Groundedness in Set Theory, Part I

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1. Groundedness and Related Notions
2. Iterative Conception and Groundedness
3. Groundedness and Logic
4. Conclusion

The cumulative hierarchy

- Levels V_α of the hierarchy usually defined as follows using ordinal recursion:

$$
\begin{aligned}
V_0 &= \varnothing; \\
V_{\alpha+1} &= \mathcal{P}(V_\alpha); \\
V_\lambda &= \bigcup_{\alpha<\lambda} V_\alpha \text{ if } \lambda \text{ is a limit ordinal.}
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- This definition needs to be justified; instead, define the V_αs explicitly by synthetic means:

$$\begin{aligned} x = V_\alpha \Leftrightarrow & \exists f(\mathrm{Dom}(f) = \alpha + 1 \wedge \forall \beta \leqslant \alpha \forall y(y \in f(\beta) \leftrightarrow \\ & \exists \lambda < \beta(y \subseteq f(\lambda))) \wedge f(\alpha) = x) \end{aligned}$$

Groundedness

Definition

A set is *grounded* iff it's a subset of some V_α.

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Every set is grounded.

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Every set is grounded.

- The received view is that Groundedness holds.
- But what is needed, at the formal level, to ensure this (taking, for the moment, Z minus the Axiom of Regularity as our base theory)?

Well-foundedness of **V** and $\in$-induction

- Possible initial thought: we need to ensure that the set-theoretic universe is *well-founded*.
- This is natural interpreted as: we want to ensure that every nonempty *class* has an $\in$-minimal element.

Well-foundedness of **V** and $\in$-induction

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- This is natural interpreted as: we want to ensure that every nonempty *class* has an $\in$-minimal element.It implies:

$\in$-induction

$\forall x(\forall y \in x\ \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x)$.

- Conversely, by $\in$-induction, we can argue that every nonempty class has an $\in$-minimal element.

Niceness

- Can we get $\in$-induction by requiring that *sets* have some property in the ballpark of well-foundedness?
- Idea: we want to rule out sets that are at the beginning of an infinite descending $\in$-chain.

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- Idea: we want to rule out sets that are at the beginning of an infinite descending $\in$-chain.

Definition

A set a is *nice* iff there is no infinite descending $\in$-chain starting with a.

Niceness

Every set is nice.

Regularity

Theorem

Assume DC *and 'Every set has a transitive closure'. Then, if every set is nice, $\in$-induction holds.*

- To dispense with DC, we are led to the standard property required of sets to get $\in$-induction: regularity.

Regularity

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- To dispense with DC, we are led to the standard property required of sets to get $\in$-induction: regularity.

Definition

A set is *regular* iff it has an $\in$-minimal element.

Regularity

Every set is regular.

Regularity and existence of transitive closures

THEOREM

Assume 'Every set has a transitive closure'. Then, if every set is regular, $\in$-induction holds.

Regularity and existence of transitive closures

Theorem

Assume 'Every set has a transitive closure'. Then, if every set is regular, $\in$-induction holds.

- One way of looking at the situation:
  - if we add an axiom asserting that every set has a transitive closure to Z_2 it follows that the universe is well-founded;
  - but there are non-well-founded models of Z_2.
- Thus, we might consider adding 'Every set has a transitive closure' as an axiom.

Regularity and Groundedness

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- Consider the theory obtained by replacing Z_2's Axiom of Infinity with an axiom to the effect that V_ω exists and an axiom to the effect that every set has a transitive closure.
- Uzquiano (1999) has shown that this theory does not characterize the models of the form $\langle V_\lambda, \in \cap (V_\lambda \times V_\lambda) \rangle$ for λ a limit ordinal greater than ω.
- On the other hand, the theory obtained by replacing Z_2's Axiom of Regularity with Groundedness does characterize these models (as well as implying $\in$-induction and proving that every set has a transitive closure).

Groundedness vs. Replacement

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Groundedness vs. Replacement

- Upshot seems to be: if we want to enforce the cumulative hierarchy structure on the set-theoretic universe, we need a different strategy.
- Two options immediately come to mind:

1. Replace Regularity with Groundedness.
2. Add Replacement, which implies Groundedness.

Is Groundedness natural?

- A worry for option 1 might be that Zermelo set theory with Groundedness instead of Regularity is not a 'natural extension of the Zermelo axioms' (Uzquiano 1999: 301).
- And this might mean that we are left with option 2, since

> *replacement may very well be the only natural principle about sets whose addition to the Zermelo axioms delivers a system of axioms which contains an implicit description of a cumulative hierarchy of levels and stages. (Uzquiano 1999: 301)*

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- To neutralize this worry, one might respond by pointing out that Groundedness becomes very natural once we switch to axiomatizations *à la* Scott-Potter which revolve around the notion of a level.

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2 Iterative Conception and Groundedness

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4 Conclusion

The iterative conception

- According to the iterative conception of set, every set occurs at one level or another of the cumulative hierarchy.
- Thus, according to the iterative conception, Groundedness holds.
- But is the iterative conception *correct*?

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- Thus, according to the iterative conception, Groundedness holds.
- But is the iterative conception *correct*?
- Two main answers: *substantial* approach and *minimalist* approach.

The substantial approach

- Argues that the fact that sets can be arranged in a cumulative hierarchy divided into levels follows from general considerations about the nature of sets.
- Claims that 'there is a fundamental relation of presupposition, priority or [...] *dependence* between collections' (Potter 2004: 36).

The substantial approach

- Argues that the fact that sets can be arranged in a cumulative hierarchy divided into levels follows from general considerations about the nature of sets.
- Claims that 'there is a fundamental relation of presupposition, priority or [...] *dependence* between collections' (Potter 2004: 36).
- Crucial question for the substantial approach: what does this relation amount to?
- Most popular answer (nowadays): the relation of priority between a set and its members is a relation of *metaphysical dependence*.

Metaphysical dependence and necessity

- Idea: whilst singleton of Socrates metaphysically depends on Socrates, there's no dependence in the reverse direction.
- But what does it mean to say that a set metaphysically depends on its members?

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- But what does it mean to say that a set metaphysically depends on its members?
- It's tempting to characterize the dependence relation in terms of necessity, and say that x depends on y iff, necessarily, x exists only if y exists.
- This won't serve the dependency theorist's purposes: for although she accepts that necessarily the singleton of Socrates exists only if Socrates does, she also seems to be committed to the converse: necessarily, Socrates exists only if his singleton does (Fine 1995).

Metaphysical dependence as primitive and unanalyzable

- Potter (2004: 39–40) concludes that 'priority is a modality distinct from that of time or necessity, a modality arising in some way out of the manner in which a collection is constituted from its members'.
- The idea seems to be that the dependence relation of a set upon its members is primitive and unanalyzable, although this, the thought goes, does not prevent us from determining some of its structural properties.

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- The idea seems to be that the dependence relation of a set upon its members is primitive and unanalyzable, although this, the thought goes, does not prevent us from determining some of its structural properties.
- This must be the case because, according to the dependency theorist, the claim that the hierarchy covers all sets follows from the existence of the said relation of metaphysical dependence between a set and its members.
- And for this to be the case this relation must have certain structural properties.

Metaphysical dependence and Groundedness

- In particular, the dependence relation has to be *irreflexive*, so that no set can depend upon itself.
- It has to be *antisymmetric*, so that although a set always depend upon its members, the members never depend upon the set.
- And, crucially, it has to be *well-founded*.

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- In particular, the dependence relation has to be *irreflexive*, so that no set can depend upon itself.
- It has to be *antisymmetric*, so that although a set always depend upon its members, the members never depend upon the set.
- And, crucially, it has to be *well-founded*.
- Thus, the dependency theorist needs arguments to the effect that the metaphysical dependence relation, which she takes to be primitive and unanalyzable, does have these structural properties.

Metaphysical dependence and Groundedness

- Notice, though, that even if she succeeds in doing that, it would only seem to follow that all sets are regular.
- In the absence of Replacement, this is not enough to vindicate Groundedness (even if transitive closures exist).

Metaphysical dependence and Groundedness

- Notice, though, that even if she succeeds in doing that, it would only seem to follow that all sets are regular.
- In the absence of Replacement, this is not enough to vindicate Groundedness (even if transitive closures exist).
- Two ways out come to mind:

1. Argue that Groundedness holds because Replacement is justified on the dependency account.
2. Argue directly for Groundedness, via some principle maximizing the width of the hierarchy.

Maximizing principles

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Maximizing principles

- Dependency theorists doubting that the iterative conception sanctions Replacement might favour the second way out.
- The second way out would make use of some principle like the following:

Width-Maximizing

If a level can contain a set a, then it does contain a.

- The obvious challenge is then to give an account of the modality involved here which does not raise the problems that one wanted to avoid by resorting to dependence.

The minimalist approach

- An alternative approach to the substantial one is the *minimalist* approach:

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- Consists of two main components:

1. A view of how the iterative conception is to be understood (*minimalist account* of the iterative conception);
2. A view of why the iterative conception so understood is to be regarded as correct (*indirect strategy*).

The minimalist account

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- Analogy with idea that our conception of the natural number structure can be conveyed by saying that the natural numbers are 0, and its successor, and *its* successor, and *ITS* successor, and so on.
- On the minimalist account, the iterative conception of the universe of (pure) sets can be conveyed by saying that the sets are the empty set and the set containing the empty set, sets of those, sets of *those*, sets of *THOSE*, and so on.
- Iterative conception as a conception of sets as sets *of* something: sets as what one obtains by iterating the *set of* operation starting from the individuals (or nothing).

The minimalist account and Groundedness

- The minimalist account is meant to sidestep the difficulties involved in establishing the structural features of the metaphysical dependence relation (and in its lack of explanatory role).
- But we have seen that this is not the only problem that the dependence account faces, as Regularity doesn't deliver Groundedness in the absence of Replacement.

The minimalist account and Groundedness

- The minimalist account is meant to sidestep the difficulties involved in establishing the structural features of the metaphysical dependence relation (and in its lack of explanatory role).
- But we have seen that this is not the only problem that the dependence account faces, as Regularity doesn't deliver Groundedness in the absence of Replacement.
- On the other hand, Groundedness is part of the very characterization of the minimalist account.
- Thus, if successful, the minimalist approach seems to avoid the need to resort to something like Width-Maximizing. (But issue still open about the possible need for Height-Maximizing.)

The indirect strategy

- Main idea: regard the iterative conception as correct because it's the most satisfactory conception of set that we have — i.e. because better than its rivals with regards to certain virtues that a conception of set may have.

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- Main idea: regard the iterative conception as correct because it's the most satisfactory conception of set that we have — i.e. because better than its rivals with regards to certain virtues that a conception of set may have.
- Idea arises quite naturally out of the minimalist's emphasis that the appeal of the iterative conception is due to its explanation of the paradoxes and capacity to show how (most of) the axioms of standard set theory can be true.
- These and perhaps other virtues of the iterative conception make it the most satisfactory conception of set that we have, and for this reason the conception should be regarded as correct.
- Thus, a full defence of the iterative conception will involve looking at the viability of conceptions of set other than the iterative one.

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Intuitionistic set theory and Regularity

- Briefly something about the connection between Groundedness and the logic of set theory.
- Suppose we're interested in developing an intuitionistic set theory based on the iterative conception.

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- Suppose we're interested in developing an intuitionistic set theory based on the iterative conception.
- Let Z_0 be the theory in which the Axioms of Empty Set, Extensionality, Separation and Unordered Pairs hold. Then we have:

Theorem (Myhill)

Regularity intuitionistically implies LEM *in the presence of the axioms of* Z_0.

Intuitionistic set theory and Regularity

Proof. Consider the sets $0 = \emptyset$, $1 = \{0\}$, and $2 = \{0, 1\}$, whose existence is guaranteed by the Axioms of Empty Set and Unordered Pairs, and use the Axiom of Separation to define the set $a = \{x \in 2 | (x = 0 \wedge \varphi) \vee x = 1\}$. Clearly, a is non-empty, since $1 \in a$. Hence, by the Axiom of Foundation, it has an $\in$-minimal element, i.e. $\exists y((y \in 2 \wedge ((y = 0 \wedge \varphi) \vee y = 1)) \wedge \neg\exists z(z \in a \wedge z \in y))$. If the $\in$-minimal element of a is 0, φ must hold. If, on the other hand, the $\in$-minimal element is 1, $\neg\exists z(z \in a \wedge z \in 1)$, and so $\neg(0 \in a \wedge 0 \in 1)$. Since $0 \in 1$ and $\neg(\varphi \wedge \psi) \wedge \varphi \rightarrow \neg\psi$ is intuitionistically valid, we can conclude $0 \notin a$. But, by definition of a, this is equivalent to $\neg\varphi$. Hence, LEM must hold. $\square$

Intuitionistic set theory and the substantial approach

- For this reason, in intuitionistic ZF, $\in$-induction is used instead than Regularity.
- Thus, the dependency theorist's strategy to get the hierarchy by establishing Regularity is problematic in an intuitionistic setting.

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- For this reason, in intuitionistic ZF, $\in$-induction is used instead than Regularity.
- Thus, the dependency theorist's strategy to get the hierarchy by establishing Regularity is problematic in an intuitionistic setting.
- Possible ways out:

1. Ditch (or weaken) some of the axioms of Z_0 so that the proof does not go through.
2. Try and argue directly for $\in$-induction.

Intuitionistic set theory and the substantial approach

- For obvious reasons, the option usually chosen is the second one.
- In particular, the second option is used in standard formulations of both intuitionistic ZF and constructive ZF (where the main difference between the two is that in the latter, Separation is restricted to bounded formulae).

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- For obvious reasons, the option usually chosen is the second one.
- In particular, the second option is used in standard formulations of both intuitionistic ZF and constructive ZF (where the main difference between the two is that in the latter, Separation is restricted to bounded formulae).
- But if the second option is chosen, the issue is what kind of argument the dependency theorist can offer for $\in$-induction.
- In particular, it is unclear whether the dependency theorist's focus on the dependence relation can provide the basis for such an argument.

Intuitionistic set theory and the minimalist approach

- Note, on the other hand, that the minimalist approach seems very well suited to offer the basis of an argument for $\in$-induction.
- For, as we pointed out earlier, on the minimalist account sets are the objects obtained by iterating the *set of* operation.

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- Note, on the other hand, that the minimalist approach seems very well suited to offer the basis of an argument for $\in$-induction.
- For, as we pointed out earlier, on the minimalist account sets are the objects obtained by iterating the *set of* operation.
- In other words: the minimalist offers an analogy between the way numbers are inductively generated from 0 by repeated applications of the successor operation and the way sets are generated by iterating the *set of* operation.
- One way to characterize this feature is via a suitable induction principle concerning sets.

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Summing up

- In the absence of Replacement, Regularity (even if transitive closure exist) is not enough to secure Groundedness.
- This is relevant for the debate concerning the best way to understand the iterative conception (also in connection with issues surrounding the logic of iterative set theory).

Summing up

- In the absence of Replacement, Regularity (even if transitive closure exist) is not enough to secure Groundedness.
- This is relevant for the debate concerning the best way to understand the iterative conception (also in connection with issues surrounding the logic of iterative set theory).
- There are set theories based on the idea of a set being depicted by a graph which include assumptions violating Groundedness.
- If one wants to pursue the minimalist approach, it becomes interesting to ascertain whether these theories embody a conception of set and how such a conception fares with respect to the iterative one: we'll make some initial steps in this direction in the next lecture.

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