices. Recall that the variables $\ln DJ$ and $\ln AO$ contain the natural logarithms of the respective indices and that the first differences of both these variables appear plausibly to be white noise. Recall also that if $P_t$ denotes the closing value on day $t$ of either of the indices then $X_t \equiv \Delta \ln P_t$ is the “return”.

(i) After generating and examining the cross-correlation function of the two series of first differences, i.e., the returns, comment on what kind of relationship there appears to exist between them.

(ii) Fit a VAR(1) model to the first differences and obtain forecasts of the next five values of $\ln DJ$ and $\ln AO$. Write down explicitly in terms of the returns the equations of the fitted model and the estimated covariance matrix of the underlying bivariate white noise process. (Try fitting the model with and without constant terms.)

(iii) Find an appropriate transfer function model to fit to the data and obtain forecasts of the next five values of $\ln DJ$ and $\ln AO$. Write down explicitly the equation of the fitted transfer function model in terms of the returns. (You should find that the fitted model is a very simple one. Try fitting models with and without constant terms.)