1. (a) (i) The state space is the set of all non-negative integers. The instantaneous transition rates are given by

\[ \text{transition rate} \quad i \to i + 1 \quad \lambda \quad (i \geq 0). \]

(ii) \( N(t) \) has the Poisson distribution with parameter \( \lambda t \)

\[ [ \text{i.e. } p_n(t) = e^{-\lambda t}(\lambda t)^n/n! \quad (n \geq 0) ] \]

\[ G(z) = \exp[\lambda t(z-1)]. \]

(b) (i) Let \( X \) denote the total number of arrivals by time \( t \). Conditional upon \( n \) groups having arrived by time \( t \), we are adding together \( n \) i.i.d. r.v.s and hence multiplying together their p.g.f.s.

\[ E(z^X|N(t) = n) = [K(z)]^n. \]

(ii) Unconditionally, the p.g.f. of the total number of arrivals by time \( t \) is given by

\[
E(z^X) = \sum_{n=0}^{\infty} E(z^X|N(t) = n) \Pr(N(t) = n) \\
= \sum_{n=0}^{\infty} [K(z)]^n e^{-\lambda t}(\lambda t)^n/n! \\
= e^{-\lambda t} \sum_{n=0}^{\infty} [\lambda tK(z)]^n/n! \\
= \exp[\lambda t(K(z) - 1)].
\]

(iii) Write \( \Pi(z) = E(z^X) \).

\[
E(X) = \Pi'(z)|_{z=1} = \lambda tK'(z) \exp[\lambda t(K(z) - 1)]|_{z=1} = \lambda t \mu.
\]

\[
\text{var}(X) = \Pi''(z)|_{z=1} + E(X) - [E(X)]^2 \\
= [\lambda tK'(z)]^2 \exp[\lambda t(K(z) - 1)]|_{z=1} + \lambda tK''(z) \exp[\lambda t(K(z) - 1)]|_{z=1} \\
+ \lambda t \mu - (\lambda t \mu)^2 \\
= (\lambda t \mu)^2 + \lambda t(\sigma^2 - \mu + \mu^2) + \lambda t \mu - (\lambda t \mu)^2 \\
= \lambda t(\sigma^2 + \mu^2).
\]
2. (a) The state space is \( \{0, 1, 2\} \). The instantaneous transition rates are given by

<table>
<thead>
<tr>
<th>Transition</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 \rightarrow 1</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>1 \rightarrow 2</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>2 \rightarrow 1</td>
<td>( \mu )</td>
</tr>
<tr>
<td>1 \rightarrow 0</td>
<td>( \mu )</td>
</tr>
</tbody>
</table>

(b) The detailed balance equations are

\[
\begin{align*}
\lambda \pi_0 &= \mu \pi_1 \\
\lambda \pi_1 &= \mu \pi_2
\end{align*}
\]

Hence

\[
\begin{align*}
\pi_1 &= \rho \pi_0 \\
\pi_2 &= \rho \pi_1 = \rho^2 \pi_0
\end{align*}
\]

Using the normalization condition \( \pi_0 + \pi_1 + \pi_2 = 1 \), we obtain

\[
(\pi_0, \pi_1, \pi_2) = \left( \frac{1}{1 + \rho + \rho^2}, \frac{\rho}{1 + \rho + \rho^2}, \frac{\rho^2}{1 + \rho + \rho^2} \right).
\]

In equilibrium the probability that an arriving customer is lost is given by the probability that an arriving customer finds 2 people already in the queue, i.e., by \( \pi_2 \).

(c) The probability that an arriving customer joins the queue is \( \pi_0 + \pi_1 \). The probability that an arriving customer who joins the queue finds no customers ahead of him is

\[
\frac{\pi_0}{\pi_0 + \pi_1} = \frac{1}{1 + \rho} = \frac{\mu}{\lambda + \mu}
\]

and the probability that he finds one customer ahead of him is

\[
\frac{\pi_1}{\pi_0 + \pi_1} = \frac{\rho}{1 + \rho} = \frac{\lambda}{\lambda + \mu}.
\]

If he finds no customers ahead of him, his waiting time has the exponential distribution with parameter \( \mu \), p.d.f. \( \mu e^{-\mu t} \) \((t \geq 0)\). If he finds one customer ahead of him, using the memoryless property of the exponential distribution, his waiting time, the sum of two i.i.d. exponentials, has the gamma distribution with parameters \( \mu \) and 2, p.d.f. \( \mu^2 t e^{-\mu t} \) \((t \geq 0)\).

Hence, unconditionally, the probability density function of the waiting time is given by

\[
\frac{\mu}{\lambda + \mu} \mu e^{-\mu t} + \frac{\lambda}{\lambda + \mu} \mu^2 t e^{-\mu t} = \frac{\mu^2 (1 + \lambda t)}{\lambda + \mu} e^{-\mu t} \quad (t \geq 0).
\]
3. (a) (i) The forward equations are
\[
\frac{dp_0}{dt} = -\lambda p_0(t) + \mu p_1(t)
\]
\[
\frac{dp_1}{dt} = -\mu p_1(t) + \lambda p_0(t)
\]
with initial conditions \( p_0(0) = 1, p_1(0) = 0 \).

Using the fact that the Laplace transform of the derivative of a function \( f \) is given by \( sf^*(s) - f(0) \), the Laplace transforms of the forward equations are as given in the question,
\[
sp_0^*(s) - 1 = -\lambda p_0^*(s) + \mu p_1^*(s),
\]
\[
sp_1^*(s) = -\mu p_1^*(s) + \lambda p_0^*(s).
\]

(ii) From equation (2),
\[
p_1^*(s) = \frac{\lambda}{\mu + s} p_0^*(s).
\]
Substituting into equation (1), we obtain
\[
p_0^*(s) = \frac{\mu + s}{s(\lambda + \mu + s)},
\]
\[
p_1^*(s) = \frac{\lambda}{s(\lambda + \mu + s)}.
\]

Rewriting the expression for \( p_1^*(s) \) in terms of partial fractions,
\[
p_1^*(s) = \frac{\lambda}{\lambda + \mu} \left( 1 - \frac{1}{s - \frac{\lambda}{\mu} + \frac{1}{s - \frac{\mu}{\lambda + s}}} \right).
\]

Hence the given expression for \( p_1(t) \). The expression for \( p_0(t) \) follows, using the fact that \( p_0(t) + p_1(t) = 1 \). [Alternatively, express \( p_0^*(s) \) in terms of partial fractions.]

(b) (i) The lifetime distribution is the sum of two independently distributed exponential variates, with parameters \( \lambda \) and \( \mu \), respectively (the distributions of the length of time spent in State 0 before a transition into State 1 and of the length of time spent in State 1 before a transition into State 0). The Laplace transform of the lifetime distribution is
\[
f^*(s) = \left( \frac{\lambda}{\lambda + s} \right) \left( \frac{\mu}{\mu + s} \right) = \frac{\lambda \mu}{\mu - \lambda} \left( \frac{1}{\lambda + s} - \frac{1}{\mu + s} \right).
\]

Hence the given expression for \( f(t) \).
(ii) According to the renewal-reward theorem, the long-term average cost of the process per unit time is the ratio of the expected total cost in any renewal cycle to the expected length of any renewal cycle.
In the present case, the expected length of the renewal cycle, i.e., the mean of the lifetime distribution, is $1/\lambda + 1/\mu$. The expected length of time that the lift is out of order is $1/\mu$. Hence the required ratio is

$$\frac{c/\mu + k}{1/\lambda + 1/\mu} = \frac{\lambda(c + \mu k)}{\lambda + \mu}.$$
4. (a) (i) The identify statement generates the listing and plot of the acf and pacf of the process ($y_t$) after differencing and seasonal differencing at lag 4. The acf of the differenced data appears to be that of a stationary process. Only the autocorrelations at lags 1, 3 and 4 are clearly significant, suggesting a multiplicative seasonal model, ARIMA(0, 1, 1) × (0, 1, 1)4 or perhaps ARIMA(0, 1, 4) × (0, 1, 0)4, for ($y_t$). The pacf is less clear cut, with cut-off point at lag 10 and an earlier cut-off point at lag 4. So perhaps an ARIMA(4, 1, 0) × (0, 1, 1)4 might be considered.

(ii) ARIMA(4, 1, 0) × (0, 1, 1)4:

$$\begin{align*}
(1 - L)(1 - L^4)(1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3 - \phi_4 L^4) y_t &= \epsilon_t \\
\text{ARIMA}(0, 1, 4) \times (0, 1, 0)4: \\
(1 - L)(1 - L^4) y_t &= (1 - \theta_1 L - \theta_2 L^2 - \theta_3 L^3 - \theta_4 L^4) \epsilon_t \\
\text{ARIMA}(0, 1, 0) \times (0, 1, 1)4: \\
(1 - L)(1 - L^4) y_t &= (1 - \Theta_1 L^4) \epsilon_t \\
\text{ARIMA}(0, 1, 1) \times (0, 1, 1)4: \\
(1 - L)(1 - L^4) y_t &= (1 - \theta_1 L)(1 - \Theta_1 L^4) \epsilon_t
\end{align*}$$

(iii) For the first 3 models, some of the p-values of the portmanteau statistics are significant, whereas for the fourth model none are significant. So the fourth model gives the best fit in this sense. The fourth model also has the smallest value of the AIC and the smallest value of the SBC. So it gives the best fit in this sense too.

The fitted equation for this model is

$$\begin{align*}
(1 - L)(1 - L^4) y_t &= (1 - 0.50119 L)(1 - 0.72547 L^4) \epsilon_t \\
i.e.
\quad y_t &= y_{t-1} + y_{t-4} - y_{t-5} + \epsilon_t - 0.50119 \epsilon_{t-1} - 0.72547 \epsilon_{t-4} + 0.36360 \epsilon_{t-5}
\end{align*}$$

(b) (i) A vector autoregressive model, a VAR($p$) model, is being fitted to the differenced data. Writing $Y_t = ((1 - L)(1 - L^4)x_t, (1 - L)(1 - L^4)y_t)^t$, the model satisfies the equation

$$Y_t - \mu = \Phi_1(Y_{t-1} - \mu) + \ldots + \Phi_p(Y_{t-p} - \mu) + \epsilon_t \quad (-\infty < t < \infty),$$

where $\mu$ is a 2×1 mean vector, $\Phi_1, \ldots, \Phi_p$ are 2×2 matrices of autoregressive coefficients and $(\epsilon_t)$ is a white noise process. In fact, in the present case, $\mu$ is taken to be 0.
(ii) The `minic` option produces a table containing the values of the AIC for the VAR(\(p\)) models for \(p\) in the range from 0 to 5. It fits the model for the value of \(p\) that minimizes the AIC, in this case \(p = 4\).

(iii)

\[
(1 - L)(1 - L^4)x_t = -0.437(1 - L)(1 - L^4)x_{t-4} + \epsilon_t \\
(1 - L)(1 - L^4)y_t = -0.392(1 - L)(1 - L^4)y_{t-1} - 0.279(1 - L)(1 - L^4)y_{t-2} \\
-0.404(1 - L)(1 - L^4)y_{t-4} + \eta_t
\]

A vector autoregressive model is fitted so that the lagged values of either series may be used in the model equation for the other series. In the present case, it appears that none of the coefficients for the lagged values of either series in the model equation for the other series is significant. So there appears to be no significant evidence that either series is of use in predicting the other series. The model for the series \((y_t)\) here is essentially the same as the ARIMA\((4, 1, 0) \times (0, 1, 0)_4\) model noted in part (a)(ii).

(iv) The output statement produces forecasts of production and imports for the next 4 quarters, i.e., for 2010.