Large Cardinals, Inner Models and Determinacy

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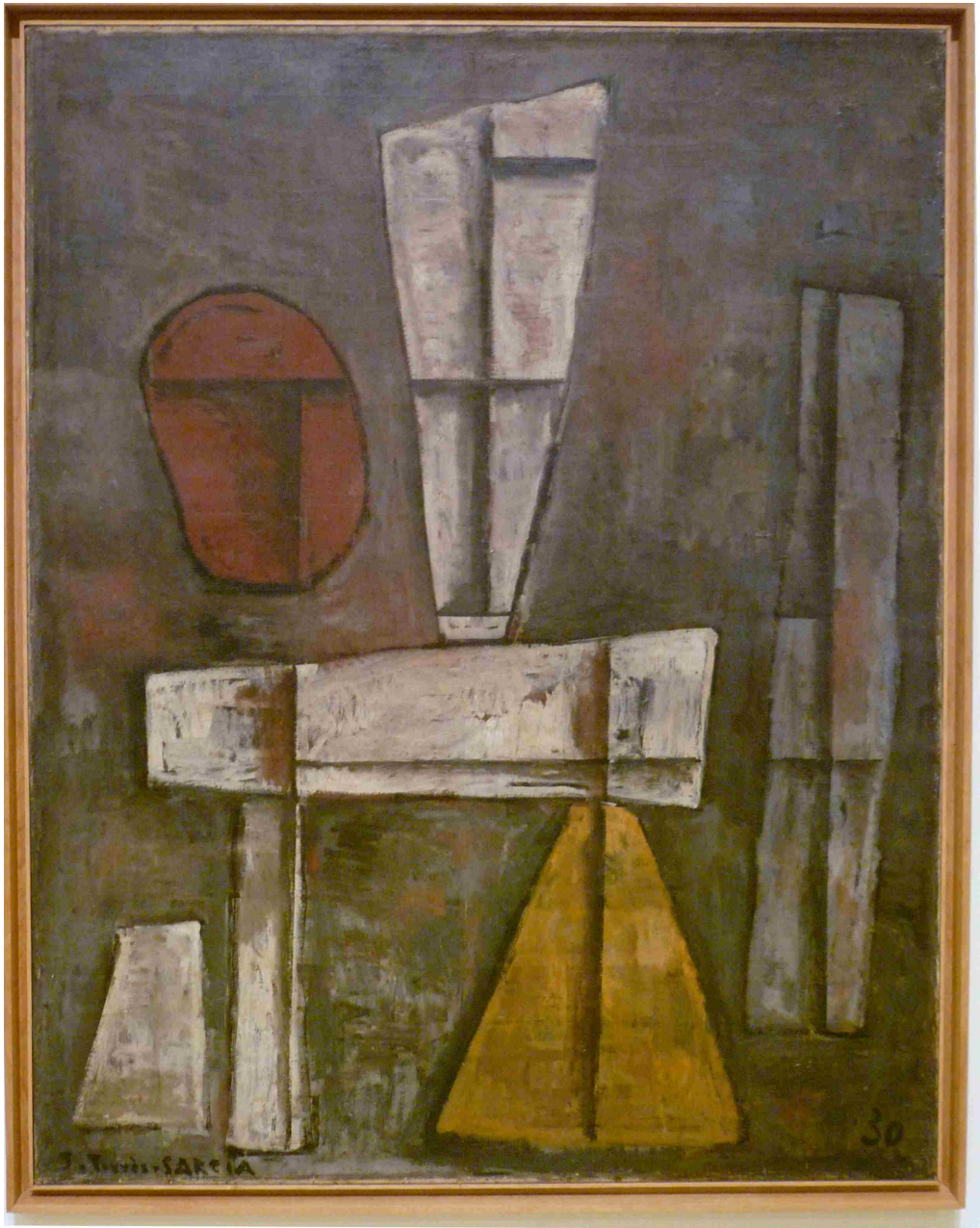

Overview

Part I: Inner Models, Elementary Embeddings and Covering Lemmas
Part II: Determinacy
Part III: Large Cardinals

Overarching Theorem

We use the following theorem to direct and anchor our discussion

Theorem

(ZF) TFAE

(a) $\exists j : L \longrightarrow_e L$.

(b) $\exists \gamma(\omega_2 \leq \gamma \in \text{SingCard} \longrightarrow (\gamma \notin \text{SingCard})^L)$;

(c) $\text{Determinacy}(\Pi^1_1)$

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(a) is an example of an *elementary embedding* of an *inner model*. This assertion is sometimes abbreviated as "$0^\#$ *exists*."
(b) is an example of the *negation* of a *Covering Lemma*, in this case over the inner model L; the negation would assert that the cardinality structure of V is deeply connected with that of L, in that every V-singular cardinal, is singular in L.
(c) is an assertion that two person perfect information games played on Baire space, ω^ω with (lightface) co-analytic payoff sets are determined.

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((a) $\leftrightarrow$ (b) is Jensen (b)$\rightarrow$ (c) is Martin, (b)$\leftarrow$ (c) is Harrington-Martin)

The Main Theorem: (a) $\leftrightarrow$ (b)

- We explore the background here.
- First we have to define some terms:

Definition (Inner Model of ZF)

$IM(M) \leftrightarrow \mathrm{Trans}(M) \wedge \mathrm{On} \subseteq M \wedge (ZF)^M$.

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$IM(M) \leftrightarrow \mathrm{Trans}(M) \wedge \mathrm{On} \subseteq M \wedge (ZF)^M$.

- $IM(M)$ actually has a first order formalisation: it is well known that

$$\mathrm{ZF} \vdash IM(M) \leftrightarrow \forall u \subseteq M \exists v \supseteq u(\mathrm{Trans}(v) \wedge \mathrm{Def}(\langle v, \in \rangle) \subseteq M).$$

Definition (Elementary Embeddings)

Let M, N be inner models of ZF, $j : M \longrightarrow N$ *is an* elementary embedding *if* the function j *takes elements $x \in M$ to elements $j(x) \in N$ in a 'truth preserving way': for any formula $\varphi(v_0, \ldots, v_{n-1})$ and any $\vec{x} = x_0, \ldots, x_{n-1} \in M$, then*

$$\varphi(\vec{x})^M \leftrightarrow \varphi(\overrightarrow{j(x)})^N$$

In this case we write: $j : M \longrightarrow_e N$ and $cp(j)$ for the critical point: the least ordinal α with $j(\alpha) > \alpha$, if it exists.

(ii) If the above holds, but with the formulae restricted to a certain class, eg. Σ_k formulae, then we write $j : M \longrightarrow_{\Sigma_k} N$.

• In the above scheme, we have assumed that the models M, N satisfy $IM(M), IM(N)$ above and are given by terms of our basic set theoretical language, and the same holds true for j. Our embeddings will all have critical points.

• It is an easy consequence of the ZF axioms (using the definition of the rank function, the V_α hierarchy, and Replacement) that if $j : M \longrightarrow_{\Sigma_1} N$, then by a (meta-theoretic) induction on k we may prove $j : M \longrightarrow_{\Sigma_k} N$ for any $k \in \omega$.

Extracting ultrafilters from embeddings

If $\exists j : L \longrightarrow_e L$ then we may define a measure $U = U_j$ on $\kappa = \mathrm{cp}(j)$ and U_j is a normal measure on $\mathcal{P}(\kappa)^L$ as follows: we set

$$X \in U_j \leftrightarrow X \in \mathcal{P}(\kappa)^L \wedge \kappa \in j(X).$$

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More generally: Suppose we have $j : M \longrightarrow_e N$; $\mathrm{cp}(j) = \kappa$, then we may derive a measure $U = U_j$:

$$X \in U_j \leftrightarrow X \in \mathcal{P}(\kappa)^M \wedge \kappa \in j(X).$$

Measures (often) yield embeddings

• Given an ultrafilter U on $\mathcal{P}(\kappa)^M$ for some $IM(M)$ the ultrapower construction yields a map $i : M \to \mathrm{Ult}(M, U)$ where

$$|\,\mathrm{Ult}(M, U)| = \{[f]_\sim \,:\, f \in {}^\kappa M \cap M\}$$

and we define an equivalence as a pseudo-identity:

$$f \sim g \leftrightarrow \{\alpha \,:\, f(\alpha) = g(\alpha)\} \in U$$

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If $U = U_j$ has come from an embedding, we are guaranteed E is wellfounded on $\mathrm{Ult}(M, U)$ and in fact we have: [Diag.]

• We can, and often do, have $(M, \in) = (V, \in)$.
• Also *starting* from a U on $\mathcal{P}(\kappa)^M$, $(\kappa > \omega)$, then we can define $\mathrm{Ult}(M, U)$ as above. If the measure is ω_1-complete we are guaranteed that $\mathrm{Ult}(M, U)$ will be wellfounded.

Definition (Measurable Cardinal)

κ *is a* measurable cardinal *if there is a* κ*-complete non-principle ultrafilter on* $\mathcal{P}(\kappa)$.

• In wellfounded cases we may define by recursion along E a transitivising collapse map $\pi : \mathrm{Ult}(M, U) \longrightarrow (\overline{N}, \in)$ isomorphism.

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• In wellfounded cases we may define by recursion along E a transitivising collapse map $\pi : \mathrm{Ult}(M, U) \longrightarrow (\overline{N}, \in)$ isomorphism.
• Taking $M = V$ then by composition we have:

Theorem

(ZFC) TFAE: Let $\kappa > \omega$:
(a) κ is a measurable cardinal.
(b) $\exists j : V \longrightarrow_e N$ with $cp(j) = \kappa$.

• In (b) $\rightarrow$ (a) because $dom(j) = V$ we obtain κ-completeness of U because $\forall \delta < \kappa \; {}^{\delta}U \subset dom(j)$. Non-principledness is trivial.

An early result on L

Theorem (Scott)

(Scott) (ZF) $\exists\kappa(\kappa$ *a measurable cardinal)*$\longrightarrow V \neq L$.

Proof: If $V = L$, let κ be the least such MC, form the ultrapower and so the embedding above. Then from $j : V \longrightarrow_e N$, and elementarity we have: $(V = L)^V \longrightarrow (V = L)^N$; so (Trans(N)), and so $N = V = L$. But

"κ is the least MC"$\longrightarrow$ "$(j(\kappa)$ is the least MC$)^N$".

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• The assumption implies $\exists j : V \longrightarrow N$, but by Gödel $L^N = L$, so $j \restriction L : L \longrightarrow_e L$. Note that in this case no first order formula $\varphi(v_0)$ can differentiate between κ and $j(\kappa)$: $\varphi(\kappa)^L \leftrightarrow \varphi(j(\kappa))^L$. They are *indiscernible.*

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• Work of Kunen shows that if $\exists j : L \longrightarrow_\varepsilon L$, then a number of consequences follow:

Consequences of $j : L \to L, j \neq id$

(i) Then there is such a j: $L \longrightarrow L$ with $\mathrm{cp}(j) < \omega_1$. Moreover defining U_0 from such a j with critical point κ_0 least, we are guaranteed wellfoundedness of *iterated ultrapowers*: that is we may define $j_{01} : L \longrightarrow_e L$ by taking the ultrapower of L by U_0; define U_1 on $\kappa_1 =_{\mathrm{df}} j_{01}(\kappa_0)$, and then $\mathrm{Ult}(L, U_1)$ will also be wellfounded. We thus may take its transitive isomorph and then have the ultrapower map $j_{12} : L \longrightarrow L$ with critical point κ_1. The process may be iterated without breaking down, forming a directed system $\langle\langle M_\alpha\rangle, j_{\alpha\beta}, \kappa_a, U_\alpha\rangle_{\alpha\leq\beta\in\mathrm{On}}$ with (in this case) all $M_\alpha = L$ and elementary maps into direct limits at limit stages λ, and the κ_a forming a class C of L-inaccessibles, which is closed and unbounded below any uncountable cardinal.

(ii) The iteration points of such ultrapowers enjoy full-blooded indiscernibility properties in L: if $\varphi(v_0, \ldots, v_n)$ is any formula of $\mathcal{L}$ and $\vec{\gamma}$,$\vec{\delta}$ any two ascending sequences from $[C]^{n+1}$ then $(\varphi(\vec{\gamma}) \leftrightarrow \varphi(\vec{\delta}))^L$.

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• NB in (i) really all the action of $j_{\alpha,\alpha+1}$ takes place where the subsets of κ_α are in L: we don't need the whole of L to make sense of this, only the $L_{\kappa_\alpha^{+L}}$. This leads to:

Mice!

Definition (Dodd-Jensen)

Let $j_{\alpha\beta}$ etc. be as above. Let $M_0 = \langle L_{\kappa_0^{+L}}, \in, U_0 \rangle$. This is called the "$0^{\#}$-mouse" which itself has iterated ultrapowers using maps that are the restrictions of the

$$j_{\alpha\beta} : M_\alpha \longrightarrow M_\beta \quad \text{where } M_\alpha = \langle L_{\kappa_\alpha^{+L}}, \in, U_\alpha \rangle \text{ etc.}$$

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By these means we argue for

Theorem (Kunen (a) $\rightarrow$ (b))

(ZF) If $\exists j : L \longrightarrow L$ *then* $\forall \gamma((\gamma \in \text{SingCard}) \longrightarrow (\gamma\,\text{Inacc})^L)$.

Proof: The above implies that $C \cap \gamma$ is unbounded below γ. But C is closed, so $\gamma \in C$. Each $\gamma \in C$ is inaccessible in L. Q.E.D.

Strong Covering Lemma

Theorem (Jensen)

($ZF + \neg 0^{\#}$) For any $X \subseteq \mathrm{On}$, if $|X| > \omega$ then there exists $Y \in L$ with (a) $|Y| = |X|$ and (b) $Y \supseteq X$.

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Corollary

($ZF + \neg 0^{\#}$) (a) Let $(\tau \in \mathrm{Reg})^L$ with $\tau \geq \omega_2$, then $\mathrm{cf}(\tau) = |\tau|$.
(b) Let $\tau \in \mathrm{SingCard}$. Then $\tau^+ = \tau^{+L}$.

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• The Corollary above is sometimes called WCL the Weak Covering Lemma. For other inner models M we may have $\mathrm{WCL}(M)$ provable (obtained by replacing L by M in the Corollary's statements) whilst the strong $\mathrm{CL}(M)$ is not.

Generalizations

If $0^\# = M_0$ exists as above, perhaps there is no non-trivial $j : L[0^\#] \longrightarrow_e L[0^\#]$ and then we have a $\mathrm{CL}(L[0^\#])$? This is indeed the case; if the assumption fails then we have "$(0^\#)^\#$". We then get a theorem along the lines of $\mathrm{CL}(L[0^\#])$ iff $\neg j : L[0^\#] \longrightarrow_e L[0^\#]$. $(0^\#)^\#$ is again a countable object and we can repeat this process. After we have done this uncountably often our #-like mouse objects are no longer countable and we have to resort to uncountable M.

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Theorem (Dodd-Jensen)

(ZF) If there is no IM(M) with $(\exists\kappa(\kappa$ measurable$))^M$ then there is an inner model K^{DJ}, so that there is no non-trivial embedding $j; K^{\mathrm{DJ}} \longrightarrow K^{\mathrm{DJ}}$and $\mathrm{CL}(K^{\mathrm{DJ}})$.

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Theorem (Steel)

(ZFC) If there is no IM(M) with $(\exists\kappa(\kappa$ Woodin$))^M$ then there is a model K^{Steel}, which is again rigid, and over which $\mathrm{WCL}(K^{\mathrm{Steel}})$ *holds.*