Algebraic Quantum Mechanics, Algebraic Spinors and Hilbert Space.

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Abstract.

The orthogonal Clifford algebra and the generalised Clifford algebra, C^n, (discrete Weyl algebra) is re-examined and it is shown that the quantum mechanical wave function (element of left ideal), density operator (element of a two sided ideal) and mean values (algebraic trace) can be constructed from entirely within the algebra. No appeal to Hilbert space is necessary. We show how the GNS construction can be obtained from within both algebras. The limit of C^n as n is shown to be the extended Heisenberg algebra. Finally the relationship to the usual Hilbert space approach is discussed.

1. Introduction.

In my study of the structure of Clifford algebras particularly in regard to the role they play in quantum mechanics through Pauli spinors, the Dirac spinors and Penrose's twistors, I was struck by how all the relevant results could be accounted for from within the algebra. There was no need to leave the algebra and to construct an external Hilbert space upon which the elements of the algebra were deemed to act when they play the role of observables. Of course all of these algebras are finite and non-nilpotent so that they are easily treated by standard algebraic analysis. My original thoughts on these topics can be found in Frescura and Hiley (1980, 1984 and 1987).

However when it comes to the Heisenberg algebra these techniques appear not to be applicable because this algebra is nilpotent. However the earlier work of Schönberg (1957) and the later ideas of Frescura and Hiley (1989) and Hiley (1991) suggested there was such a possibility provided one extended the algebra (Hiley 2001). Considerable

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clarification of the meaning of this specific extension has been achieved by exploring the finite discrete Weyl algebra[1] C^n. (Morris 1967 and Hiley and Monk 1993). The importance of these algebras for topic of discussion in this paper is that in the limit as n the algebra contains the Heisenberg algebra. These algebras are finite so that the same techniques used in Clifford algebras can be directly applied. Thus we have a unified way of dealing with the basic algebras that lie at the heart of quantum mechanics.

These results have only intensified my curiosity as to why most if not all of the results can be obtained without seemingly the need to resort to Hilbert space. This goes against the prevailing orthodoxy that appears to insist that quantum mechanics cannot be done except in the context of a Hilbert space. Yet there have been other voices raised against the necessity of Hilbert space. Von Neumann himself wrote to Birkoff (1966) writing "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more." (A detailed discussion of why von Neumann made this comment can be found in Rédei 1996).

But there are more important reasons why an algebraic approach has advantages. As Dirac (1965) has stressed, when algebraic methods are used for systems with an infinite number of degrees of freedom (e.g., field theory), we can obtain solutions to some physical problems that give no solution in the usual Schrödinger picture. This possibility arises because in these systems the equivalence between the Schrödinger representation and the Heisenberg representation no longer holds. Indeed in field theory there is no longer one unique representation. There exist many inequivalent representations and these give rise to more general structures as has been pointed out by Emch (1972) and more recently in Haag (1992).

These works are of a more mathematical nature but from the point of view of physics what emerges is that the density operator, ρ, rather than the wave function, plays a fundamental role. This means we have a natural way of describing more general quantum processes that involve, for example, thermal systems. Systems described by wave functions then correspond to the special case when $\rho^2 = \rho$ which means the system is effectively at zero temperature. But it is not merely a question of the appearance of non-zero temperatures. As Prigogine (1994) has already pointed out, nonintegrable systems give rise to diffusive terms that correspond to the symmetry breaking in time. Such systems cannot be described in terms of wave functions. The only possible way of describing such systems is through the density operator.

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[1] These algebras are called generalised Clifford algebras by Morris (1967)

While all the important new features that arise in the algebraic approach involve an infinite number of degrees of freedom even in the case of systems with a finite number of degrees of freedom we find new features that are rarely discussed in the physics literature. One such example, which we shall discuss later, is the appearance of algebraic spinors. We show that these spinors are a generalisation of the ordinary spinor, which have new mathematical implications that are ignored in the usual approach.

This generalisation is not only an enrichment of the mathematics. Following on from the work of Frescura and Hiley (1980a, 1980b), Monk and Hiley (1998) have pointed out that the algebra provides a very different perspective on the nature of quantum processes. In fact by emphasising the purely algebraic aspect of the approach we remove the distinction between operator and operand and this allows us to provide an interpretation in terms of the notion of process rather than in terms of particles and/or fields in interaction. These ideas were developed to explore connections with non-commutative geometry but we will not develop these ideas here (See Hiley 1991).

Indeed it is not the purpose of this paper to attempt to justify this more general and speculative position here. Rather we want to explore the mathematical consequences of adopting this more general position by bringing out its relationship to the usual approach, a feature that is generally lost in the many details provided, for example, in excellent two volume treatise by Bratteli and Robinson (1979). In particular we want to present a simpler approach through which we can see what results depend only on the structure of the abstract algebra, what features depend on a specific representation, usually a matrix representation, and what features require the construction of a Hilbert space.

To maintain generality we start by recalling the main properties of an abstract * algebra. We then show a how a measure, , called a weight is introduced, which plays the role of the state. is a functional, mapping elements of the algebra onto the real field. It is not difficult to show that this is equivalent to introducing the density operator in the usual approach

To motivate the approach we first explore the orthogonal Clifford algebra by showing how primitive idempotents play a crucial role in this approach. We use these idempotents can be used to construct minimal left and right ideals. These ideals are spanned by algebraic spinors and these elements play the role of 'wave functions' in the Hilbert space formalism. But it must be emphasised these elements are contained entirely within the algebra and no external vector space is needed. We have already stressed this point in Frescura and Hiley (1980a). In this paper we show how these same ideals are generated in the standard algebraic approaches described in Haag (1992), and in Emch

(1972). As the aim of this paper is mainly pedagogic, we rely heavily on the theorems proved in those volumes, particularly the latter and will not reproduce them here. We then go on to illustrate how these methods can be to specific example that use two of the simplest Clifford algebra $R_{1,1}$ and H. $R_{1,1}$ is the Clifford algebra of a relativistic space-time with one space dimension while H is the quaternion algebra. We choose these not for physical reasons but because they are the simplest examples in which to demonstrate the principles involved. We also show how the density operator is constructed and the role it plays in the whole structure.

We then widen the discussion to the discrete Weyl algebra C^n (Weyl 1931 and Monk and Hiley 1993) ultimately generalising to the symplectic Clifford algebra (Crumeyrolle 1990). This enables us to extend the discussion to include the Heisenberg algebra.

We bring out clearly what is involved in this structure by first exploring C^3. In the mathematics literature this gives the structure of the nonions first introduced as a generalisation of the quaternion many years ago by Sylvester (1884). As the generalisation to higher n is straight forward but tedious we finally discuss the limit as *n* to include the Heisenberg algebra to make contact with standard quantum mechanics. In doing this we show how the algebraic approach is related to the Hilbert space approach.

Finally we show how the density matrix can be written as a vector in a higher dimensional space, which can be mapped into a Hilbert space. This is the so-called GNS construction, which arises in a very straightforward manner in the algebraic approach. We are then immediately able to connect up with the thermal field theory approach of Umezawa (1993) that exploits the rich properties of bialgebras. Physically this approach allows a description of the thermal properties of quantum processes.

2. **The Algebra**.

Since our main intention of the paper is pedagogical we begin by outlining the properties of the general algebra within which we will work. Recall that a linear algebra, A, is a vector space with (i) addition $A + B$ A, (ii) multiplication by a scalar, λA A, λ is an element of the real or complex field, (iii) together with a product AB A. We will assume the product to be associative. Furthermore we will initially consider the algebra to be of finite rank, *n*, with a finite basis $\{e_i\}$.

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Since we require the analogue of a Hermitian conjugate we assume the algebra, A, is equipped with an involutive anti-automorphism defined by *: A → A with the following properties :

For A, B A

$$(A^*)^* = A, \qquad\qquad (\lambda A)^* = \lambda^* A^* \quad \lambda \quad C$$

$$(A + B)^* = A^* + B^* \qquad (AB)^* = B^* A^*$$

We will further assume the following two requirements are satisfied,

    (1) For each A A, B in A such that $A^*A = B$

    (2) $A^*A = 0$ $A = 0$.

So as not worry too much about the abstract nature of this algebra, it is useful to keep in mind a matrix algebra with Hermitian conjugation defined.

3. The Density Operator.

3.1 The Algebraic preliminaries.

We need some way of obtaining numbers from this algebra that will allow us to identify with expectation values used in quantum mechanics. We use a analogous method to that used in set theory by introducing a functional ω such that

    $\omega: A$ or C A A   such that $\omega(A) = \alpha,$ α or C

ω is a positive linear functional (called the expectation or state functional) satisfying,

    $\omega(\alpha A + \beta B) = \alpha\omega(A) + \beta\omega(B);$ $\omega(\lambda A) = \lambda\omega(A)$ and $\omega(A^*A)$ 0

If the algebra has a unit, I, then we choose $\omega(I) = 1$. This means that the state is normalsed. The collection of all linear functionals over A forms the dual space A* of the algebra.

Now for A A, we have a real or complex number $\omega(A)$ depending linearly on A with $\omega(A^*A)$ 0. This enables us to define a Hermitian scalar product through

$$\langle A|B \rangle = \omega \left(A\, B \right) \qquad\qquad A, B \quad \text{A}. \qquad\qquad (1)$$

The collection of all positive linear functionals over A is the positive cone $\text{A}*^{(+)}$ of the dual space $\text{A}*$ of the algebra.

In $\text{A}*$ ω is called 'extremal' if it cannot be decomposed into a linear combination of two others. These extremal functionals are called 'pure states'. Thus in general we can form

$$\omega(A) = \sum_{i=1}^{r} \lambda_i \omega_i(A) \qquad\qquad \text{with} \qquad\qquad \sum_i \lambda_i = 1$$

where ω_i are a set of extremal states.

As the algebras we consider in this paper are non-nilpotent, they have a set of idempotents. We can relate the extremal states to the primitive idempotents of the algebra in the following way. If $\{\varepsilon_i\}$ are a set of primitive idempotents[2], we can write $1 = \sum_{i=1}^{r} \varepsilon_i$ where $\varepsilon_i \varepsilon_j = \delta_{ij}\varepsilon_j$. Since $\omega(1) = 1$ we have

$$\omega(1) = \sum_{i=1}^{r} \lambda_i \omega_i(1) = \sum_{i=1}^{r} \lambda_i \omega_i \left(\sum_{j=1}^{r} \varepsilon_j \right) = \sum_{i,j=1}^{r} \lambda_i \omega_i(\varepsilon_j)$$

This defines the condition $\omega_i(\varepsilon_j) = \delta_{ij}$ so that the condition $\sum_i \lambda_i = 1$ is satisfied. In this way we establish a one-to-one correspondence between extremal states and primitive idempotents in the algebra.

We emphasise that through out this discussion we remain entirely within the algebra A even though the precise choice of the functional ω is left open at this stage. However if we are thinking in terms of a matrix algebra, then we can think of the linear functional as defining the trace. However it is not necessary to find a matrix representation of an algebra to define the invariant concept of a trace. For a finite algebra the trace can be evaluated by examining the coefficients of the characteristic polynomial of the algebra. A detailed discussion of this technique as applied to the algebras treated in this paper will be found in Frescura and Hiley (1981).

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[2] An idempotent ε is primitive iff there does not exist two idempotents ε_1, ε_2 such that $\varepsilon = \varepsilon_1 + \varepsilon_2$. An idempotent is primitive iff $\varepsilon A \varepsilon = \lambda \varepsilon$ where A A and λ Z the centre of A.

3.2 The Physics.

In quantum mechanics we are interested in expectation values. If the system is described by a pure state with wave function $\psi(r, t)$ then the expectation value is defined through the inner product

$$\langle A \rangle = \langle \psi \,|A|\, \psi \rangle = \int \psi^*(r,t) A \psi(r,t) d^3 r$$

where the assumption of the continuity of the wave function $\psi(r,t)$ forces us into an abstract Hilbert space. In what follows we will not make this assumption and will work completely in the algebra. Thus Hilbert space in not essential to deal with spin systems such as the Pauli spinor, the Dirac spinor and the twistor.

In the usual approach a system in a mixed state requires a density matrix. In this case the expectation value $\langle A \rangle = \sum_{i=1}^{r} \lambda_i \langle \psi_i \,|A|\, \psi_i \rangle$ can be written as

$$\langle A \rangle = Tr(\rho A) \qquad\qquad (2)$$

where ρ is the density operator with the properties that $\rho^\dagger = \rho$, $Tr(\rho A^* A) \geq 0$ and $Tr(\rho) = 1$.

Now in the usual approach using Hilbert space we always have to make a separation between the algebra of observables and the vector space on which the observables operate, but when the system is described by a finite algebra $\mathbb{A}$, there is no need to introduce this extra external space. Everything can be done within the algebra as there is a suitable vector space already contained completely within the algebra. Indeed there are a set of subspace of this vector space and these correspond to the minimal left ideals of the algebra. There is also a corresponding dual vector subspace, namely, the minimal right ideals. It is through these ideals that one can describe the physical properties of the physical system without leaving the algebra. (See Frescura and Hiley 1980a, 1980b, Benn and Tucker 1983 and Monk and Hiley 1998)

Let us recall how this can be done. Recall that a left ideal, I_L consists of a set of elements K such that

$$I_L = \{K \in \mathbb{A} \,|\, AK \in I_L \,|\, \forall A \in \mathbb{A}\} \qquad\qquad (3)$$

These are just the algebraic spinors discussed in Frescura and Hiley (1980a). They are generated by the primitive idempotents introduced above. For example, if ε_i is a primitive idempotent, then a minimal left ideal I_L is generated as follows;

$$K = A\varepsilon_i \qquad\qquad A \in \mathbb{A} \text{ and } K \in I_L$$

Now a right ideal I_R is defined through

$$I_R = \{K \in \mathbb{A} \mid KA \in I_R \mid A \in \mathbb{A}\} \tag{4}$$

So that a right ideal I_R can also be generated by ε_i giving

$$K' = \varepsilon_i A \qquad\qquad A \in \mathbb{A} \text{ and } K' \in I_R$$

Since $(\varepsilon_i)^* = \varepsilon_i$ we can generate a right ideal that is dual to the left ideal and is defined through

$$K^* = \varepsilon_i A^* \qquad\qquad A^* \in \mathbb{A}$$

Thus if $A \in I_L$ and $B^* \in I_R$ then equation (1) becomes

$$\omega(B\,A) = \langle B | A \rangle \tag{5}$$

where $\langle B | A \rangle \in \mathbb{C}$.

In order to make contact with the expectation values used in physics, the pure state 'wave function' is thus replaced by an element of the minimal left ideal and its Hermitian conjugate is replaced by an appropriate element of the right ideal. Then if $B_\psi \in I_L$ and $B_\psi^* \in I_R$,

$$\langle A \rangle = \omega_\psi\left(B^*_{\,\psi} A B_\psi\right) \quad \langle \psi | A | \psi \rangle = Tr\left(\rho_\psi A\right) \tag{6}$$

where ρ_ψ is the density operator for the pure state. We can now construct a density operator as a element in the algebra as follows:

$$\langle A \rangle = \omega_\psi\left(B^*_{\,\psi} A B_\psi\right) = \omega_\psi\left(A B_\psi B^*_{\,\psi}\right) = \omega_\psi\left(A\rho_\psi\right) = \omega_\psi\left(\rho_\psi A\right)$$

Here the density operator is

$$\rho_\psi = B_\psi B_\psi^* \qquad\qquad\qquad (7)$$

so that the density operator can be written as an element of the algebra, $\mathbb{A}$. We can also regard B as the 'square root' of the density operator, $B = \sqrt{\rho}$.

Let us make explicit the role of the primitive idempotent by writing

$$\rho_i = A\varepsilon_i A^* \qquad\qquad A \quad \mathbb{A}$$

Thus we see that the density operator is an element of a two-sided ideal subject to the conditions that ρ_i is a positive and $\omega(\rho_i) = 1$. This puts a normalisation constraint on A.

We can check that this density operator actually does correspond to a pure state by showing directly that ρ_i itself is idempotent. In fact

$$\rho_i^2 = (A\varepsilon_i A^*)(A\varepsilon_i A^*) = A(\varepsilon_i A^* A \varepsilon_i)A = \lambda A \varepsilon_i A^* = \lambda \rho_i$$

where λ or C. ρ_i can always be re-defined to absorb λ.

In the case of the mixed state we have

$$\langle A \rangle = \sum_{i=1}^{r} \lambda_i \omega_i (B_i\ AB_i) = \sum_{i=1}^{r} \lambda_i Tr(\rho_i A) \quad Tr(\rho A) \qquad\qquad (8)$$

where the density operator takes the form

$$\rho = \sum_{i=1}^{r} \lambda_i B_i B_i^* = \sum_{i=1}^{r} \left(\sqrt{\lambda_i} B_i \right) \left(\sqrt{\lambda_i} B_i^* \right) \qquad\qquad (9)$$

If we are able to write the density operator in the form $\rho = DD^*$ and we will show how this can be done later, then we can write

$$\langle A \rangle = \omega \left(ADD^* \right) = \omega \left(D^* AD \right) \quad \langle\ |A|\ \rangle \qquad\qquad (10)$$

Thus even in a mixed state we can describe the system by a single vector D. This is equivalent to replacing the density matrix as a single 'wave function' $|\ \rangle$ in the usual Hilbert space formalism. Thus we have the possibility of describing a thermal system by a wave function implying that this wave function is a function of temperature. In other

words the wave function becomes a function of temperature. This is what is called the GNS construction (see for example Emch 1972).

From these results we see that the algebraic approach gives primary significance to the density operator and that this operator lies entirely within the algebraic structure itself. The 'wave function' is replaced by an element of the left ideal, which, as we have seen, is an element of the algebra. Thus at least for finite algebras the Hilbert space is an external auxiliary feature that is not essential to describe quantum systems. Thus we can regard the wave function simply as a device to enabling us to calculate expectation values using familiar mathematics and not as some feature describing the state of the system.

This whole approach allows us to start from the more general mixed state, seeing the pure state as a particular rather simple state. Methodologically this is much more satisfactory than the usual approach because we are not faced with the need to generalise the formalism when we find we have to deal with more general systems that require 'outside' factors to somehow destroy coherence and form mixed states. Indeed the physicist's emphasis on the wave function as being a description of the state of a quantum system and therefore a primitive element of the description may be too restrictive (See Prigogine 1994).

A more general approach based on the algebra as suggested by Monk and Hiley (1998) has the possibility of avoiding some of the interpretation problems in quantum mechanics. However this comment should not be taken to imply that this new approach will enable us to solve all the problems of interpretation. It does not. What it does is to open up another way of looking at quantum phenomena in which the algebra itself is playing a primary role. In other words singling out the density operator as the primary feature of the mathematics opens up the possibility of giving a different meaning to the formalism (See Fernandes and Hiley 1997 and Brown and Hiley 2000).

As I have argued elsewhere, this approach suggests that a notion of process rather than particles/fields-in-interaction should be taken as basic (Hiley 1995). In a process based philosophy there are no separately existing objects. They are invariants of the total unfolding process. Here the wave function is regarded as a transition probability amplitude rather than a function of state of some system. As Bohr insisted, there is no sharp separation between subject and object so we should have no sharp separation between operator and operand. The mathematics of the algebraic approach reflects these notions and makes no such distinction. Thus we do not have 'operators' playing the role of 'observables' which are to 'act' on vectors in a different abstract space. All aspects of the process are united in the same structure described by elements of the same algebraic

structure, elements that only have meaning in the context of the total algebra. This approach has some interesting consequences that have already been discussed in some detail by Hiley and Monk (1993) and will not be discussed further here. The emphasis in this paper will be on those aspects of the algebraic structure that are relevant to this general discussion.

4. Minimal Left Ideals, Algebraic Spinors.

4.1 Pure States.

Let us now illustrate how these ideas work by considering a finite, linear *algebra on which a state ω is defined. We require to find a set of minimal left ideals. We will not follow the techniques based on the characteristic polynomial which have already been outlined in Frescura and Hiley (1978). Rather we follow the method that used by Emch (1972) and is essentially based on the fact that $\omega_i(\varepsilon_j) = \delta_{ij}$ as discussed earlier. They produce exactly the same results but this method is the one conventionally used in algebraic quantum mechanics. (See, for example, Haag 1992)

We can construct left ideals by solving the equation $\omega(B^*K) = 0$ for K and for all $B^* \in$ A. Here we are using the definition

$$I_L = \left\{ K \quad \middle| \omega\left(B^* K\right) = 0, \quad B \right\} \tag{11}$$

to construct the left ideals. To show this definition does in fact produce a left ideal take any A and B in A and any $K \in I_L$ and use associativity to establish

$$\omega\left(A^*\left(BK\right)\right) = \omega\left(\left(B^* A\right)^* K\right) = 0 \tag{12}$$

showing that I_L is a left ideal in A. The left ideal generated in this way is sometimes called the Gel'fand ideal.

Similarly a right ideal can be generated by the sub-set

$$I_R = \left\{ R \quad \middle| \omega\left(R B^*\right) = 0, \quad B \right\} \tag{}$$

4.2. A Simple Example to illustrate these Ideas.

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As the main purpose of this paper is pedagogic, I will now demonstrate how these ideas work in a simple real Clifford algebra $R_{1,1}$. This example is not without physical meaning as it forms the basis of relativity in one spatial dimension, and one time dimension. It is the background structure to spin networks and relativity discussed by Kauffman (1991).

The algebra is characterised by the multiplication table below

1	e_1	e_2	e_{12}
e_1	1	e_{12}	e_2
e_2	$-e_{12}$	-1	e_1
e_{12}	$-e_2$	$-e_1$	1

with $\qquad e_1{}^* = e_1 \; ; \; e_2{}^* = -e_2 \; ; \; e_{12}{}^* = e_{12}.$

Define the state ω_1 by

$$\omega_1\left(\alpha 1 + \beta e_1 + \gamma e_2 + \delta e_{12}\right) = \alpha - \delta \tag{13}$$

It is easy to show using $\omega_1(B{*}A) = 0$, that

$$\omega_1\left(\left(\alpha 1 + \beta e_1^* + \gamma e_2^* + \delta e_{12}^*\right)\left(a\left(1 + e_{12}\right) + b\left(e_1 + e_2\right)\right)\right) = 0$$

confirming that ω_1 produces an element in a left ideal Ψ_{L1} defined by

$$_{L1} = a\varepsilon_{11} + b\varepsilon_{21} \tag{14}$$

where $\varepsilon_{11} = \tfrac{1}{2}(1 + e_{12})$ and $\varepsilon_{21} = \tfrac{1}{2}(e_1 + e_2)$.

The element in the corresponding right ideal can be shown to be

$$_{R1} = a^*\varepsilon_{11} + b^*\varepsilon_{12} \tag{15}$$

where $\varepsilon_{11} = \tfrac{1}{2}\left(1 + e_{12}\right)$ and $\varepsilon_{12} = \tfrac{1}{2}\left(e_1 - e_2\right)$

Then we can form

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$$\omega_1\left(\psi_{R1}^{*}\psi_{L1}\right) = \langle\psi_{R1}|\psi_{L1}\rangle = |a|^2 + |b|^2$$

which is unity for a normalised state.

We now construct a density operator from elements in these ideals. Recalling the definition of the density operator defined in section 4.0, we find that this density operator is

$$\rho_1 = \psi_{L1}\psi_{R1} = \tfrac{1}{2}\left[\left(|a|^2 + |b|^2\right)1 + \left(ab^{*} + a^{*}b\right)e_1 + \left(a^{*}b - ab^{*}\right)e_2 + \left(|a|^2 - |b|^2\right)e_{12}\right]$$

$$= |a|^2\varepsilon_{11} + ab^{*}\varepsilon_{12} + a^{*}b\varepsilon_{21} + |b|^2\varepsilon_{22} \qquad (16)$$

We can generate a second set if elements in second left ideal by using the state

$$\omega_2\left(\alpha 1 + \beta e_1 + \gamma e_2 + \delta e_{12}\right) = \alpha + \delta \qquad (17)$$

so that

$$\omega_2\left(\left(\alpha 1 + \beta e_1^{*} + \gamma e_2^{*} + \delta e_{12}^{*}\right)\left(c\left(1 - e_{12}\right) + d\left(e_1 - e_2\right)\right)\right) = 0$$

producing the element of the left ideal, ψ_{L2} defined by

$$\psi_{L2} = c\varepsilon_{22} + d\varepsilon_{12}. \qquad (18)$$

where $\varepsilon_{22} = \tfrac{1}{2}\left(1 - e_{12}\right)$ and $\varepsilon_{12} = \tfrac{1}{2}\left(e_1 - e_2\right)$.

The corresponding element in the right ideal is

$$\psi_{R2} = c^{*}\varepsilon_{22} + d^{*}\varepsilon_{21} \qquad (19)$$

where $\varepsilon_{22} = \tfrac{1}{2}\left(1 - e_{12}\right)$ and $\varepsilon_{21} = \tfrac{1}{2}\left(e_1 + e_2\right)$.

Then we find

$$\omega_2\left(\psi_{R2}^{*}\psi_{L2}\right) = \langle\psi_{R2}|\psi_{L2}\rangle = |c|^2 + |d|^2$$

while the density operator is

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$$\rho_2 = \quad_{L2}\quad_{R2} = \tfrac{1}{2}\left[\left(|c|^2 + |d|^2\right)1 + \left(cd^* + c^*d\right)e_1 + \left(cd^* - c^*d\right)e_2 - \left(|c|^2 - |d|^2\right)e_{12}\right]$$

$$= |d|^2 \varepsilon_{11} + c^* d\varepsilon_{12} + cd^* \varepsilon_{21} + |c|^2 \varepsilon_{22} \tag{20}$$

We see immediately that the two left ideals are generated by the idempotents

$$\varepsilon_{11} = \tfrac{1}{2}(1 + e_{12}) \qquad \text{and} \qquad \varepsilon_{22} = \tfrac{1}{2}(1 - e_{12})$$

so that $\qquad \varepsilon_{11} + \varepsilon_{22} = 1 \quad$ and $\quad \varepsilon_{11}. \, \varepsilon_{22} = 0.$

Since these idempotents are primitive we have generated two minimal left ideals. These are the results presented in Frescura and Hiley (1980a). A more lengthy discussion of the Pauli Clifford will be found in Frescura and Hiley (1978).

4.3 A Matrix Representation.

These results can be made clearer if we consider the following matrix representation although these matrix representations are not necessary in general;

$$\pi(e_1) = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\;; \qquad\qquad \pi(e_2) = \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}\;; \qquad\qquad \pi(e_{12}) = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \tag{21}$$

In terms of matrices, the elements of the minimal left ideals become

$$\pi(\;_{L1}) = \begin{matrix} a & 0 \\ b & 0 \end{matrix} \qquad\qquad \text{and} \qquad\qquad \pi(\;_{L2}) = \begin{matrix} 0 & d \\ 0 & c \end{matrix} \tag{22}$$

The appearance of two minimal left ideals may at first sight seems surprising since in physics there is only one column vector for spin[3]. In section 8 we will show exactly how these two spinors are merged into one. At this stage we merely remark that the two spinors are equivalent under the inner automorphsim defined by

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[3] It should be noted that minimal left ideals are not always represented as single column matrices. For example $\pi_2\left(\;_{L1}\right) = \tfrac{1}{2} \begin{matrix} a - b & a - b \\ a + b & a + b \end{matrix}$ is a minimal left ideal.

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$$\pi(e_1) = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

The corresponding elements of the minimal right ideals become

$$\pi(\ _{R_1}) = \begin{matrix} a^* & b^* \\ 0 & 0 \end{matrix} \quad \text{and} \quad \pi(\ _{R_2}) = \begin{matrix} 0 & 0 \\ d^* & c^* \end{matrix} \qquad (23)$$

Finally let us return to the definition of the state ω_1 and see how it fits in with the matrix representation. In this case the density operator takes the special form ρ_1 shown above so that

$$\omega_1(\alpha 1 + \beta e_1 + \gamma e_2 + \delta e_{12}) = Tr(\rho_1(\alpha 1 + \beta e_1 + \gamma e_2 + \delta e_{12})) = \alpha - \delta \qquad (24)$$

whereas for ω_2 the density operator takes the form ρ_2 so that

$$\omega_2(\alpha 1 + \beta e_1 + \gamma e_2 + \delta e_{12}) = Tr(\rho_2(\alpha 1 + \beta e_1 + \gamma e_2 + \delta e_{12})) = \alpha + \delta \qquad (25)$$

Thus we find consistency since equations (24) and (25) are identical to equations (13) and (17). This means that the state defined by ω_1 generates the left ideal generated by ε_2, and vice versa. The reason for this is as follows. Since $\omega_1(B^*K) = 0$ for B A, we must have $\omega_1(K^*K) = 0$ for K I_{L1}. Since $\omega_1(\varepsilon_1^*\varepsilon_1)$ 0, ε_1 I_{L1}. Thus the left ideal generated by $\omega_i(B^*K) = 0$ is always the complement of the ideal generated by ε_i. We will discuss this point again in section 8.1.

4.4 Mean Values.

As we have seen above we can write

$$\langle A \rangle = \omega_1(\ _{R1} A\ _{L1})$$

Now if we write

$$A = a_0 1 + a_1 e_1 + a_2 e_2 + a_{12} e_{12}$$

Then using the above expressions for Ψ_{L1} and Ψ_{R1} above we find

$$(\ _{R1} A\ _{L1}) = \left[(|d|^2 + |b|^2) a_0 + (a^*b + b^*a) a_1 + (b^*a - a^*b) a_2 + (|d|^2 - |b|^2) a_{12} \right] \tfrac{1}{2} (1 + e_{12})$$

while

$$\langle A \rangle = \omega_1 \left({}_{R1}A{}_{L1} \right) = \left[\left(|a|^2 + |b|^2 \right)a_0 + \left(a^*b + b^*a \right)a_1 + \left(b^*a - a^*b \right)a_2 + \left(|a|^2 - |b|^2 \right)a_{12} \right] \quad (26)$$

We can also write

$$\langle A \rangle = \omega_1 \left({}_{R1}A{}_{L1} \right) = \omega_1 \left(A{}_{L1}{}_{R1} \right) = \omega_1 \left(A\rho_1 \right) = Tr\left(\rho_1 A \right)$$

Indeed one can evaluate $\omega_1(A\rho_1)$ and show it is equal to the result calculated in the last equation.

5. Mixed States and the GNS Construction.

In order to illustrate how to deal with mixed states for the algebra $R_{1,1}$, we start with the matrix representation defined in section 4.3 and choose the specific density matrix

$$\pi(\rho) = \begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}$$

It can quickly be checked that ρ in fact corresponds to mixed state by showing $\rho^2 \neq \rho$. Remembering that ω is equivalent to taking the trace, it is straight forward to show that

$$\omega_\rho (A) = \omega_\rho \left(a_0 1 + a_1 e_1 + a_2 e_2 + a_{12} e_{12} \right) = \left(\lambda_1 + \lambda_2 \right)a_0 + \left(\lambda_1 - \lambda_2 \right)a_{12} \quad (27)$$

Let us now try to find the Gel'fand ideal using the technique outlined in section 4.1 and 4.2. We start by writing this ideal in the general form

$$_L = z + z_1 e_1 + z_2 e_2 + z_{12} e_{12}$$

Then we need to solve the equation

$$\omega_\rho \left(\left(a_0 + a_1 e_1 + a_2 e_2 + a_{12} e_{12} \right)\left(z + z_1 e_1 + z_2 e_2 + z_{12} e_{12} \right) \right) = 0$$

with

$$\omega_\rho \left(a_0 1 \right) = a_0\left(\lambda_1 + \lambda_2 \right); \quad \omega_\rho \left(be_1 \right) = 0; \quad \omega_\rho \left(ce_2 \right) = 0; \quad \omega_\rho \left(de_{12} \right) = d\left(\lambda_1 - \lambda_2 \right)$$

We can then show that the equation we have to solve is

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$$a_0\left[(\lambda_1 + \lambda_2)z + (\lambda_1 - \lambda_2)z_{12}\right] + a_1\left[(\lambda_1 + \lambda_2)z_1 + (\lambda_1 - \lambda_2)z_2\right] +$$
$$a_2\left[(\lambda_1 + \lambda_2)z_2 + (\lambda_1 - \lambda_2)z_1\right] + a_{12}\left[(\lambda_1 + \lambda_2)z_{12} + (\lambda_1 - \lambda_2)z\right] = 0 \; .$$

This equation has no non-trivial solution except for $\lambda_1 = 1$, $\lambda_2 = 0$; and $\lambda_1 = 0$, $\lambda_2 = 1$ which are the results we obtained in section 4.2. Hence we cannot extend these methods to general values of λ and therefore we have to use an alternative method, which we will discuss below. However before going on to explore a more general method let us first explore how the mixed state can be handles within $R_{1,1}$.

In section 3.2 equation (9), we showed that the density matrix could be written in the form

$$\rho = \sum_{i=1}^{r} \sqrt{\lambda_i} B_i \sqrt{\lambda_i} B_i^*$$

which in this example can be written as

$$\rho = \rho_1 + \rho_2 = \,_{R1}\,_{L1} + \,_{R2}\,_{L2}$$

where

$$_{L1} = \sqrt{\lambda_1}\tfrac{1}{2}(1 + e_{12}) = \sqrt{\lambda_i}\,\varepsilon_{11} \qquad \text{with} \qquad _{R1} = \sqrt{\lambda_1}\tfrac{1}{2}(1 + e_{12}) = \sqrt{\lambda_i}\,\varepsilon_{11}$$

and

$$_{L2} = \sqrt{\lambda_2}\tfrac{1}{2}(1 - e_{12}) = \sqrt{\lambda_2}\,\varepsilon_{22} \qquad \text{with} \qquad _{R2} = \sqrt{\lambda_2}\tfrac{1}{2}(1 - e_{12}) = \sqrt{\lambda_2}\,\varepsilon_{22}$$

so that

$$\rho = \rho_1 + \rho_2 = \lambda_1 \varepsilon_{11} + \lambda_2 \varepsilon_{22} \qquad\qquad (28)$$

Then

$$\langle A \rangle = Tr(\rho A) = \omega_1(\rho_1 A) + \omega_2(\rho_2 A)$$

gives

$$\langle A \rangle = (\lambda_1 + \lambda_2)a_0 + (\lambda_1 - \lambda_2)a_{12} \qquad\qquad (29)$$

This then agrees with the result calculated using the matrix representation above. Thus we see that we use both minimal ideals in constructing the density operator. This result has already been used for the Dirac Clifford algebra by Frescura and Hiley (1987)

5.1 GNS Construction.

As we indicated in section 3.2 we would like to construct a D so that we can write the density operator as $\rho = DD^*$ which in turn allows us to write

$$\langle A \rangle = \omega\left(D^* A D\right) = \omega\left(DD^* A\right) = \omega\left(\rho A\right) = \langle \ |A| \ \rangle$$

so that D is the algebraic equivalent of $| \ \rangle$. This would enables to find the algebraic background to the GNS construction.

From equation (28) together with the fact that ε_{11}, ε_{22} are the orthogonal primitive idempotents satisfying $\varepsilon_{ii}\varepsilon_{jj} = \delta_{ij}\varepsilon_{jj}$, we can immediately write

$$D = \sum_{i=1}^{2} \sqrt{\lambda_i}\varepsilon_{ii} \tag{30}$$

so that

$$\langle A \rangle = \omega\left(\sum_{r-1}^{2} \sqrt{\lambda_i}\varepsilon_{ii} A \sum_{r-1}^{2} \sqrt{\lambda_i}\varepsilon_{ii}\right) = \left(\lambda_1 + \lambda_2\right)a_0 + \left(\lambda_1 - \lambda_2\right)a_{12}$$

which is the result produced in equation (29). Thus we have obtained exactly the same result as that obtained by matrix representation used in section 4.2. It is now obvious how to generalise this. We simply write

$$D = \sum_{i=1}^{r} \sqrt{\lambda_i}\varepsilon_{ii} \tag{31}$$

which immediately gives

$$\langle A \rangle = \omega\left(D^* A D\right) = \omega\left(\rho A\right) \quad \langle \ |A| \ \rangle$$

5.2 The Matrix Representation

The GNS construction is usually presented as a matrix representation (See Emch 1972). We can immediately construct a matrix representation if we first notice that any matrix transformation of the form $C\rho D$ can be written in the form $(C \quad D)\hat{\rho}$ where $\hat{\rho}$ is the $n \times n$ density matrix written as a column with n^2 entries. But we are interested in forming

$$\langle A \rangle = Tr(\rho A) = Tr(DD^*A) = Tr(D^*AD) = Tr\left(\pi(D^*)\pi(A \quad 1)\pi(D)\right) \quad (32)$$

where $\pi(D)$ is a column with n^2 entries. This can be written as

$$\pi(D) = \sum_{j=1}^{r} \sqrt{\lambda_j}\,\pi(\varepsilon)_{jj} = \sum_{j=1}^{r} \left(B_j\right) \quad\quad (33)$$

In the matrix representation we can write

$$\pi(B_j) = \begin{matrix} 0 \\ . \\ 0 \\ \sqrt{\lambda_j} \\ 0 \\ . \\ 0 \end{matrix}$$

That is where $\pi(B_j)$ is the $1 \times n^2$ column matrix with $\sqrt{\lambda_j}$ appearing in the $s = [j(n+1)+1]$ element. Here λ_j is the jth eigenvalue of the density matrix in the representation used in section 5. This enables us to introduce the column matrix $\pi(D)$ with $\sqrt{\lambda_j}\delta_{js}$ in the sth element satisfying $s = [j(n+1)+1]$ as j runs from 1 to r. Then we can write

$$\langle A \rangle = \omega_\rho(A) = Tr(\rho A) = \omega\left(\pi(D^*)(\pi(A) \quad \pi(1))\pi(D)\right) \quad\quad (34)$$

5.3 Specific example

Let us illustrate the above discussion by returning to the example in section 5.1. Here we find that $\pi(D)$ becomes

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$$\pi(D) = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & 0 & \\ & 0 & \\ & & \sqrt{\lambda_2} \end{pmatrix} \tag{35}$$

so that

$$\langle A \rangle = \langle \, | \hat{A} | \, \rangle = \left(\sqrt{\lambda_1} \quad 0 \quad 0 \quad \sqrt{\lambda_2} \right) \begin{pmatrix} a_0 + a_{12} & a_1 - a_2 & & \\ a_1 + a_2 & a_0 - a_{12} & & \\ & & a_0 + a_{12} & a_1 - a_2 \\ & & a_1 + a_2 & a_0 - a_{12} \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1} \\ 0 \\ 0 \\ \sqrt{\lambda_2} \end{pmatrix}$$

$$\langle A \rangle = \left(\lambda_1 + \lambda_2 \right) a_0 + \left(\lambda_1 - \lambda_2 \right) a_{12}$$

which is identical to equation (27) showing that we can get exactly the same result as the standard method using the density matrix. The advantage of this method lies in the fact that when generalised to a Hilbert space construction, which we will discuss later, we have 'wave functions' depending upon temperature. This leads us to a more generalised quantum mechanics discussed by Umezawa (1993),

5.4 A more Physical Example.

To this point we have been considering mathematical structures per se. Let us now turn to see how these techniques in a physical problem. Consider a thermal spin system in an external magnetic field. A typical density operator can be written in the form

$$\rho = \frac{\exp[-\beta H]}{Tr \exp[-\beta H]} \tag{36}$$

where $\beta = 1/kT$ and H is the Hamiltonian $H = -B\sigma_3$. Since we are using Pauli spin matrices we must use the standard multiplication rules for Pauli spin matrices, i.e. the quaternion Clifford algebra, H and not $R_{1,1}$ as used above. Using the standard representation for the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we find

$$\rho = \begin{array}{cc} \dfrac{1+\tanh\gamma}{2} & 0 \\ 0 & \dfrac{1-\tanh\gamma}{2} \end{array} \qquad (37)$$

then it is straightforward to show

$$\omega_\rho\left(a1 + b\sigma_x + c\sigma_y + d\sigma_z\right) = a + d\tanh\gamma \qquad (38)$$

Let us now try to find the Gel'fand ideal using the technique outlined above. We start by writing this ideal in the general form

$$_L = z + z_1\sigma_1 + z_2\sigma_2 + z_3\sigma_3$$

Then we need to solve the equation

$$\omega\left(\left(a + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3\right)\left(z + z_1\sigma_1 + z_2\sigma_2 + z_3\sigma_3\right)\right) = 0$$

with

$$\omega(a1) = a; \quad \omega(b\sigma_1) = 0; \quad \omega(c\sigma_2) = 0; \quad \omega(d\sigma_3) = d\tanh\gamma$$

It is straightforward to show that the equation we eventually have to solve is

$$\left(a - a_3\tanh\gamma\right)z + \left(a_1 + a_2\tanh\gamma\right)z_1 + \left(a_2 - a_1\tanh\gamma\right)z_2 + \left(a\tanh\gamma + a_3\right)z_3 = 0.$$

Once again this equation has no non-trivial solution so we cannot used the methods outlined above in section 4 and we are forced to use the GNS construction. To do this we need a pair of primitive idempotents from the algebra H. These are

$$\varepsilon_{11} = \tfrac{1}{2}\left(1 + \sigma_3\right) \qquad \text{and} \qquad \varepsilon_{22} = \tfrac{1}{2}\left(1 - \sigma_3\right) \qquad (39)$$

so that

$$D = \frac{\sqrt{\lambda_1}}{2}\left(1 + \sigma_3\right) + \frac{\sqrt{\lambda_2}}{2}\left(1 - \sigma_3\right) \qquad (40)$$

If we write $A = a1 + b\sigma_1 + c\sigma_2 + d\sigma_3$ then

$$\langle A \rangle = \omega(A\rho) = \omega(D^*AD) = a + d\tanh\gamma$$

which is the value found in equation (38). In the matrix representation discussed in 5.2 we find

$$\pi(D) = |\ \rangle = \begin{matrix} \sqrt{\dfrac{1+\tanh\gamma}{2}} \\ 0 \\ 0 \\ \sqrt{\dfrac{1-\tanh\gamma}{2}} \end{matrix} \qquad \text{with} \quad \pi(\sigma) = \begin{matrix} \sigma & 0 \\ 0 & \sigma \end{matrix} \qquad (41)$$

Again it is straight forward to check that

$$\omega_\rho(\pi(A)) = \langle\ |\pi(A)|\ \rangle = a + d\tanh\gamma$$

6. **Umezawa's Approach**.

In the above generalisation to mixed states we need to consider a product representation $A \quad 1$. It is of interest to compare the above approach with that introduced by Umezawa (1993).

Suppose we start with a general thermodynamic system with the partition function given by

$$Z(\beta) = Tr(\exp[-\beta H])$$

where $\beta = 1/kT$ and $H = H_0 - \mu N$. Here H_0 is the particle Hamiltonian and μ is the chemical potential. The ensemble average is given by

$$\langle A \rangle = \frac{Tr(\exp[-\beta H]A)}{Z(\beta)}$$

We want to find a state $|\ (\beta)\rangle$ such that

$$\langle A \rangle = \frac{Tr\left(\exp[-\beta H]A\right)}{Z(\beta)} = \langle \ (\beta)|\pi(A)| \ (\beta)\rangle$$

Suppose $H|n\rangle = E_n|n\rangle$, then

$$\langle A \rangle = Z^{-1}(\beta) \sum_n \langle n| A |n\rangle \exp[-\beta E_n] = \langle \ (\beta)|\pi(A)| \ (\beta)\rangle$$

Can we find $|\ (\beta)\rangle$? To this effect let us write $|\ (\beta)\rangle = \sum_n |n\rangle f_n(\beta)$ so that

$$\langle \ (\beta)|\pi(A)| \ (\beta)\rangle = \sum_{n,m} f_m^*(\beta)f_n(\beta)\langle m| A |n\rangle = Z^{-1}(\beta) \sum_n \langle n| A |n\rangle \exp[-\beta E_n]$$

which will be satisfied only if

$$f_m^*(\beta)f_n(\beta) = Z^{-1}(\beta)\exp[-\beta E_n]\delta_{mn}$$

However it is impossible to satisfy this relation if $f_n(\beta)$ are simply complex numbers. Umezawa suggests we write

$$f_n(\beta) = Z^{-\frac{1}{2}}(\beta)\exp[-\beta E_n/2]|n\rangle$$

Here $|n\rangle$ is a copy of the original set of kets. Then

$$f_m^*(\beta)f_n(\beta) = Z^{-1}(\beta)\exp[-\beta\left(E_n + E_m\right)/2]\delta_{mn}$$

This means we can write

$$|\ (\beta)\rangle = Z^{-\frac{1}{2}} \sum_n \exp[-\beta E_n/2]|n\rangle \ |n\rangle. \qquad (42)$$

We can show this is exactly the same vector as is obtained from the GNS construction. We will simply illustrate the similarity using the example of section 5.4.

It is straightforward to show that

$$Z^{-\frac{1}{2}}\exp[\beta E_1/2]|1\rangle \ |1\rangle = \sqrt{\frac{1 + \tanh\gamma}{2}}|1\rangle \ |1\rangle$$

and

$$Z^{-1/2} \exp[\beta E_2/2]|2\rangle \quad |2\rangle = \sqrt{\frac{1 - \tanh\gamma}{2}} \, |2\rangle \quad |2\rangle$$

Since

$$|1\rangle \quad |1\rangle \quad \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} = \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix}$$

and

$$|2\rangle \quad |2\rangle \quad \begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix} = \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix}$$

We find

$$| \quad (\beta)\rangle = \begin{matrix} \sqrt{\dfrac{1 + \tanh\gamma}{2}} \\ 0 \\ 0 \\ \sqrt{\dfrac{1 - \tanh\gamma}{2}} \end{matrix}$$

which is identical to equation (41), the result we obtained from the GNS construction.

What we see from this approach to the GNS construction is that we need to "double up" the algebra since we need two copies of the vector space. To put it more formally we need to form the bialgebra A A. Indeed this is exactly what Umezawa (1993) proposed in his approach to thermal field theory. Starting from the boson algebra generated by the set $\{1, a, a^\dagger\}$ where a is the boson destruction operator and $a^\dagger$ the corresponding creation operator, he introduced an additional pair of annihilation and creation operators $\tilde{a}$ and $\tilde{a}^\dagger$ which he attributes to a "ghost" field (Umezawa 1993a). Of course the idea of a ghost field is not very appealing from the physics point of view, however this approach opens up the possibility of using this doubling to account for quantum dissipation (Celeghini, Rasetti, and Vitiello 1992). Hiley and Fernandes (1997), following from Bohm and Hiley (1981), have shown this doubling of the number of degrees of freedom is also required in going from two point functions in configuration

space description to an algebraic phase space description. This gives an alternative view on dissipation. These ideas will be developed further in a forthcoming paper.

7. Extension to the Heisenberg Algebra.

All of the previous examples have used the orthogonal Clifford algebra in some form or other. What we would like to do now is to show how the whole procedure can be extended to the Heisenberg algebra. As pointed out in a previous paper (Hiley 2001), the main problem in following the above approach arises from the fact that the Heisenberg algebra is a nilpotent algebra of degree three under the product $[A, B]$. As a consequence of a well-known theorem, nilpotent algebras do not contain any non-trivial idempotents so it is not possible to construct any non-trivial left ideals.

We want to approach Heisenberg algebra in two steps. Rather than go to the full algebra immediately, we will start by considering the finite algebra introduced by Weyl (1930) which contains the Heisenberg algebra in the continuum limit. The Weyl algebra has an extremely simple structure. It is a finite polynomial algebra C^2_n generated by the set of elements $\{1, e_1, e_2\}$ subject to

$$e_1 e_2 = \omega\, e_2 e_1 \qquad\qquad e_1^n = 1, \qquad e_2^n = 1, \qquad \text{with } \omega = \exp[2\pi i/n] \quad (43)$$

where n is an integer. This algebra has a long history being first explored by Sylvester in 1884 as an example of a generalisation of the quaternions. He called the elements nonions for $n = 3$ and n-ions more generally. What Weyl shows is that we can write $e_1 = \exp[i\xi P]$ and $e_2 = \exp[i\eta X]$ where $\xi = 2\pi/\delta pn$ and $\eta = 2\pi/\delta xn$ so that we begin to see the beginnings of the Heisenberg algebra. Indeed in the limit as $n \to \infty$ this algebra does approaches the Heisenberg algebra with X and P representing the position and momentum operators..

The importance of the algebra from our point of view is that it is not nilpotent. In order to find the idempotents using the method described in the section 4, equation (11) we need to generate the sub-set satisfying

$$\mathcal{I}_\omega = \left\{ K \quad \middle| \omega\left(B^* K\right) = 0, \quad B \quad \right\}$$

In order to see how this works in this example we again illustrate the procedure with an example.

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We now take the simplest example of the structure described in the last sub-section, namely, $n = 3$, the nonions, and write down the explicit multiplication table explicitly.

1	e_1	e_2	e_{11}	e_{12}	e_{22}	e_{112}	e_{122}	e_{1122}
e_1	e_{11}	e_{12}	1	e_{112}	e_{122}	e_2	e_{1122}	e_{22}
e_2	$\omega^2 e_{12}$	e_{22}	ωe_{112}	$\omega^2 e_{122}$	1	ωe_{1122}	$\omega^2 e_1$	ωe_{11}
e_{11}	1	e_{112}	e_1	e_2	e_{1122}	e_{12}	e_{22}	e_{122}
e_{12}	$\omega^2 e_{112}$	e_{122}	ωe_2	$\omega^2 e_{1122}$	e_1	ωe_{22}	$\omega^2 e_{11}$	$\omega 1$
e_{22}	ωe_{122}	1	$\omega^2 e_{1122}$	ωe_1	e_2	$\omega^2 e_{11}$	ωe_{12}	$\omega^2 e_{112}$
e_{112}	$\omega^2 e_2$	e_{1122}	ωe_{12}	$\omega^2 e_{22}$	e_{11}	ωe_{122}	$\omega^2 1$	ωe_1
e_{122}	ωe_{1122}	e_1	$\omega^2 e_{22}$	ωe_{11}	e_{12}	$\omega^2 1$	ωe_{112}	$\omega^2 e_2$
e_{1122}	ωe_{22}	e_{11}	$\omega^2 e_{122}$	$\omega 1$	e_{112}	$\omega^2 e_1$	ωe_2	$\omega^2 e_{12}$

with $\qquad e_1{}^* = e_{11}; \qquad e_2{}^* = e_{22}; \qquad e_{12}{}^* = \omega^2 e_{1122}; \qquad e_{112}{}^* = \omega e_{112},$

and $\qquad\qquad (e_i{}^*)^* = e_i; \qquad \omega^* = \omega^{-1}.$

We define the state by

$$\omega_1(a1) = a; \qquad \omega_1(be_2) = b; \qquad \omega_1(ce_{22}) = c; \qquad \text{Rest} = 0.$$

Now we need to solve the equation $\quad \omega_1(B^*K) = 0, \quad$ where

$$B = \alpha 1 + \beta e_2 + \gamma e_{22} + \delta e_1 + \varepsilon e_{12} + \eta e_{122} + \theta e_{11} + e_{112} + e_{1122}$$
$$= \alpha 1 + \beta e_{22} + \gamma e_2 + \delta e_{11} + \varepsilon \omega^2 e_{1122} + \eta \omega e_{112} + \theta e_1 + \omega e_{122} + \omega^2 e_{12}$$

and

$$K = z1 + z_2 e_2 + z_{22} e_{22} + z_1 e_1 + z_{12} e_{12} + z_{122} e_{122} + z_{11} e_{11} + z_{112} e_{112} + z_{1122} e_{1122}.$$

It is tedious but straight forward to show that the solution of the equation $\omega_1(B^*K) = 0$ is

$$(\alpha + \beta + \delta)(z + z_2 + z_{22}) + (\delta + \varepsilon + \eta)(z_1 + z_{12} + z_{122}) + (\theta + \; + \;)(z_{11} + z_{112} + z_{1122}) = 0$$

which gives solution (A) as

$$z = 1; \qquad z_2 = \omega; \qquad z_{22} = \omega^2$$

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$$z_1 = 1; \qquad z_{12} = \omega; \qquad z_{122} = \omega^2$$
$$z_{11} = 1; \qquad z_{112} = \omega; \qquad z_{1122} = \omega^2$$

so that the left ideal is

$$3\ _{L1}(A) = \left(1 + \omega e_2 + \omega^2 e_{22}\right)a + \left(e_1 + \omega e_{12} + \omega^2 e_{122}\right)b + \left(e_{11} + \omega e_{112} + \omega^2 e_{1122}\right)c \qquad (44)$$

Solution (B) is

$$z = 1; \qquad z_2 = \omega^2; \qquad z_{22} = \omega$$
$$z_1 = 1; \qquad z_{12} = \omega^2; \qquad z_{122} = \omega$$
$$z_{11} = 1; \qquad z_{112} = \omega^2; \qquad z_{1122} = \omega$$

$$3\ _{L1}(B) = \left(1 + \omega^2 e_2 + \omega e_{22}\right)a + \left(e_1 + \omega^2 e_{12} + \omega e_{122}\right)b + \left(e_{11} + \omega^2 e_{112} + \omega e_{1122}\right)c \qquad (45)$$

We see from these two solutions we have two idempotents

$$\varepsilon_{11} = \tfrac{1}{3}\left(1 + \omega e_2 + \omega^2 e_{22}\right) \qquad \text{and} \qquad \varepsilon_{22} = \tfrac{1}{3}\left(1 + \omega^2 e_2 + \omega e_{22}\right). \quad \text{We can find the third from}$$

$$1 = \varepsilon_{00} + \varepsilon_{11} + \varepsilon_{22}$$

So that $\varepsilon_{00} = \tfrac{1}{3}\left(1 + e_2 + e_{22}\right)$ from which we can generate the third left ideal by multiplying this idempotent from the left by each element of the algebra. For completeness we find this third ideal is explicitly

$$3\ _{L1}(C) = \left(1 + e_2 + e_{22}\right)a + \left(e_1 + e_{12} + e_{122}\right)b + \left(e_{11} + e_{112} + e_{1122}\right)c \qquad (46)$$

It is now straight forward to use these three left ideals to construct density operators and mean values for both pure and mixed states as described in section 4 and 5. It is also possible to construct the GNS representation in a straight forward manner. We will not go into the details here.

7.2 Generalisation to arbitrary n.

The generalisation to arbitrary n is also very straight forward. It is easy to obtain a general expression for the family of idempotents defined in the previous section. They are

$$\varepsilon_{ii} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{-ji} (e_2)^j \qquad i, j = 0,1,\ldots\ldots n-1. \qquad (47)$$

Each one of these idempotents can be used to generate a set of n left ideals so that the techniques discussed in section 4 and 5 can also easily be generalised to arbitrary n.

8. Hilbert Space and the GNS Construction.

Now we want compare our approach with the approach through Hilbert space. This means showing how the minimal left ideals are related to vectors in a Hilbert space. In order to begin we must first introduce the notion of ω-equivalence classes of elements of the algebra, $\mathbb{A}$. To do this consider two elements A and B of $\mathbb{A}$. Whenever $A - B \in \mathbb{L}$, we say A and B are ω-equivalent. Since $\mathbb{L}$ is a linear sub-space of $\mathbb{A}$ it is clearly an equivalence relation. For every $A \in \mathbb{A}$ we denote its equivalent class by $\pi(A)$.

We now equip the set of equivalent classes with a vector space structure I by defining

$$\lambda\, \pi(A) + \mu\, \pi(B) = \pi(\lambda A + \mu B) \qquad A, B \in \mathbb{A}, \quad \lambda, \mu \in \mathbb{R} \text{ or } \mathbb{C}$$

We can equip I with a pre-Hilbert space structure by defining

$$\big(\pi(A),\ \pi(B) \big) = \omega(A^* A)$$

With this we can define a norm $|\pi(A)|^2 = \omega(A^* A)$. If necessary we can use this norm to complete I thus forming a Hilbert space.

8.1. Examples.

To illustrate the implications of this structure we again examine the example $R_{1,1}$ given above. We take the algebraic spinor defined above in equation (18), namely,

$$\Phi_{L2} = \frac{a}{2}(1 - e_{12}) + \frac{b}{2}(e_1 - e_2).$$

We then see clearly that 1 and $e_{12} \in \pi(1)$, while e_1 and $e_2 \in \pi(e_1)$. The Hilbert space is thus two-dimensional so we can write

$$H = \nu\, \pi(1) + \rho\, \pi(e_1) \qquad\qquad (48)$$

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Where (1) and (e_1) are orthogonal, providing a basis for the two-dimensional Hilbert space H .

Notice that when we pass to the equivalence class we lose some of the mathematical structure in that both $\Psi_{\rho 1}$ and $\Psi_{\rho 2}$ form equivalent Hilbert spaces. In standard quantum mechanics it does not seem necessary to distinguish between these two spaces.

In the case of the nonions we can show that the three ideals I_{L1}, I_{L2} and I_{L3} belong to a single equivalence class. Thus with the obvious generalisation of equation (48) we can construct a Hilbert space by writing

$$\mathsf{H}_1 = \mu\Omega(1) + \nu\Omega(e_1) + \rho\Omega(e_{11}). \tag{49}$$

This gives a three-dimensional Hilbert space as expected. Notice that in this case we are dealing with a pure state so that the GNS construction gives us an irreducible representation.

We could generalise to the mixed state by choosing a different state '. For example we could start with the density matrix given in equation (36) but choosing for the Hamiltonian $H = -B\pi\left(e_2\right)$. The GNS construction will go through exactly as explained in section 5. As the example chosen is not motivated by any relevant physical system we will not discuss it further here but leave it as an exercise for the interested reader.

It should also be stressed that these are not the only left ideals in the algebra. Indeed there are an infinite number of them obtained from inner automorphism. Thus we can always find another left ideal by first finding a new set of idempotents using

$$\varepsilon \ = S\varepsilon S^{-1} \qquad\qquad S \quad \mathsf{A} \tag{50}$$

It might be of some interest to note that a sub-set of these S generate a set of discrete Fourier transformations.

8.2 Hilbert Space continued.

We now define for all A A

$$\pi(A): \mathbb{H} \to \mathbb{H}, \qquad \text{so that} \qquad \pi(A)\,\Omega(B) = \Omega(AB). \qquad (51)$$

We must ensure that this definition implies that $\Omega(AB')$ is independent of B' in $\Omega(B)$ so that it does not matter which $\sim$-equivalent element B is used to label the equivalence class $\Omega(B)$. To this effect

$$\left| \Omega(AB) - \Omega(AB') \right|^2 = \left| \Omega\!\left(A(B-B')\right) \right|^2 = \left(\left(A(B-B')\right)^{*}\!\left(A(B-B')\right) \right)$$

$$= \left((B-B')^{*} A^{*} A (B-B') \right) \le \|A^{*}\| \cdot \left| \Omega(B-B') \right|^2 = 0$$

This is zero because $B-B' \in I_{\wp}$. Thus it does not matter which element in the equivalence class is used to label the class.

Now we need to verify that $\pi_\omega(A)$ is, indeed, a representation.

$$\pi_\omega\!\left(\lambda A + \mu B\right)\Omega(C) = \Omega\!\left((\lambda A + \mu B)C\right) = \lambda\,\Omega(AC) + \mu\,\Omega(BC)$$

$$= \left(\lambda \pi_\omega(A) + \mu \pi_\omega(B)\right)\Omega(C)$$

$$\pi_\omega(AB)\,\Omega(C) = \Omega\!\left((AB)C\right) = \Omega\!\left(A(BC)\right) = \pi_\omega(A)\pi_\omega(B)\,\Omega(C)$$

$$\left(\pi_\omega(A)^{*}\,\Omega(C),\,\Omega(D)\right) = \left(\Omega(C),\pi_\omega(A)\,\Omega(D)\right) = \omega\!\left(C^{*}AD\right)$$

$$= \omega\!\left((A^{*}C)^{*}D\right) = \left(\Omega(A^{*}C),\,\Omega(D)\right) = \left(\pi_\omega(A^{*})\,\Omega(C),\,\Omega(D)\right)$$

Thus π_ω is a homorphism of the involutive algebra $\mathbb{A}$ to the set of operators acting on the pre-Hilbert space $I_{\wp}$. It is not difficult to show that, if necessary, we can extend $\pi_\omega(A)$ to a bounded operator on $\mathbb{H}$ in a unique manner. Furthermore since $\pi_\omega(A^{*}) = \pi_\omega(A)^{*}$, $\pi_\omega(A)$ is self-adjoint for every $A \in \mathbb{A}$.

8.3. **Return to the Example**.

Returning to $R_{1,1}$, in the orthogonal basis $\Omega(1)$, $\Omega(e_1)$ we find

$$\pi_\omega(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad \pi_\omega(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\pi_\omega(e_2) = \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \qquad\qquad \pi_\omega(e_{12}) = \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}$$

Thus we have produced an irreducible representation and we can write

$$\pi_\omega(\alpha 1 + \beta e_1 + \gamma e_2 + \delta e_{12}) = \begin{array}{cc} \alpha + \delta & \beta - \gamma \\ \beta + \gamma & \alpha - \delta \end{array} \qquad (52)$$

while

$$(1) = \begin{array}{c} 1 \\ 0 \end{array} \qquad \text{and} \qquad (e_1) = \begin{array}{c} 0 \\ 1 \end{array} \qquad\qquad (53)$$

Thus in a pure state defined by $\omega_1(A)$ given above, we find that the representation for Ω is the familiar real Pauli spinor.

In the case of the nonions equation (53) is generalised to

$$(1) = \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \qquad (e_1) = \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \quad \text{and} \qquad (e_{11}) = \begin{array}{c} 0 \\ 0 \\ 1 \end{array}$$

Giving us the basis for the three-dimension representation discussed in section 8.1. It is not difficult to show that the corresponding matrix representation can be written as

$$\pi_{\omega_1}(e_1) = \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \ ; \ \pi_{\omega_1}(e_2) = \begin{array}{ccc} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{array} \text{ and } \pi_{\omega_1}(e_{12}) = \begin{array}{ccc} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{array}$$

The rest of the matrixes can be found using the multiplication table above.

Notice also that this approach enables us to attribute to each $A \quad \mathbb{A}$, a vector

$$(A) = \pi_\omega(A) \ (1) \ .$$

When generalising to include an infinite algebra we require the representation space H to be defined in terms of a strong closure of these vectors. This means that $\Omega(1)$ is cyclic for π. A vector is cyclic relative to the algebra if the set { $(A)\Omega \,|\, A \quad A$} is dense in H. This means that for any vector $\psi \quad \mathcal{H} \quad \|\psi - \pi_\omega(A)\|$ can be made as small as desired.

In conclusion then our construction associates to every state ω on A a cyclic representation π of A. We can then write the expectation value as

$$\omega(A) = \langle \quad |\pi_\omega(A)| \quad \rangle$$

where we have written $(1) = | \quad \rangle$.

8.4 The Limit n $\rightarrow \infty$

The simplest way to proceed to generalise the n-ions is to go to the limit n . This limit has already been discussed by Hiley and Monk (1993) and Hiley (2001), their argument being based on earlier work by Weyl (1931). We will briefly outline the argument again here for the sake of completeness.

We have written $e_1 = \exp[i\xi P]$ and $e_2 = \exp[i\eta X]$ where $\xi = 2\pi/\delta pn$ and $\eta = 2\pi/\delta xn$ so that we can regard e_1 as a translation in space and e_2 as a translation in momentum space. Thus we can regard $(e_1)^{-s}$ as translating from $x_k \quad x_{k-s}$. In the limit this corresponds to the transformation $\psi(x) \quad \psi(x-s)$. Similarly $(e_2)^t$ takes $x_k \quad \omega^{kt}x_k$, which in the limit corresponds to $\psi(x) \quad \exp[itx].\psi(x)$. This shows how the Schrödinger representation emerges from the discrete Weyl algebra. In this case the discrete Fourier transformation mentioned above in section 7.1 becomes the usual Fourier transformation that takes the *x*-representation to the *p*-representation.

If we now examine the limiting process on the primitive idempotent ε_{00} we find

$$\varepsilon_{00}(n) = \frac{1}{n} \sum_\beta \exp[i\beta X] \quad \frac{1}{2\pi} \int d\beta \exp[i\beta X] \quad \delta(x) \tag{53}$$

Thus we see that in the limit the Weyl idempotent $\varepsilon_0(n)$ plays the role of the Dirac delta function in the continuum algebra. If we start directly from the Heisenberg algebra defined through the standard commutation relations $[X, P] = i$, we miss the idempotent,

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the Dirac delta function which we must add later when we have constructed the Hilbert space.

If we remain in the algebraic structure then we can formally write $\varepsilon_{00}(n) = \quad$ then we find the relations

$$X\Delta = 0, \qquad\qquad \Delta P = 0, \qquad\qquad \text{and} \qquad {}^2 = \qquad (54)$$

If we identify $E = {}^{\dagger}$ introduced by Frescura and Hiley (1984) then the following relation result:

$$EX = 0, \qquad\qquad PE = 0, \qquad\qquad \text{and} \quad E^2 = E \qquad (56)$$

The set $\{1, X, P, E\}$ then generates what we have called the extended Heisenberg algebra. Further details of this structure can be found in Hiley (2001).

9. Conclusions.

We have shown how an approach to quantum mechanics can be built from the algebraic structure of the Clifford algebra and the discrete Weyl algebra (or the generalised Clifford algebra). These algebras can be treated by the same techniques that do not require Hilbert space yet enable us to calculating mean values required in quantum mechanics. In the appendix we compare the two approaches in table form. We have also shown how these techniques are related to the Hilbert space, which effectively requires additional structure that involves the space-time continuum. Yet these algebras already contain geometric feature not only of space-time itself, but of a generalised phase space as already pointed out in Bohm and Hiley (1981 and 1983). The full implications of this geometric structure will be discussed in another publication.

10. Appendix.

In this appendix we list the correspondence between the algebraic approach to quantum mechanics which we call "The Algebra of Process" and the usual approach which we call "The Algebra of Observables". We also include the extra step which involves a projection of the ket vector space into $L^2(x)$.

Algebra of Process.	**Algebra of Observables**.
*Algebra A	† Algebra A
Left ideal, $L \quad I_L$	**External vector space**, $\quad$ V $$x \psi = \psi(x)$$
Generated by idempotents ε_j, $\varepsilon_j^2 = \varepsilon_j \quad A\varepsilon_j.$	
Basis of ideal ε_{ij} with $\varepsilon_j \quad \varepsilon_{jj}$ $[1 \quad i \quad n,$ AND $1 \quad j \quad n]$	Basis i $[1 \quad i \quad n]$
Right ideal, $R \quad I_R$	**Dual vector space**, $\quad$ V* $$\psi x = \psi (x)$$
Generated by idempotents ε_j $\varepsilon_j^2 = \varepsilon_j \quad \varepsilon_j A.$	
$[*: L \quad R, \ \varepsilon_i^{\ *} = \varepsilon_i \ ]$	$[\ \dagger : \qquad]$
Inner product. $\qquad (RL) = R\,L$	**Inner product** $$\psi^{\ *}(x)\psi (x)d^3x$$
$\omega : \varepsilon_{ij} \quad \delta_{ij}, \quad \varepsilon_{ij},$	
[Trace : coeff of λ^{r-1} of characteristic poly.]	
$\omega(L*L) = \omega(\varepsilon_i A \varepsilon_i) = \omega(\lambda_A \varepsilon_i) = \lambda_A \quad R$	
Outer product, $L.R \quad B\varepsilon_{ii}C$ $\qquad\qquad L.L^* \quad B\varepsilon_{ii}B^*$	**Outer product**, Projection operator $P_i = i \quad i$
Completeness relation.	**Completeness relation**.
$$1 = \sum_i \varepsilon_{ii}$$	$$1 = \sum_i |i\rangle\langle i|$$ $$\sum_i \langle \psi_i|x \rangle\langle x|\psi_i\rangle = \delta(x - x)$$

Density operator, $\rho = \sum_i B\varepsilon_{ii}B^*$ **Density operator**, $\rho,$

$$\rho = \sum_i \mu_{ii}\varepsilon_{ii} + \sum_{i,j\,(i<j)} \mu_{ij}\left(\varepsilon_{ij} + \varepsilon_{ji}\right)$$ $$\rho = \sum_i \mu_{ii}|\psi_i\rangle\langle\psi_i| + \sum_{i,j(i<j)} \mu_{ij}\left(|\psi_i\rangle\langle\psi_j| + |\psi_j\rangle\langle\psi_i|\right)$$

$[\ = \dagger,\ \ Tr(\ ) = 1,\ \ +ve\ and\ linear]$

For diagonal ρ For diagonal ρ

$$\rho = \sum_i \lambda_i\varepsilon_{ii}$$ $$\rho = \sum_i \lambda_i|i\rangle\langle i|$$

$$\rho(x,x) = \sum_i \lambda_i\psi^*(x)\psi(x)$$

Expectation values.   **Expectation values.**

$$\langle A\rangle = \omega(\rho A) = \sum_i \varepsilon_{ii}B^*AB\varepsilon_{ii}$$ $$\langle A\rangle = Tr(\rho A) = \sum_r \sum_i \langle r|\lambda_i A|i\rangle\langle i|r\rangle = \sum_r \langle r|A|r\rangle$$

$$\langle A\rangle = \sum_i \lambda_i \int \psi_i^*(x)\psi_i(x)d^3x$$

GNS construction   **GNS construction**

$$\rho = D*D = D*1D, \quad i.e. \quad D = \sum_i \sqrt{\lambda_i}\varepsilon_{ii}$$ a cyclic vector

$$\langle A\rangle = \omega\left(D^*AD\right)$$ $$\langle A\rangle = \langle\ |\pi(A)|\ \rangle$$

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