Model theory of second order logic

Lecture 1

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Outline

1. Introduction
   - Introduction
   - Number theory
   - Analysis
   - Set theory
2. Second order characterizable structures
   - Second order characterizable structures

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2 Second order characterizable structures
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Our framework

- Usual set theoretic framework.
- In the third lecture a more general approach.

- $\mathcal{M} = (M, R_1, ..., R_n, f_1, ..., f_m, c_1, ..., c_k)$.
- First order logic: for every element a of the domain of the model ...
- Second order logic: for every subset of (or relation on) the domain of the model ...
- Third order logic: for every family of subsets of (or relation between relations on) the domain of the model ...
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- Sort logic: for every superset M' of the domain M and every relation on M' ...

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- $\mathcal{M}$ is characterizable by a sentence (or theory) ϕ in *some* logic if for all $\mathcal{M}'$:

$$\mathcal{M}' \cong \mathcal{M} \iff \mathcal{M}' \models \phi.$$

- $\mathcal{M}$ characterizable in first order logic iff M is finite.
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- Can an infinite $\mathcal{M}$ be characterized in second order logic?
  - For many: Yes.
  - For *all* e.g. countable: Depends on set theory.
  - Depends on model theoretic properties of $\mathcal{M}$.
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- There is a more general notion of a structure: Henkin structure $(\mathcal{M}, \mathcal{G})$, where $\mathcal{G}$ is a set of subsets of and relations on M, for the range of second order quantifiers, such that certain comprehension axioms are satisfied. Every usual structure $\mathcal{M}$ gives rise to a Henkin structure $(\mathcal{M}, \mathcal{P}(M) \cup \mathcal{P}(M^2) \cup ...)$, called a full structure.
- Some results about all full models hold also for Henkin models.
- Sometimes we cannot get a full model and have to settle with a Henkin model.
- “Henkin model" is an auxiliary concept.
- Typically, full models “lurk" inside clouds of Henkin models.
- Truth in Henkin models can be axiomatized, truth in full models not.

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A rough correspondence

first order logic	$\Delta_1^{KPU-Inf}$-part of set theory
first order logic with the well-ordering quantifier	Δ_1-part of set theory
first order logic with the Härtig quantifier	$\Delta_1(Cd)$-part of set theory
second order logic	Δ_2-part of set theory
sort logic	set theory
Boolean valued sort (or second order) logic	Ω-logic (Ongoing work with D. Ikegami)

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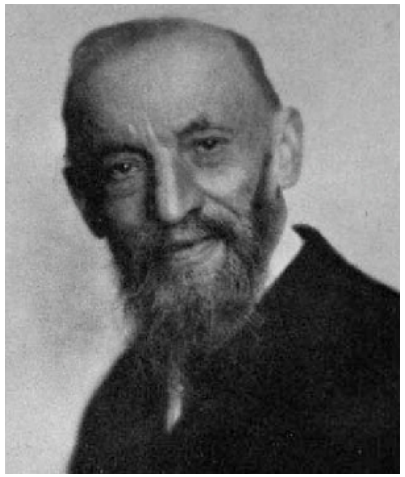
Second order axioms of number theory, P^2

Dedekind analyzed and Peano axiomatized number theory in 1888 and 1889. Also Peirce 1881.



- Constant symbol c for zero, function symbol S for successor function.
- $\forall y \neg Sy = c$,
  $\forall y \forall z (Sy = Sz \rightarrow y = z)$,
  $\forall X((Xc \wedge \forall y(Xy \rightarrow XSy)) \rightarrow \forall y Xy)$ (Induction)

Second order axioms of number theory, P^2

Theorem (Peirce 1881 (?), Dedekind 1888, Peano 1889 (?))

There is, up to isomorphism, only one model of P^2, namely the standard model $(\mathbb{N}, S, 0)$.

Proof.

If $(M, s, a) \models P^2$, interpret Y as $\{a, sa, ssa, sssa, ...\}$. □

Note

- In first order logic there is no theory T such that $(\mathbb{N}, S, 0)$ is, up to isomorphism, the only model of T.
- Addition, multiplication, etc can be defined in second order logic from 0 and S, by recursion. Not so in first order logic.

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An axiomatization of real numbers, R^2

- Karl Weierstrass brought the theory of the real numbers (from geometry) to algebra and set theory.
- David Hilbert axiomatized the ordered field of real numbers in 1902.

Real numbers, R^2

Second order axioms of real numbers, R^2:

Constant symbols for zero and one, function symbols $+, \times$, relation symbol $<$.

- Field axioms for $+$, $\cdot$, 0 and 1
- $x < y \rightarrow x + z < y + z$
- $0 < x, 0 < y \rightarrow 0 < xy$
- $\forall X \Big((\exists y Xy \wedge \exists x \forall y (Xy \rightarrow y \leq x))) \rightarrow$
  $\exists x \forall y \big(\forall z (Xz \rightarrow z \leq y) \leftrightarrow x \leq y \big) \Big)$

Characterization of the reals

Theorem

There is, up to isomorphism, only one model of R^2, namely $(\mathbb{R}, +, \cdot, 0, 1, \leq)$.

Proof.

Given two models, find integers, then rationals in both. Map rationals to rationals. Extend to isomorphism using completeness. □

First order versus second order

- In second order logic natural numbers can be defined on $\mathbb{R}$. Not so in first order logic.
- Tarski 1948: The ordered field of reals is decidable in first oder logic. Not so in second order logic.

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Second order axioms of set theory, ZFC^2, Zermelo 1908

- Usual ZFC axioms except replacement and separation are written in second order form. A finite set of axioms ZFC^2 obtains.
- Models of ZFC^2 are well-founded and the order-type of their ordinals is a strongly inaccessible cardinal.

Second order axioms of set theory, ZFC^2

Theorem (Zermelo 1930)

For any strongly inaccessible cardinal $> \omega$ there is, up to isomorphism, exactly one model of ZFC^2 with the set of ordinals of order-type κ, namely $(V_\kappa, \in)$.

Proof.

Transfinite induction on the rank. Suppose a well-founded (M, E) is a model of ZFC^2 of inaccessible height κ. W.l.o.g. (M, E) is a transitive structure $(M, \in)$, hence $M \subseteq V_\kappa$. Easily, M is supertransitive (Suppose $X \subseteq x \in M$. Then $X \subseteq M$, as M is transitive. By separation, $X \in M$). Hence $M = V_\kappa$. □

- In second order set theory mathematical questions outside set theory are decided.
- Continuum Hypothesis CH $\forall A \subseteq \mathbb{R}(|A| \leq |\mathbb{N}| \vee |A| = |\mathbb{R}|)$ is decided by ZFC^2, but we do not (yet) know in which way ...
- ... just as we do not (yet) know how $(V, \in)$ decides CH in set theory.

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Second order characterizable structures

- A model M is second order characterizable if there is a second order sentence θ_M such that for all M':

$$M' \models \theta_M \iff M' \cong M.$$

- Characterizability by a theory is defined similarly.
- We already know many second order characterizable structures. However, there are only countably many such.

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1. Natural numbers $(\mathbb{N}, +, \cdot, 0, 1, <)$
2. The field of real numbers $(\mathbb{R}, +, \cdot, 0, 1, <)$
3. The field of complex numbers $(\mathbb{C}, +, \cdot, 0, 1)$
4. The Euclidean space $(\mathbb{R}^n, ||\cdot||)$
5. The Banach space $(\ell^p, ||\cdot||^p)$
6. The Hilbert space $(L^p, ||\cdot||^p)$
7. The Euclidean geometry, consisting of points and lines
8. The ordered set $(\omega_n, <)$
9. The pure set $(\aleph_n)$
10. The Boolean algebra $(\mathcal{P}(\omega), \cap, \cup, \emptyset, -)$
11. Any finite structure
12. Any recursive countable structure
13. The random graph
14. The free group of $\aleph_n$ generators
15. The level of the cumulative hierarchy $(V_{\omega+n}, \in)$
16. The level of the cumulative hierarchy $(V_{\omega_n}, \in)$
17. The level of the cumulative hierarchy $(V_\kappa, \in)$, where κ is the first inaccessible (Mahlo, weakly compact, Ramsey) cardinal.
18. Every consistent recursive first order theory has a second order characterizable model in each second order characterizable infinite cardinality.

Non-second order characterizable structures

Not second order characterizable:

- $(\mathbb{R}, +, \times, 0, 1, <, \prec)$, where $\prec$ is a well-order, assuming large cardinals. (Martin-Steel 1989)
- $(\mathbb{N}, <, P)$, where P is the set of Gödel numbers of valid second order sentences.

As to $(\mathbb{N}, <, P)$, we give a proof of a more general result:

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We give two proofs of:

Theorem

There is no second order characterizable structure $\mathcal{M}$ such that the set of Gödel numbers of valid second order sentences is Turing-reducible to truth in $\mathcal{M}$.

Proof.

The theory of any second order characterizable structure is Δ_2 (see below). The set of Gödel numbers of valid second order sentences is Π_2-complete (see below). A Π_2-complete set cannot be Turing reducible to a Δ_2-set, by the Hierarchy Theorem of the Levy-hierarchy. □

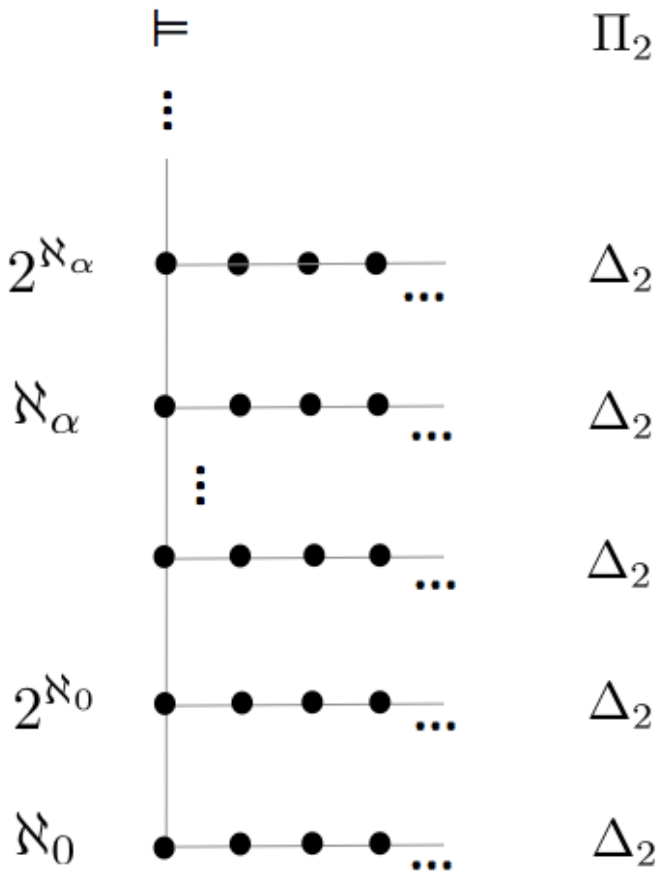

Figure: The hierarchy of second order characterizable structures

We prove two fact that we used in the above proof:

Theorem

If $\mathcal{A}$ is a second order characterizable structure, then the theory of $\mathcal{A}$ is Δ_2-definable.

Theorem

The set of second order ϕ such that $\models \phi$, is Π_2-complete.

Proof elements

- L a finite vocabulary, $\mathcal{A}$ a second order characterizable L-structure.
- σ = the conjunction of a large finite part of ZFC.
- $\mathrm{Sut}(M)$ a Π_1-formula which says that M is supertransitive.
- $\mathrm{Voc}(x)$ = the definition of “x is a vocabulary".
- $\mathrm{SO}(L, x)$ = the set-theoretical definition of the class of second order L-formulas.
- $\mathrm{Str}(L, x)$ = the set-theoretical definition of L-structures.
- $\mathrm{Sat}(\mathcal{A}, \phi)$ = the inductive truth-definition of second order logic written in the language of set theory.

Two ways to say y is true in the z-structure x

$$
\begin{aligned}
P_1(z,x,y) &= \mathrm{Voc}(z) \wedge \mathrm{Str}(z,x) \wedge \mathrm{SO}(z,y) \wedge \\
&\quad \exists M(z,x,y \in M \wedge \sigma^{(M)} \wedge \mathrm{Sut}(M) \\
&\quad \wedge (\mathrm{Sat}(z,x,y))^{(M)}) \quad (\Sigma_2) \\
P_2(z,x,y) &= \mathrm{Voc}(z) \wedge \mathrm{SO}(z,y) \wedge \mathrm{Str}(z,x) \wedge \\
&\quad \forall M((z,x,y \in M \wedge \sigma^{(M)} \wedge \mathrm{Sut}(M)) \\
&\quad \rightarrow (\mathrm{Sat}(z,x,y))^{(M)}) \quad (\Pi_2).
\end{aligned}
$$

- $ZFC \vdash \forall z \forall x \forall y (P_1(z,x,y) \leftrightarrow P_2(z,x,y))$
- $\mathcal{A} \models \phi$ iff $\exists x(P_1(L,x,\phi) \wedge P_1(L,x,\theta_{\mathcal{A}}))$ iff $\forall x(P_1(L,x,\theta_{\mathcal{A}}) \rightarrow P_2(L,x,\phi))$. This shows that the second order theory of a second order characterizable $\mathcal{A}$ is Δ_2.
- If L is a vocabulary and ϕ a second order L-sentence, then $\models \phi \iff \forall x(\mathrm{Str}(L,x) \rightarrow P_1(L,x,\phi))$. This shows that "$\phi$ is valid" is Π_2. Now we show it is Π_2-complete.

- Suppose $\exists x \forall y P(x, y, n)$ is a Σ_2-predicate. Let ϕ_n be a second order sentence the models of which are, up to isomorphism, exactly the models $(V_\alpha, \in)$, where $\alpha = \beth_\alpha$ and $(V_\alpha, \in) \models \exists x \forall y P(x, y, n)$. If $\exists x \forall y P(x, y, n)$ holds, we can find a model for ϕ_n by means of the Levy Reflection principle. On the other hand, suppose ϕ_n has a model. W.l.o.g. it is of the form $(V_\alpha, \in)$. Let $a \in V_\alpha$ such that $(V_\alpha, \in) \models \forall y P(a, y, n)$. Since in this case $H_\alpha = V_\alpha$, $(H_\alpha, \in) \models \forall y P(a, y, n)$, where H_α is the set of sets of hereditary cardinality $< \alpha$. By another application of the Levy Reflection Principle we get $(V, \in) \models \forall y P(a, y, n)$, and we have proved $\exists x \forall y P(x, y, n)$.