Model Theory of Second Order Logic

Lecture 2

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Outline

1. Second order characterizable structures
   - Recap: Second order characterizable structures
   - Ajtai's results
   - Recent results
2. Henkin models
   - Henkin models
   - Completeness Theorem
   - Significance of the Completeness Theorem

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We give two proofs of:

Theorem

There is no second order characterizable structure $\mathcal{M}$ such that the set of Gödel numbers of valid second order sentences is Turing-reducible to truth in $\mathcal{M}$. This is the case even if we study just the valid Σ^1_1-sentences of second order logic.

Proof.

The theory of any second order characterizable structure is Δ_2 (see below). The set of Gödel numbers of valid second order sentences is Π_2-complete (see below). A Π_2-complete set cannot be Turing reducible to a Δ_2-set, by the Hierarchy Theorem of the Levy-hierarchy. □

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We prove two facts that we used in the above proof:

Theorem

If $\mathcal{A}$ is a second order characterizable structure, then the theory of $\mathcal{A}$ is Δ_2-definable.

Theorem

The set of second order (even just Σ^1_1) ϕ such that $\models \phi$, is Π_2-complete.

Proof elements

- L a finite vocabulary, $\mathcal{A}$ a second order characterizable L-structure.
- σ = the conjunction of a large finite part of ZFC.
- $\mathrm{Sut}(M)$ a Π_1-formula which says that M is supertransitive.
- $\mathrm{Voc}(x)$ = the definition of "x is a vocabulary".
- $\mathrm{SO}(L, x)$ = the set-theoretical definition of the class of second order L-formulas.
- $\mathrm{Str}(L, x)$ = the set-theoretical definition of L-structures.
- $\mathrm{Sat}(\mathcal{A}, \phi)$ = the inductive truth-definition of second order logic written in the language of set theory.

Two ways to say y is true in the z-structure x

$$
\begin{aligned}
P_1(z,x,y) &= \mathrm{Voc}(z) \wedge \mathrm{Str}(z,x) \wedge \mathrm{SO}(z,y) \wedge \\
&\quad \exists M(z,x,y \in M \wedge \sigma^{(M)} \wedge \mathrm{Sut}(M) \\
&\quad \wedge (\mathrm{Sat}(z,x,y))^{(M)}) \quad (\Sigma_2) \\
P_2(z,x,y) &= \mathrm{Voc}(z) \wedge \mathrm{SO}(z,y) \wedge \mathrm{Str}(z,x) \wedge \\
&\quad \forall M((z,x,y \in M \wedge \sigma^{(M)} \wedge \mathrm{Sut}(M)) \\
&\quad \rightarrow (\mathrm{Sat}(z,x,y))^{(M)}) \quad (\Pi_2).
\end{aligned}
$$

- $ZFC \vdash \forall z \forall x \forall y (P_1(z,x,y) \leftrightarrow P_2(z,x,y))$
- $\mathcal{A} \models \phi$ iff $\exists x(P_1(L,x,\phi) \wedge P_1(L,x,\theta_{\mathcal{A}}))$ iff $\forall x(P_1(L,x,\theta_{\mathcal{A}}) \rightarrow P_2(L,x,\phi))$. This shows that the second order theory of a second order characterizable $\mathcal{A}$ is Δ_2.
- If L is a vocabulary and ϕ a second order L-sentence, then $\models \phi \iff \forall x(\mathrm{Str}(L,x) \rightarrow P_1(L,x,\phi))$. This shows that "$\phi$ is valid" is Π_2. Now we show it is Π_2-complete.

- Suppose $\exists x \forall y P(x, y, n)$ is a Σ_2-predicate. Let ϕ_n be a Π^1_1-second order sentence the models of which are, up to isomorphism, exactly the models $(V_\alpha, \in)$, where $\alpha = \beth_\alpha$ and $(V_\alpha, \in) \models \exists x \forall y P(x, y, n)$. If $\exists x \forall y P(x, y, n)$ holds, we can find a model for ϕ_n by means of the Levy Reflection principle.

  On the other hand, suppose ϕ_n has a model. W.l.o.g. it is of the form $(V_\alpha, \in)$. Let $a \in V_\alpha$ such that $(V_\alpha, \in) \models \forall y P(a, y, n)$. Since in this case $H_\alpha = V_\alpha$, $(H_\alpha, \in) \models \forall y P(a, y, n)$, where H_α is the set of sets of hereditary cardinality $< \alpha$. By another application of the Levy Reflection Principle we get $(V, \in) \models \forall y P(a, y, n)$, and we have proved $\exists x \forall y P(x, y, n)$.

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An alternative proof

Suppose " $\models \phi$" is Turing-reducible to truth in $\mathcal{A}$, and $\mathcal{A}$ is second order characterizable. Then truth in any second order characterizable structure is Turing-reducible to truth in $\mathcal{A}$. There is a second order characterizable structure $\mathcal{B}$ of size $2^{|A|}$. Hence truth in $\mathcal{B}$ is reducible to truth in $\mathcal{A}$. We show that this is impossible:

Theorem

If $\mathcal{A}$ and $\mathcal{B}$ are any infinite second order characterizable structures such that $2^{|A|} \leq |B|$, then truth in $\mathcal{B}$ is not Turing reducible to truth in $\mathcal{A}$.

Let $|A| = \kappa$, $|B| = \lambda$. Both are second order characterizable, and so are $\mathcal{B}' = (\lambda \cup \mathcal{P}(\kappa), \lambda, <, \mathcal{P}(\kappa), \kappa, \pi, \mathbb{N})$ and $\mathcal{A}' = (\kappa, <, \pi, \mathbb{N})$, where π is a bijection of $\kappa \times \kappa$ onto κ. We show $\mathcal{B}'$ is not Turing-reducible to $\mathcal{A}'$.
L = the vocabulary of $\mathcal{B}'$, $L' \subset L$ that of $\mathcal{A}'$. Suppose for all second order L-sentences ϕ: $\mathcal{B}' \models \phi \iff \mathcal{A}' \models \phi^*$. Can write a second order L-sentence $\Theta(x, y)$ such that for all L'-formulas $\phi(x)$ and any $n \in \mathbb{N}$

$$\mathcal{B}' \models \Theta(\underline{\ulcorner \phi \urcorner}, \underline{n}) \iff \mathcal{A}' \models \phi(\underline{n}), \text{ thus}$$

$$\mathcal{A}' \models \phi(n) \iff \mathcal{A}' \models \Theta(\underline{\ulcorner \phi \urcorner}, \underline{n})^*.$$

Let $\psi(x)$ be the L'-formula which says $\neg\Theta(\underline{a}, \underline{a})^*$ for the "natural number" a, in the formal sense, that is the value of x. Let $k = \ulcorner \psi(x) \urcorner$. Now

$$\kappa \models \Theta(\underline{k}, \underline{k})^* \iff \mathcal{A}' \models \psi(\underline{k}) \iff \mathcal{A}' \models \neg(\Theta(\underline{k}, \underline{k}))^*,$$

a contradiction.

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- In summary: Validity of sentences of even “low level" second order logic cannot be analyzed in terms of truth in a particular second order characterizable structure.
- In set theory: There is a Σ_n-truth definition for Σ_n-formulas. There are arbitrarily large α such that $V_\alpha \prec_n V$. So for a Σ_n-sentence ϕ :

$$\phi \text{ is true } \iff V_\alpha \models \phi.$$

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- Significance of the Completeness Theorem

- $\mathcal{M} = (M, R_1, ..., R_n, f_1, ..., f_m, c_1, ..., c_k)$, M countable.
- Does the second order theory of $\mathcal{M}$ determine $\mathcal{M}$ up to isomorphism?
- Ajtai showed in 1979 that this cannot be decided on the basis of CA (or ZFC) alone.
- We show: The non-categoricity phenomenon is quite general.

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Ajtai's First Theorem (1979)

Theorem

If $V = L$ (or there is a second order definable well-order of $\mathbb{R}$), then every countable model in a finite vocabulary is second order characterizable by a theory.

- A second order definable well-order of $\mathbb{R}$ means a well-order $\prec$ of $\mathcal{P}(\mathbb{N})$ such that for some second order formula $\phi(<, P, Q)$ for all $A \subseteq \mathbb{N}, B \subseteq \mathbb{N}$:

$$A \prec B \iff (\mathbb{N}, <, A, B) \models \phi(<, P, Q)\}.$$

- If $V = L$, then there is a Σ^1_2 well-order of $\mathbb{R}$.

Ajtai's First Theorem (1979)

Proof.

Assume $V = L$ via $\phi(<, P, Q)$. Suppose $\mathcal{M}$ and $\mathcal{M}'$ are second order equivalent. W.l.o.g. $M = M' = \mathbb{N}$ and $\mathcal{M}$ is minimal in $\prec$ (mod coding) among models $\cong \mathcal{M}$, same with $\mathcal{M}'$. We show $\mathcal{M} = \mathcal{M}'$.

Suppose S is in the vocabulary of $\mathcal{M}$ and $\mathcal{M} \models S(a_1, ..., a_n)$. $\mathcal{M}$ satisfies the sentence "For some linear order $<$ of order-type ω, the $\prec$-minimal structure isomorphic to me satisfies $S(\underline{a_1}, ..., \underline{a_n})$". Hence $\mathcal{M}'$ has a linear order $<'$ of order-type ω such that the $\prec$-minimal structure isomorphic to $\mathcal{M}'$ satisfies $S(\underline{a_1}, ..., \underline{a_n})$. Extend $\pi : (M', <') \cong (\mathbb{N}, <)$ to $\pi : (\mathcal{M}', <') \cong (\mathcal{M}^*, <)$. By minimality, $\mathcal{M}^* = \mathcal{M}'$. Then $\mathcal{M}' \models S(a_1, ..., a_n)$ follows. □

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Ajtai's Second Theorem

Theorem (Ajtai 1979)

It is consistent, relative to the consistency of ZF, that there is a countable model in a finite vocabulary which is not second order characterizable by a theory.

Ajtai's Second Theorem 1979

Proof.

Let $\mathbb{P}$ be Cohen forcing for adding a generic set $G \subseteq \mathbb{N}$. Let F^G be the set of $A \subseteq \mathbb{N}$ with $A \Delta G$ finite. We show that $\mathcal{M} = (\omega \cup F^G, \omega, <, E)$, where $nEx \iff n \in X$, is not characterizable by a second order theory.

Let $\mathcal{M}' = (\omega \cup F^{-G}, \omega, <, E)$. Since $\mathcal{M} \not\cong \mathcal{M}'$, it suffices to prove that $\mathcal{M}$ and $\mathcal{M}'$ are second order equivalent. In fact more is true: If $\Phi(x)$ is any formula of set theory, then $\Phi(\mathcal{M}) \iff \Phi(\mathcal{M}')$. Suppose $p \Vdash \Phi(\dot{\mathcal{M}})$. Let G be a generic set extending p. Then $V[G] \models \Phi(\mathcal{M})$. Let G' be another generic, which agrees with G on p but is the complement of G elsewhere. Clearly $\mathcal{M}^{G'} = \mathcal{M}'^{G}$. Thus $V[G'] \models \Phi(\mathcal{M}')$. But $V[G] = V[G']$, so $V[G] \models \Phi(\mathcal{M}')$. $\square$

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Non-categoricity in uncountable models

Lauri Keskinen, Amsterdam 2011:

- $L^2_{\kappa\omega}$: Add to second order logic infinite conjunctions and disjunctions of length $< \kappa$.
- Ajtai(κ) says "the $L^2_{\kappa\omega}$-theory of any model of cardinality κ in a finite vocabulary determines the model up to isomorphism among models of cardinality κ" i.e. "$L^2_{\kappa\omega}$-equivalent models of cardinality κ in a finite vocabulary are isomorphic."

Uncountable models

Lauri Keskinen, Amsterdam 2011:

- $V = L$ implies Ajtai(κ) for all κ.
- For any κ there is a forcing extension in which κ is preserved and Ajtai(κ) fails.
- For any finite set of regular cardinals, Ajtai(κ) can hold in the set and fail for regular cardinals outside.
- Ajtai(ω) is consistent with n Woodin cardinals. Third order Ajtai(ω) is consistent even with a supercompact cardinal.
- A proper class of Woodin cardinals implies that Ajtai(ω) fails.

Hyttinen-Kangas-V. 2011: Extension of Ajtai's results to models of first order theories, using stability theory and Shelah's Classification Theory.

T countable complete first order theory

Classifiable	Non-classifiable
• Every model of size κ is infinitary second order characterizable. • Converse holds, too. κ regular $\kappa = \aleph_\alpha$, $\beth_{\omega_1}(\lvert\alpha\rvert + \omega) \leq \kappa$ $2^\lambda < 2^\kappa$ *for all* $\lambda < \kappa$. $L^2_{\kappa\omega}$	• For large κ, Cohen(κ) forces T to have models of size κ that are not infinitary second order characterizable. • For unstable T, λ can be ω. • With Henkin models can avoid cardinality assumptions and forcing. $\kappa = \lambda^+ = 2^\lambda$, $\lambda^{<\lambda} = \lambda > 2^\omega$ $L^2_{\kappa\omega}$

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Henkin structures

Definition

A Henkin model is a pair $(M, \mathcal{G})$, where M is a structure and $\mathcal{G}$ is a collection of relations on M, i.e. $\mathcal{G} \subseteq \bigcup_{n<\omega} \mathcal{P}(M^n)$. Full, if "$=$" instead of "$\subseteq$".

Truth definition for Henkin structures

- $\mathcal{M} \models_s \exists X_n^m \phi \iff \mathcal{M} \models_{s(P/X_n^m)} \phi$ for some $P \in \mathcal{P}(M^m)$
- $(\mathcal{M}, \mathcal{G}) \models_s \exists X_n^m \phi \iff (\mathcal{M}, \mathcal{G}) \models_{s(P/X_n^m)} \phi$ for some $P \in \mathcal{P}(M^m) \cap \mathcal{G}$

The rationale behind Henkin models

- Some results about full models hold even for Henkin models.
- Some results can only be obtained for Henkin models.
- Some results obtain for Henkin models under weaker assumptions than for full models.
- Truth in Henkin models can be axiomatized.

- The Henkin models are assumed to satisfy all instances of the Comprehension Axioms:

$$\phi(\psi(\vec{z})/Y) \rightarrow \exists Y \phi,$$

or equivalently,

$$\exists Y \forall x_1 \ldots \forall x_m (Yx_1 \ldots x_m \leftrightarrow \psi)$$

for any second order ψ not containing Y free.

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Completeness Theorem

There are obvious axioms and rules for second order logic, introduced by Hilbert-Ackermann 1928, among them the Comprehension Axioms and Axioms of Choice.

Theorem (Henkin 1951)

If T is consistent (does not prove a contradiction), then T has a Henkin model.

Sketch.

One describes a winning strategy of the “second player" in a game, called the Model Existence Game (see *Models and Games* monograph CAP 2011). The strategy of *II* is to play so that at each point the set of played sentences is consistent with T (i.e. does not prove a contradiction). The winning strategy, when played against the best strategy of the “first player", yields a Henkin model of T. □

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- Henkin models are everywhere where there is consistency. This does not tell us so much about the mathematical objects we are usually interested in, as about the nature of consistency.
- Some properties of the Henkin model cannot be expressed: countable, uncountable, finite, infinite. Henkin models are a tool, and this tool is very flexible.
- The axiom P^2 has non-standard Henkin models. Whatever we prove about the standard model will be true in the non-standard models, too. This is what provability means.

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To keep in mind about Henkin models:

- Second order sentences are not full or Henkin, only models are.
- The sentences of second order logic do not "know" what semantics is being used.

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The usefulness of Henkin models:

- To give a (finite) proof of a second order ϕ from a second order theory T one can use the semantical method:
  - Suppose $(\mathcal{M}, \mathcal{G}) \models T$. We show $(\mathcal{M}, \mathcal{G}) \models \phi$.
- We have proved not only that ϕ is true in the intended model of T (if there was one), but that ϕ is true in the entire cloud inside which the intended model of T is lurking.
- One can hardly find a convincing argument for ϕ logically following from T unless there was an actual formal proof of ϕ from T and the Comprehension (and Choice) axioms.

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- The transfer from

$$\forall \mathcal{M}(\mathcal{M} \models T \Rightarrow \mathcal{M} \models \phi)$$

to

$$\forall \mathcal{M}, \mathcal{G}((\mathcal{M}, \mathcal{G}) \models T \Rightarrow (\mathcal{M}, \mathcal{G}) \models \phi)$$

is in practice insignificant, but it has important philosophical content.

- Paying attention to $\mathcal{G}$ in the proof gives a reduction from an infinitistic inference to a finitist one.