5 Factorial Models

5.1 Introduction

In the experimental designs that have concerned us so far, we have been chiefly interested in how the levels of a single factor affect the variation in the response variable. However, in some situations, it may be suspected that the responses may be affected by two or more factors. We can investigate to what extent this is indeed the case by setting up the appropriate experimental design, and performing a statistical analysis with respect to a model to be introduced later.

Example 5.1 (Battery Operating Conditions (c.f. Montgomery p.234-235))
An engineer is investigating whether battery life is affected by:
- the material from which it is made;
- the prevailing temperature in its end-use environment

There are 3 different types of material ($M_1, M_2, M_3$), and three temperatures that are of particular interest (1 for $-9.4^\circ C$, 2 for $21.1^\circ C$, and 3 for $51.7^\circ C$).

The design of the experiment is such that 4 different batteries are tested at each combination of levels of material and temperature, and all $3 \times 3 \times 4 = 36$ runs are tested in random order.

The battery lifetime data (in hours) arising from the experiment are given in the table below:

<table>
<thead>
<tr>
<th>Material</th>
<th>Temperature</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$M_1$</td>
<td>130</td>
<td>155</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td></td>
<td>74</td>
<td>180</td>
<td>80</td>
</tr>
<tr>
<td></td>
<td>$M_2$</td>
<td>150</td>
<td>188</td>
<td>136</td>
</tr>
<tr>
<td></td>
<td></td>
<td>159</td>
<td>126</td>
<td>106</td>
</tr>
<tr>
<td></td>
<td>$M_3$</td>
<td>138</td>
<td>110</td>
<td>174</td>
</tr>
<tr>
<td></td>
<td></td>
<td>168</td>
<td>160</td>
<td>150</td>
</tr>
</tbody>
</table>

This is an example of a two-factor factorial design.
5.2 The Completely Randomized Two-Factor Factorial Design

A more general representation of the design corresponding to Example 5.1 may be given as follows:

<table>
<thead>
<tr>
<th>Factor B</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>j</th>
<th>...</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>j</td>
<td>...</td>
<td>b</td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>...</td>
<td>j</td>
<td>...</td>
<td>b</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>...</td>
<td>j</td>
<td>...</td>
<td>b</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>i</td>
<td>1</td>
<td>...</td>
<td>j</td>
<td>...</td>
<td>b</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>a</td>
<td>a11</td>
<td>...</td>
<td>a1n</td>
<td>...</td>
<td>abn</td>
</tr>
</tbody>
</table>

The completely randomized two-factor factorial design is characterized by:

- a collection of responses \( \{y_{ijk}\} \), where \( y_{ijk} \) represents the response corresponding to the \( k \)-th replicate at the \( i \)-th level of \( A \), and the \( j \)-th level of \( B \), with
  - \( i = 1, \ldots, a \)
  - \( j = 1, \ldots, b \)
  - \( k = 1, \ldots, n \)
- \( n \geq 2 \) so that there are at least 2 replications in each cell of the table;
- the \( abn \) responses are taken in random order (thus completely randomized).

It is usual to associate the following linear statistical model to this design:

\[
y_{ijk} = \mu + \tau_i + \beta_j + (\tau \beta)_{ij} + \epsilon_{ijk} \quad i=1, \ldots, a \\
\quad \quad \quad j=1, \ldots, b \\
\quad \quad \quad k=1,2,\ldots,n
\]  

where

- \( \mu \) is the overall mean\(^1\),
- \( \tau_i \) is the effect at the \( i \)-th level of Factor \( A \),
- \( \beta_j \) is the effect at the \( j \)-th level of Factor \( B \),
- \( (\tau \beta)_{ij} \) is the interaction effect at the \( i \)-th level of factor \( A \) and \( j \)-th level of factor \( B \), and
- \( \epsilon_{ijk} \sim \text{NID}(0, \sigma^2) \) is a random error term.

We further suppose that all of the effects defined above are fixed. Thus, we impose the convenient (and intuitively reasonable) constraints:

\[
\sum_{i=1}^{a} \tau_i = 0 \quad (2) \\
\sum_{j=1}^{b} \beta_j = 0 \quad (3) \\
\sum_{i=1}^{a}(\tau \beta)_{ij} = 0 \quad j=1, \ldots, b \quad (4) \\
\sum_{j=1}^{b}(\tau \beta)_{ij} = 0 \quad i=1, \ldots, a. \quad (5)
\]

\(^1\)when using the sum-to-zero constraints.
Differences between the various levels of the two factors are of equal interest this time, and so we test between the following pairs of hypotheses:

\[ H_{0A} : \tau_1 = \tau_2 = \ldots = \tau_a = 0 \]
\[ H_{1A} : \tau_i \neq 0 \text{ for some } i \]

\[ H_{0B} : \beta_1 = \beta_2 = \ldots = \beta_b = 0 \]
\[ H_{1B} : \beta_j \neq 0 \text{ for some } j \]

\[ H_{0AB} : (\tau \beta)_{ij} = 0 \text{ for all } i, j \]
\[ H_{1AB} : (\tau \beta)_{ij} \neq 0 \text{ for some } i, j \]

### 5.3 Some Notation

For our particular model, let us introduce the following (sub-) totals:

\[ y_{i..} = \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk} \quad i=1,\ldots,a \]

\[ y_{.j.} = \sum_{i=1}^{a} \sum_{k=1}^{n} y_{ijk} \quad j=1,\ldots,b \]

\[ y_{ij.} = \sum_{k=1}^{n} y_{ijk} \quad i=1,\ldots,a, \quad j=1,\ldots,b \]

\[ y_{...} = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk} \]

with

\[ \overline{y}_{i..} = y_{i..}/bn \quad \overline{y}_{.j.} = y_{.j.}/an \quad \overline{y}_{ij.} = y_{ij.}/n \quad \overline{y}_{...} = y_{...}/abn. \]

### 5.4 Parameter Estimation

We derive parameter estimates for our model based upon the principle of least squares. The form of these estimates will help to motivate our decomposition of the total sum-of-squares when we consider Analysis of Variance.

First we construct the quantity

\[ L = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (y_{ijk} - \mu - \tau_i - \beta_j - (\tau \beta)_{ij})^2 \]
which is a measure of the distance between the data points \( \{y_{ijk}\} \) and their expected values under the model (1).

Our least squares estimates will be those values of \( \mu, \{\tau_i\}, \{\beta_j\}, \{(\tau\beta)_{ij}\} \) that (jointly) minimize \( L \): let us denote these by \( \hat{\mu}, \{\hat{\tau}_i\}, \{\hat{\beta}_j\}, \{(\hat{\tau}\hat{\beta})_{ij}\} \), respectively.

Carrying out partial differentiations with respect to each of these parameters and equating to zero yields the following equations:

\[
-2 \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (y_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - (\hat{\tau}\hat{\beta})_{ij}) = 0 \quad (6)
\]

\[
-2 \sum_{j=1}^{b} \sum_{k=1}^{n} (y_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - (\hat{\tau}\hat{\beta})_{ij}) = 0 \quad i = 1, \ldots, a \quad (7)
\]

\[
-2 \sum_{i=1}^{a} \sum_{k=1}^{n} (y_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - (\hat{\tau}\hat{\beta})_{ij}) = 0 \quad j = 1, \ldots, b \quad (8)
\]

\[
-2 \sum_{k=1}^{n} (y_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - (\hat{\tau}\hat{\beta})_{ij}) = 0 \quad i = 1, \ldots, a, \quad j = 1, \ldots, b. \quad (9)
\]

These can be re-arranged to yield

\[
abn\hat{\mu} + bn\sum_{i=1}^{a} \hat{\tau}_i + an\sum_{j=1}^{b} \hat{\beta}_j + n\sum_{i=1}^{a} \sum_{j=1}^{b} (\hat{\tau}\hat{\beta})_{ij} = y_{..} \quad (10)
\]

\[
bn\hat{\mu} + bn\hat{\tau}_i + n\sum_{j=1}^{b} \hat{\beta}_j + n\sum_{j=1}^{b} (\hat{\tau}\hat{\beta})_{ij} = y_{..} \quad i = 1, \ldots, a \quad (11)
\]

\[
an\hat{\mu} + n\sum_{i=1}^{a} \hat{\tau}_i + an\hat{\beta}_j + n\sum_{i=1}^{a} (\hat{\tau}\hat{\beta})_{ij} = y_{..} \quad j = 1, \ldots, b \quad (12)
\]

\[
n\hat{\mu} + n\hat{\tau}_i + n\hat{\beta}_j + n(\hat{\tau}\hat{\beta})_{ij} = y_{ij} \quad i = 1, \ldots, a, \quad j = 1, \ldots, b. \quad (13)
\]

Now (10)-(13) constitute \( ab + a + b + 1 \) equations in that many unknowns. However, they do not possess a unique solution. Indeed:
- the \( a \) equations in (11) can be generated by summing (13) over \( j \) for each \( i \);
- the \( b \) equations in (12) can be generated by summing (13) over \( i \) for each \( j \);
- (10) can be generated by summing (13) over all \( i \) and \( j \).

Thus, there are \( a + b + 1 \) linear dependencies, and so we need to impose an additional \( a + b + 1 \) constraints in order to obtain unique estimates. In line with the constraints (2) to (5), we introduce the following \( a + b + 1 \) constraints on our parameter estimates:
\[
\sum_{i=1}^{a} \hat{\tau}_i = 0 \\
\sum_{j=1}^{b} \hat{\beta}_j = 0 \\
\sum_{i=1}^{a} (\hat{\tau} \hat{\beta})_{ij} = 0 \quad j = 1, \ldots, b \\
\sum_{j=1}^{b} (\hat{\tau} \hat{\beta})_{ij} = 0 \quad i = 1, \ldots, a.
\] (14) (15) (16) (17)

Notice that (14) and (15) constitute 2 independent constraints, whereas (16) and (17) constitute only \(a + b - 1\) constraints. After invoking these constraints, we obtain the following parameter estimates:

\[
\hat{\mu} = \frac{y_{..}}{abn} = \bar{y}_{..} \\
\hat{\tau}_i = \frac{y_{i.} - \hat{\mu} = \bar{y}_{i.} - \bar{y}_{..}}{bn} \quad i = 1, \ldots, a \\
\hat{\beta}_j = \frac{y_{.j} - \hat{\mu} = \bar{y}_{.j} - \bar{y}_{..}}{an} \quad j = 1, \ldots, b \\
(\hat{\tau} \hat{\beta})_{ij} = \frac{y_{ij.} - \hat{\tau} - \hat{\beta} - \hat{\mu}}{n} \\
= \bar{y}_{ij} - (\bar{y}_{i.} - \bar{y}_{..}) - (\bar{y}_{.j} - \bar{y}_{..}) - \bar{y}_{..} \\
= \bar{y}_{ij} - \bar{y}_{i.} + \bar{y}_{.j} + \bar{y}_{..} \quad i = 1, \ldots, a, \quad j = 1, \ldots, b.
\] (18) (19) (20) (21)

And the estimated or fitted value of \(y_{ijk}\) is given by:

\[
\hat{y}_{ijk} = \hat{\mu} + \hat{\tau}_i + \hat{\beta}_j + (\hat{\tau} \hat{\beta})_{ij} \\
= \bar{y}_{..} + (\bar{y}_{i.} - \bar{y}_{..}) + (\bar{y}_{.j} - \bar{y}_{..}) + (\bar{y}_{ij} - \bar{y}_{i.} + \bar{y}_{.j} + \bar{y}_{..}) \\
= \bar{y}_{ij}.
\]

for \(i = 1, \ldots, a, \quad j = 1, \ldots, b, \quad k = 1, \ldots, n.\)

Therefore, each one of the \(abn\) observations is estimated by the average of the \(n\) observations in its corresponding cell of the table. The residuals for this model are given by the \(\{e_{ijk}\}\) where

\[
e_{ijk} := y_{ijk} - \hat{y}_{ijk} = y_{ijk} - \bar{y}_{ij}.
\] (22)

\[^2\text{Why so?}\]
5.5 Analysis of Variance

Motivated by the form of the estimates for the effects, measures of the variation between levels of A, and the levels of B, may be defined as

\[
SS_A = bn \sum_{i=1}^{a} (\bar{y}_{i..} - \bar{y}_{...})^2
\]

\[
SS_B = an \sum_{j=1}^{b} (\bar{y}_{.j} - \bar{y}_{...})^2
\]

respectively.

Also, considering the form of the \( \{(\hat{\tau} \hat{\beta})_{ij}\} \), then a measure of the interaction between factors A and B is

\[
SS_{AB} = n \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{y}_{ij} - \bar{y}_{i..} - \bar{y}_{.j} + \bar{y}_{...})^2
\]

We can think of the residual sum-of-squares as literally being the sum of squares of the residuals, and so

\[
SS_R = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (y_{ijk} - \bar{y}_{ij.})^2
\]

A measure of the total variation in the data, is given by the total sum-of-squares, which is defined as

\[
SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (y_{ijk} - \bar{y}_{...})^2
\]

It turns out that \( SS_T \) can be decomposed to yield an identity which solely involves the previously defined quantities:\[3^3\]

\[
SS_T = SS_A + SS_B + SS_{AB} + SS_R
\]  

Degrees of Freedom
Since Factor A occurs at \( a \) levels, then we associate it with \( a - 1 \) degrees of freedom. Similarly Factor B occurs at \( b \) levels, so this has \( b - 1 \) degrees of freedom.

The interaction between A and B is associated with the degrees of freedom for the \( ab \) cells of the table, i.e. \( ab - 1 \), less those for A, and B: thus \( SS_{AB} \) has \( ab - 1 - (a - 1) - (b - 1) \), i.e.

---

\[3^3\]As usual, the cross terms that emerge yield zero, overall.
(a − 1)(b − 1) degrees of freedom.

As for \( SS_R \), within each of the ab cells of the design, there are \( n \) data points: thus there are \( a \times b \times (n − 1) \) degrees of freedom associated with the quantity.

But notice that the aforementioned degrees of freedom sum to \( abn − 1 \), i.e. those for \( SS_T \):

\[
abn - 1 = (a - 1) + (b - 1) + (a - 1)(b - 1) + ab(n - 1).
\]

We summarize these results in the form of an ANOVA table:

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( a - 1 )</td>
<td>( bn \sum_{i=1}^{a}(\bar{y}<em>{i.} - \bar{y}</em>{..})^2 )</td>
<td>( SS_A/(a-1) )</td>
<td>( F_A = \frac{MS_A}{MS_R} )</td>
</tr>
<tr>
<td>B</td>
<td>( b - 1 )</td>
<td>( an \sum_{j=1}^{b}(\bar{y}<em>{.j} - \bar{y}</em>{..})^2 )</td>
<td>( SS_B/(b-1) )</td>
<td>( F_B = \frac{MS_B}{MS_R} )</td>
</tr>
<tr>
<td>AB</td>
<td>( (a - 1)(b - 1) )</td>
<td>( n \sum_{i=1}^{a} \sum_{j=1}^{b}(\bar{y}<em>{ij.} - \bar{y}</em>{i..} - \bar{y}<em>{.j} + \bar{y}</em>{..})^2 )</td>
<td>( SS_{AB}/(a-1)(b-1) )</td>
<td>( F_{AB} = \frac{MS_{AB}}{MS_R} )</td>
</tr>
<tr>
<td>Residual</td>
<td>( ab(n - 1) )</td>
<td>By subtraction</td>
<td>( SS_R/ab(n - 1) )</td>
<td></td>
</tr>
</tbody>
</table>

Total \( abn - 1 \) \( \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n}(y_{ijk} - \bar{y}_{..})^2 \)

The \( F \)-statistics suggested in the final column are motivated by the fact that the expectations of the mean sum-of-squares take the following form:

\[
E[MS_A] = E \left[ \frac{SS_A}{a-1} \right] = \sigma^2 + \frac{bn \sum_{i=1}^{a} \tau_i^2}{a-1}
\]

\[
E[MS_B] = E \left[ \frac{SS_B}{b-1} \right] = \sigma^2 + \frac{an \sum_{j=1}^{b} \beta_j^2}{b-1}
\]

\[
E[MS_{AB}] = E \left[ \frac{SS_{AB}}{(a-1)(b-1)} \right] = \sigma^2 + \frac{n \sum_{i=1}^{a} \sum_{j=1}^{b} (\tau \beta)_{ij}^2}{(a-1)(b-1)}
\]

\[
E[MS_R] = E \left[ \frac{SS_R}{ab(n-1)} \right] = \sigma^2
\]

Large values of \( F_A, F_B, F_{AB} \), lead us to doubt that \( H_{0A}, H_{0B}, H_{0AB} \) hold true, respectively. Furthermore, the relevant distribution theory allows us to conclude that we:
• Reject $H_0A$ if $F_A > F_{a-1, ab(n-1), \alpha}$
• Reject $H_0B$ if $F_B > F_{b-1, ab(n-1), \alpha}$
• Reject $H_0AB$ if $F_{AB} > F_{(a-1)(b-1), ab(n-1), \alpha}$

each at the 100$\alpha$% level of significance.

Computational Issues
In order to facilitate the coding of a procedure that performs ANOVA (customized in a particular computer language/package), we express the sums-of-squares in terms of the (sub-)totals arising in the table.

First define $C_f = \frac{y^2}{abn}$. Then

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} y^2_{ijk} - C_f$$

$$SS_A = \frac{1}{bn} \sum_{i=1}^{a} y^2_{i..} - C_f$$

$$SS_B = \frac{1}{an} \sum_{j=1}^{b} y^2_{.j} - C_f$$

We can deal with the computation of $SS_{AB}$ in two stages. First define

$$SS_{Subtotals} = \frac{1}{n} \sum_{i=1}^{a} \sum_{j=1}^{b} y^2_{ij} - C_f$$

Then, secondly, we can obtain $SS_{AB}$ by subtraction, as it can be shown that

$$SS_{AB} = SS_{Subtotals} - SS_A - SS_B.$$ 

Finally, $SS_R$ can be obtained via subtraction through the identity (23):

$$SS_R = SS_T - SS_A - SS_B - SS_{AB}$$

which conveniently reduces down to

$$SS_R = SS_T - SS_{Subtotals}.$$
We now have the apparatus to carry out the statistical analysis for the data arising in our introductory example:

**Example 5.2 (Battery Operating Conditions contd.)**

To perform the analysis in S-PLUS we create 3 vectors: one containing the battery life readings, and the other two containing the corresponding levels of factor $A$, material type, and factor $B$, temperature:

```r
> battery.life<-c(130,155,74,180,150,188,159,126,138,
+ 110,168,160,34,40,80,174,120,150,
+ 139,20,70,82,58,25,70,58,45,96,104,82,60)
> material<-rep(rep(1:3,rep(4,3)),3)
> material.fac<-factor(material)
> temp<-rep(1:3,rep(12,3))
> temp.fac<-factor(temp)
> battery<-data.frame(battery.life,temp.fac,material.fac)
```

Noting that `:` is used to construct the *interaction* term in S-PLUS, then the analysis of variance can be performed as follows:

```r
> battery.aov<-aov(battery.life~material.fac+temp.fac+material.fac:temp.fac, data=battery)
> summary(battery.aov)

            Df  Sum of Sq Mean Sq  F value  Pr(>F)
material.fac  2   10683.72 5341.86   7.91137 0.001976
temp.fac     2   39118.72 19559.36  28.96769 0.000000
material.fac:temp.fac  4    9613.78 2403.44  3.55954 0.018611
Residuals   27  18230.75   675.21

> plot(fitted(battery.aov),residuals(battery.aov))
> lines(c(30,175),c(0,0))
> plot(material,residuals(battery.aov))
> lines(c(0.5,3.5),c(0,0))
> plot(temp,residuals(battery.aov))
> lines(c(0.5,3.5),c(0,0))
> plot(fitted(battery.aov),residuals(battery.aov))
> lines(c(35,170),c(0,0))
```

Here, both $F_A$ and $F_B$ are significant at the 1% level, and $F_{AB}$ is significant at the 2% level. Thus, variation in the battery life data can be accounted for by the effects of different materials, different temperatures, and interaction between the various materials and temperatures.

Before endorsing these conclusions, we should check the adequacy of the model: this can be done by inspection of the residuals.

Figure 1 shows a mild tendency for the variability of the residuals to increase with the size of the fitted values. Not very significant to be of too much concern.

The variability of the residuals for material type $M_1$ is greater than those for $M_2$ and $M_3$.
(Figure 2); similarly the variability of the residuals for temperature 1 is higher than those for temperatures 2 and 3 (Figure 3).

Upon closer inspection, we see that rather high residuals are to be found in the cell corresponding to material type $M_1$ and temperature 1, these being $-60.75$ and $45.25$:

$$\text{residuals(battery.aov)}$$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Res</td>
<td>-4.75</td>
<td>20.25</td>
<td>-60.75</td>
<td>45.25</td>
<td>-5.75</td>
<td>32.25</td>
<td>3.25</td>
<td>-29.75</td>
<td>-6</td>
<td>-34</td>
<td>24</td>
<td>16</td>
<td>-23.25</td>
</tr>
<tr>
<td>Res</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Res</td>
<td>-17.25</td>
<td>22.75</td>
<td>17.75</td>
<td>16.25</td>
<td>2.25</td>
<td>-13.75</td>
<td>-4.75</td>
<td>28.25</td>
<td>-25.75</td>
<td>4.25</td>
<td>-6.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Res</td>
<td>25</td>
<td>26</td>
<td>27</td>
<td>28</td>
<td>29</td>
<td>30</td>
<td>31</td>
<td>32</td>
<td>33</td>
<td>34</td>
<td>35</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>Res</td>
<td>-37.5</td>
<td>12.5</td>
<td>24.5</td>
<td>0.5</td>
<td>-24.5</td>
<td>20.5</td>
<td>8.5</td>
<td>-4.5</td>
<td>10.5</td>
<td>18.5</td>
<td>-3.5</td>
<td>-25.5</td>
<td></td>
</tr>
</tbody>
</table>

We should attempt to find an explanation for these high residuals, which may have resulted from an error in recording the battery life, faulty equipment etc., and then remove/replace that portion of the data.

Estimates of the mean and effects\(^4\) can be obtained as follows:

$$\text{dummy.coef(battery.aov)}$$

\$\text{"(Intercept)"}:\$

(Intercept)

105.5278

\$\text{"material.fac"}:\$

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>-22.36111</td>
<td>2.805556</td>
<td>19.55556</td>
</tr>
</tbody>
</table>

\$\text{"temp.fac"}:\$

<table>
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<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>39.30556</td>
<td>2.055556</td>
<td>-41.36111</td>
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</tbody>
</table>

\$\text{"material.fac:temp.fac"}:\$

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<th>21</th>
<th>31</th>
<th>12</th>
<th>22</th>
<th>32</th>
<th>13</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>33</td>
<td>1.777778</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

### 5.6 No Interaction Model

We could associate a linear statistical model with our experimental design that has no interaction term. In this case, the model equation becomes

$$y_{ijk} = \mu + \tau_i + \beta_j + \epsilon_{ijk} \quad i=1,...,a, \quad j=1,...,b, \quad k=1,...,n$$

One can derive parameter estimates in much the same way as for the interaction model to discover that this time round, the fitted values are

$$\hat{y}_{ijk} = \bar{y}_{i.} + \bar{y}_{.j.} - \bar{y}_{...} \quad i=1,...,a, \quad j=1,...,b, \quad k=1,...,n$$

\(^4\)under the ‘sum-to-zero’ constraints as previously described.
Analysis of Variance is performed as follows:

```r
> battery.aov<-aov(battery.life~material.fac+temp.fac, data=battery)
> summary(battery.aov)

Df Sum of Sq  Mean Sq F value  Pr(>F)
material.fac 2     10683.72   5341.86   5.9472 0.006515
temp.fac     2      39118.72  19559.36  21.7759 0.000001
Residuals   31      27844.53    898.21

```

Model still indicates significant differences due to material type and temperature. However, we must check whether the model fits adequately or not.

```r
> plot(fitted(battery.aov), residuals(battery.aov))
> lines(c(35,170),c(0,0))
```

In Figure 4 we see some kind of trend in the residuals, moving from 'high' to 'low', to 'high', then back to 'low' again, as the fitted values increase. Based on this residual analysis, we reject this no-interaction model.

![Figure 1: Plot of the residuals vs. fitted values](image-url)
Figure 2: Plot of the residuals vs. material type ($M_1=1$, $M_2=2$, $M_3=3$)

Figure 3: Plot of the residuals vs. temperature
Figure 4: Plot of the residuals vs. fitted values: no interaction model