

# An overview of predicativity

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Set Theory and Higher-Order Logic: Foundational and  
Mathematical Developments

*Birkbeck College, August 1-4, 2011*

# Two senses of 2nd models: a cautionary note

A language of second-order logic  $L_2$  is based on a first-order language  $L$ .  $L_2$  has second-order (unary) variables  $F, G, H$ , new atomic formulas of the form  $Ft$  (subsumption), where  $t$  is a (first-order) term, and second-order-quantifications  $\forall F, \exists F$ .

## Power set semantics:

Given  $M$  the domain of first-order variables, the second order (unary) variables range over  $\mathcal{P}(M)$ . Subsumption is interpreted as set membership.

The semantic consequence relation is not recursively enumerable.

## Henkin semantics:

In this semantics, the second order (unary) variables range over a given non-empty subset  $\mathcal{S}$  of  $\mathcal{P}(M)$ .

The semantic consequence relation is first-order in disguise. In particular, it is recursively enumerable (completeness theorem).

# A cautionary note, continued

Extend the original first-order fragment with two unary predicates (sorts): one  $U$  for first-order objects and the other  $S$  for sets of those elements. There is also a binary relation symbol  $E$  (for “membership”) such that:

- $\exists x U(x) \wedge \exists y S(y)$
- $\forall x (U(x) \vee S(x)) \wedge \neg \exists x (U(x) \wedge S(x))$
- for every constant  $c$  of  $L$ :  $U(c)$
- for every function symbol  $f$  of  $L$ :  $\forall x (U(x) \rightarrow U(f(x)))$
- $\forall x \forall y (E(x, y) \rightarrow U(x) \wedge S(y))$
- $\forall y, z (S(y) \wedge S(z) \wedge \forall x (E(x, y) \leftrightarrow E(x, z))) \rightarrow y = z$

Every structure  $\mathcal{N}$  of the extended first-order fragment that models the axioms above is isomorphic to a Henkin structure.

Given  $s \in \text{dom}(\mathcal{N})$  such that  $\mathcal{N} \models S(s)$ , define

$$[s] := \{x \in \text{dom}(\mathcal{N}) : \mathcal{N} \models E(x, s)\}$$

$$M := \{x \in \text{dom}(\mathcal{N}) : \mathcal{N} \models U(x)\}; \quad S := \{[s] : \mathcal{N} \models S(s)\}$$

# Frege's set theory

Frege's second-order set theory:

- first-order variables:  $x, y, z, \dots$
- second-order variables:  $F, G, H, \dots$
- equality sign '=' infixing between first-order terms
- value range VR operator:  $\phi(x) \rightsquigarrow \hat{x}.\phi(x)$

*Comprehension axiom.*

$$\exists F \forall x (Fx \leftrightarrow \phi(x))$$

*(Schematic) basic law V.*

$$\forall x (\phi(x) \leftrightarrow \psi(x)) \leftrightarrow \hat{x}.\phi(x) = \hat{x}.\psi(x)$$

Membership is *defined* between first-order objects:

## Definition

$$x \in y := \exists F (y = \hat{w}.Fw \wedge Fx).$$

# Russell's paradox

$$r \equiv \hat{w}.w \notin w$$

If  $r \in r$  then

$$\exists F(r = \hat{w}.Fw \wedge Fr)$$

$$\hat{w}.w \notin w = \hat{w}.F_0w \wedge F_0r$$

$$r \notin r$$

If  $r \notin r$  then

$$\forall F(r = \hat{w}.Fw \rightarrow \neg Fr)$$

Let  $Fw$  be  $w \notin w$ . Get,

$$r \in r$$

# Wherein lies the contradiction?

- In the extension operator and associated Basic Law V.

Note that the the  $\forall R$  operator is a procedure for type-lowering.  
Without it one should have variables of every finite type!

Get *Simple theory of types*

- In the impredicativity of the comprehension scheme.

Get *Heck's ramified second-order predicative theory*

- In both the extension operator and associated Basic Law V and the impredicativity of the comprehension scheme.

Get *Ramified theory of types*

# Digression: neologicism

*Frege arithmetic:*

Full comprehension. Cardinality operator:  $\phi(x) \rightsquigarrow Nx.\phi(x)$

(Schematic) Hume's principle:  $Nx.\phi(x) = Nx.\psi(x) \leftrightarrow \phi \approx_x \psi$

$0 := Nx.(x \neq x)$

$1 := Nx.(x = 0)$

$2 := Nx.(x = 0 \vee x = 1)$

...

$P(x, y) ::= \exists F \exists u (y = Nw.Fw \wedge Fu \wedge x = Nw.(Fw \wedge w \neq u))$

and an impredicative definition of natural number.

## Theorem

*Frege arithmetic is consistent.*

Get full second-order arithmetic (Frege's theorem).

$Nx.\phi(x) ::= \hat{z}.\exists F(z = \hat{w}.Fw \wedge F \approx_x \phi)$

(Wright:1983), (Heck:1999)

# Three impredicative definitions

- “Let  $Fw$  be  $w \notin w$ ”

$$\exists F \forall x (Fx \leftrightarrow x \notin x)$$

$$\exists F \forall x (Fx \leftrightarrow \forall G (x = \hat{w}.Gw \rightarrow \neg Gx))$$

- $\mathbb{N}x := \forall F (F0 \wedge \forall w (Fw \rightarrow F(Sw)) \rightarrow Fx)$

$$\mathbb{N}x := \forall F (F0 \wedge \forall w, u (Fw \wedge P(w, u) \rightarrow Fu) \rightarrow Fx)$$

- $\sup\{D \in \mathbb{R} : \Phi(D)\} := \{q \in \mathbb{Q} : \exists D \in \mathbb{R} (\Phi(D) \wedge q \in D)\}$

$\mathbb{N}2$ ?

Going through every single property...

$$\forall F (F0 \wedge \forall w (Fw \rightarrow F(Sw)) \rightarrow F2) \quad ?$$

$$\mathbb{N}0 \wedge \forall w (\mathbb{N}w \rightarrow \mathbb{N}(Sw)) \rightarrow \mathbb{N}2 \quad ?$$

(Carnap:1931)

# Two critiques of Poincaré

- *Vicious circle principle* (after Jules Richard)

Predicative comprehension:

$$\exists F \forall x (Fx \leftrightarrow \phi(x))$$

$\phi$  without second-order quantifications.

(Richard:1905), (Poincaré:1906)

- *Absoluteness* (after Jules Richard, again)

$$\forall x (\exists G \phi(x, G) \leftrightarrow \forall G \psi(x, G)) \rightarrow \exists F \forall x (Fx \leftrightarrow \exists G \phi(x, G))$$

$\phi$  and  $\psi$  without second-order quantifications. This is called  $\Delta_1^1$ -comprehension.

(Richard:1905), (Poincaré:1909), (Kreisel:1962),  
(Feferman:1964)

# Heck's predicative set theory (I)

Heck's system is like Frege's set theory but with predicative comprehension. In the comprehension scheme,  $\forall R$  terms must also not have bound second-order variables.

## Theorem

*Heck's predicative set theory is consistent.*

(Heck:1996)

## Theorem

*Frege's set theory restricted to  $\Delta_1^1$ -comprehension is consistent.*

(Ferreira-Wehmeier: 2002)

# Heck's predicative set theory (II)

## Proof.

Fix a denumerable infinite domain.

First, we define the denotations of first-order VR terms (together with an assignment of the free first-order variables). The rank of one such VR term is the maximum number of nested VR terms. Well-order these terms in a  $\omega^2$  sequence so that terms of smaller rank always appear before. It is easy to assign denotations to these VR terms so that Law V is met. We do this so that an infinite number of members of the domain are not denotations of these VR terms.

Second, define the second-order part of the model as the first-order (with first-order VR terms) definable sets. This determines the value ranges of VR terms containing free, but no bound, second-order variables. Law V is automatically met for these.

Third, well-order the impredicative value-range terms in a  $\omega^2$  sequence so that terms of smaller depth always appear before. The depth of VR term is the maximum number of nested impredicative VR terms. It is possible to assign denotations of these VR terms so that Law V is met using, when necessary, the vacant elements left by the assignments of the first-order VR terms.



# Problems of formalization

1.  $\mathbb{N}0$
2.  $\mathbb{N}x \wedge Pxy \rightarrow \mathbb{N}y$
3.  $\mathbb{N}x \wedge Pxy \wedge Pxz \rightarrow y = z$
4.  $\mathbb{N}x \wedge \mathbb{N}y \wedge Pxz \wedge Pyz \rightarrow x = y$
5.  $\mathbb{N}x \rightarrow \neg Px0$
6.  $\mathbb{N}x \rightarrow \exists yPxy$
7.  $\forall F[F0 \wedge \forall x, y(\mathbb{N}x \wedge Fx \wedge Pxy \rightarrow Fy) \rightarrow \forall x(\mathbb{N}x \rightarrow Fx)]$

There is a model of Frege's predicative *arithmetic* in which (6) is false.

$\Pi_1^1$ -comprehension is needed for (6) and to define sum and product with the usual recursive clauses.

(Linnebo:2004), (Walsh:ta)

# Non-Fregean moves

How about formalizing arithmetic in a non-Fregean way?

1.  $Sx \neq 0$
2.  $Sx = Sy \rightarrow x = y$
3.  $y \neq 0 \rightarrow \exists x(y = Sx)$
4.  $x + 0 = x$
5.  $x + Sy = S(x + y)$
6.  $x \cdot 0 = 0$
7.  $x \cdot Sy = (x \cdot y) + x$

Q is a very weak theory because it has no induction. It cannot prove that sum and product are commutative and associative or even that  $\forall x(Sx \neq x)$  or  $\forall x(0 + x = x)$ .

Q is an essentially undecidable theory.

# Some predicative arithmetic

Szmielew-Tarski set theory: axiom of extensionality, the existence of empty set and the existence of set adjunction (i.e., given  $x$  and  $y$ ,  $x \cup \{y\}$  exists).

Heck's predicative set theory interprets Szmielew-Tarski set theory.

Szmielew-Tarski set theory without extensionality interprets Robinson's arithmetic theory  $Q$ .

## Theorem

*Heck's predicative theory interprets  $Q$ .*

(Tarski-Mostowski-Robinson:1953), (Burgess:2005), (Heck:1996)

# Nelson's predicativism

We can define  $x < y$  as  $\exists z(x + Sz = y)$ .

A *bounded quantification* is a quantification of the form  $\forall x(x < t \rightarrow \dots)$  or  $\exists x(x < t \wedge \dots)$ .

A *bounded formula* is a formula which is built from atomic formulas using propositional connectives and bounded quantifications.

$I\Delta_0$  is the theory Q together with the scheme of induction restricted to bounded formulas  $\phi$ :

$$\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(Sx)) \rightarrow \forall x\phi(x)$$

Facts:

- The theory Q interprets  $I\Delta_0$ .
- Q does not interpret  $I\Delta_0(\text{exp})$ .
- There is a sentence of arithmetic such that Q interprets it and its negation.

(Nelson:1986), (Hájek-Pudlak:1993), (Burgess:2005)

# Digression on tameness

It is possible to interpret second-order theories in  $\mathbb{Q}$ .

A second-order theory BTFA related to polynomial time computability was shown to be interpretable in  $\mathbb{Q}$ .

In BTFA one can define the real numbers and prove that they form a field. One can also define continuous functions and prove the intermediate value theorem.

*Quintessential tame theory:* Tarski's theory of real closed ordered fields RCOF.

*Quintessential untame theory:* Robinson's  $\mathbb{Q}$ .

Fact. RCOF is interpretable in  $\mathbb{Q}$ , but not vice-versa.

Theories related to polynomial space computability can also be interpreted in  $\mathbb{Q}$ . Riemann integration can be developed in these.

(Tarski:1948), (Fernandes-Ferreira:2002), (Ferreira-Ferreira:2008),  
(Marker:2002)

# Ramified predicativity and Dedekind infinity

$$Sx \neq 0$$

$$Sx = Sy \rightarrow x = y$$

Ramified second-order (polyadic) variables:

(level 0)  $F^0, G^0, H^0, \dots$  (level 1)  $F^1, G^1, H^1, \dots$  (level 2)  $F^2, G^2, H^2, \dots$

Scheme of *ramified comprehension*:

$$\exists F^k \forall x (F^k x \leftrightarrow A(x))$$

where  $A$  contains no second-order bound variables of level greater than or equal to  $k$  and no second-order free variables of level greater than  $k$ .

RDA is the theory above.  $\text{RDA}_k$  is RDA restricted to levels less than  $k$ . Note that  $\text{RDA}_1$  is second-order predicative Dedekind arithmetic.

$$\mathbb{N}^{k+1}(x) := \forall F^k (F^k 0 \wedge \forall w (F^k w \rightarrow F^k (Sw)) \rightarrow F^k x)$$

# More predicative arithmetic

## Theorem

*In ramified predicative arithmetic, the 2-numbers form a model of  $I\Delta_0(\text{exp})$ .*

**Proof.** *k*-classes are given by monadic *k*-level second-order variables. A *k*-class is *inductive* if 0 is a member and is closed under successor.

We say that a binary relation  $F^0$  is a *computation of the sum with  $x$*  (of  $y$ , as  $z$ ) if

- $F^0$  is a function.
- 0 is in the domain of  $F^0$  and  $F^0 0 = x$
- if  $Sx$  is in the domain of  $F^0$ , then  $x$  is in the domain of  $F^0$  and  $F^0(Sx) = S(F^0 x)$
- $y$  is in the domain of  $F^0$  and  $F^0 y = z$ .

We say that the *sum of  $x$  with  $y$  is defined* if there is a computation of the sum with  $x$  of  $y$  and, moreover, that any two such computations always give the same result, denoted by  $x + y$ .

# More predicative arithmetic (continuation)

We say that  $y$  is *summable* if, for every  $x$ , the sum of  $x$  with  $y$  is defined.

## Lemma

*The 1-class of summable elements is inductive.*

We say that  $x$  is *additive* if  $x$  is summable and for all summable  $z$ ,  $z + x$  is summable, and moreover, for all  $w$ ,  $w + (z + x) = (w + z) + x$ .

## Lemma

*The 1-class of additive elements is inductive.*

## Lemma

*The 1-class of additive elements is closed under sum.*

Let  $x, y$  be additive. Clearly,  $x + y$  is summable. Given  $z$  summable,  $z + (x + y) = (z + x) + y$ , because  $y$  is additive. Moreover,  $(z + x) + y$  is summable since  $z + x$  is summable and (again)  $y$  is additive. Etc.

# More predicative arithmetic (continuation)

## Lemma

*Any inductive 1-class has contains an inductive 1-class closed under sum.*

Eventually,

## Lemma

*Any inductive 1-class has contains an inductive 1-class which is a model of  $I\Delta_0$ .*

But one only has:

## Lemma

*Any inductive 1-class contains an inductive 1-class which is a model of  $I\Delta_0$  and such that if  $x, y$  are given in the latter subclass then  $x^y$  is in the original given class.*



(Burgess-Hazen:1998), (Burgess:2005)

# Shoenfield's theorem

Given  $T$  a first-order theory, we define  $T^P$  the theory obtained from  $T$  by extending the language to (polyadic) second-order language, adding the predicative comprehension principle and replacing any (unrestricted) schemes of the original theory  $T$  by the corresponding single axioms.

If  $T$  is PA then  $T^P$  is  $ACA_0$ .

If  $T$  is ZF then  $T^P$  is BG.

## Theorem

$T^P$  is conservative over  $T$ .

**Model-theoretic proof.** Let  $\phi$  be a first-order sentence and suppose that  $T \not\vdash \phi$ . Take  $\mathcal{M}$  a model of  $T$  such that  $\mathcal{M} \models \neg\phi$ . Extend  $\mathcal{M}$  by a second-order  $Def(\mathcal{M})$  part constituted by the first-order definable subsets of the domain of  $\mathcal{M}$ . Clearly, this is a model of  $T^P$  in which  $\phi$  is false. Therefore,  $T^P \not\vdash \phi$ . □

# On cut-elimination (I)

In the Tait deductive calculus each relational symbol  $R$  has an associated opposite  $\bar{R}$ . Negation is *defined* using negation normal form where, in the atomic case, negation is given by the opposite.

$$\begin{array}{c} \Delta, R, \bar{R} \\ \frac{\Gamma_0, A_0 \quad \Gamma_1, A_1}{\Gamma_0, \Gamma_1, A_0 \wedge A_1} \\ \\ \frac{\Gamma, A}{\Gamma, A \vee B} \qquad \frac{\Gamma, B}{\Gamma, A \vee B} \\ \\ \frac{\Gamma, A(a)}{\Gamma, \forall x A(x)} \qquad \frac{\Gamma, A(t)}{\Gamma, \exists x A(x)} \\ \\ \text{(Cut Rule)} \quad \frac{\Gamma_0, C \quad \Gamma_1, \neg C}{\Gamma_0, \Gamma_1} \end{array}$$

where  $a$  is a eigenvariable.

## Theorem (Cut elimination)

*The cut rule is superfluous for the Tait calculus.*

# On cut-elimination (II)

## Theorem (Herbrand's theorem)

*If  $\exists x A(x)$  is provable in pure logic, where  $A$  is quantifier-free, then there are finitely many terms  $t_1, \dots, t_n$  such that  $A(t_1) \vee \dots \vee A(t_n)$  is provable in pure logic (it is a tautology).*

## Corollary

*Suppose that a universal theory  $T$  proves a quantifier-free sentence  $C$ . Then  $C$  is a propositional consequence of (finitely many) instances of the universal axioms.*

## Proof.

We have  $\forall x A(x) \rightarrow C$ , i.e.,  $\exists x (A(x) \rightarrow C)$ , provable in pure logic. By Herbrand's theorem, there are terms  $t_1, \dots, t_n$  such that  $A(t_1) \wedge \dots \wedge A(t_n) \rightarrow C$  is a tautology. □

(Tait:1968), (Schwichtenberg:1977)

# On cut elimination (III)

$$\frac{\Gamma, A(H)}{\Gamma, \forall F A(F)}$$

$$\frac{\Gamma, A(\{u : \phi(u)\})}{\Gamma, \exists F A(F)}$$

where  $H$  is an eigenvariable and  $\phi$  has no second-order quantifications. The set notation above (an abstract) is a meta-device. Note that the Tait predicative calculus proves predicative comprehension.

## Theorem (Predicative cut-elimination)

*The cut rule is superfluous for the Tait predicative calculus.*

## Theorem

*If  $\vdash_P \exists F A(F)$ , then there is an arithmetical formula  $\phi(u, z)$  such that  $\vdash_P \exists y A(\{u : \phi(u, y)\})$ .*

(Takeuti:1987)

# Predicative comprehension (proof-theoretically)

**Proof-theoretic proof.** Let  $Eq(F)$  be  $\forall F \forall x, y (x = y \wedge Fx \rightarrow Fy)$ . Let  $AXIOMS$  be the non-schematic axioms of  $T$  and  $\forall F S_1(F), \dots, \forall F S_n(F)$  be the single axioms replacing the schema of  $T$ . Suppose that  $T^P \vdash C$ , where  $C$  is a (first-order) sentence. Then,

$$\vdash_P AXIOMS \wedge \forall F S_1(F) \wedge \dots \wedge \forall F S_n(F) \rightarrow C$$

$$\vdash_P \exists F (AXIOMS \wedge S_1(F) \wedge \dots \wedge S_n(F) \rightarrow C)$$

$$\vdash_P \exists z (AXIOMS \wedge S_1(\{u : \phi(u, z)\}) \wedge \dots \wedge S_n(\{u : \phi(u, z)\}) \rightarrow C)$$

by cut-elimination (again),

$$\vdash \exists z (AXIOMS \wedge S_1(\{u : \phi(u, z)\}) \wedge \dots \wedge S_n(\{u : \phi(u, z)\}) \rightarrow C)$$

Therefore,  $T \vdash C$ . □

(Shoenfield:1954), (Takeuti:1987)

# $\Sigma_1^1$ axiom of choice

We say that a second-order model satisfies modified  $\Sigma_1^1$ -choice if, for every arithmetical formula (possibly with parameters), there is  $n$  such that:

$$\forall x \exists F A(F, x) \rightarrow \exists R \forall x \exists \bar{y} A(R_{x, \bar{y}}, x)$$

where  $\bar{y}$  is  $y_1, \dots, y_n$  and  $R_{x, \bar{y}}(u)$  stands for  $R(u, x, \bar{y})$ .

## Lemma

*Models of  $T^P$  satisfying modified  $\Sigma_1^1$ -choice also satisfy  $\Delta_1^1$ -comprehension.*

## Proof.

Suppose  $\forall x (\forall G B(G, x) \leftrightarrow \exists F A(F, x))$ . In particular,

$$\forall x \exists G, F (B(G, x) \rightarrow A(F, x))$$

$$\exists R, Q \forall x \exists y, z (B(R_{x, y}, x) \rightarrow A(Q_{x, z}, x))$$

Then,  $\exists F A(F, x)$  is equivalent to  $\exists z A(Q_{x, z}, x)$ . □

# $\Sigma_1^1$ axiom of choice: a model-theoretic proof (I)

Let  $T^D$  be like  $T^P$  but with  $\Delta_1^1$ -comprehension instead of predicative comprehension.

## Theorem

$T^D$  is conservative over  $T$ .

This follows from the fact that if  $\mathcal{M}$  is a *recursively saturated structure* than  $Def(\mathcal{M})$  is a model of modified  $\Sigma_1^1$ -choice.

A (first-order) structure is recursively saturated if every recursive type is realized. That is, for every recursive set of formulas  $\{\phi_i(x) : i \in \omega\}$  (with a fixed number of parameters) the following holds in  $\mathcal{M}$ :

$$\forall n \in \omega \exists x \bigwedge_{i \leq n} \phi_i(x) \rightarrow \exists x \bigwedge_{i \in \omega} \phi_i(x)$$

or, equivalently,

$$\forall x \bigvee_{i \in \omega} \phi_i(x) \rightarrow \exists n \in \omega \forall x \bigvee_{i \leq n} \phi_i(x)$$

## Theorem

*Every structure is elementarily equivalent to a recursively saturated structure.*

Now, suppose  $\mathcal{M}$  is recursively saturated and that the following holds in  $\text{Def}(\mathcal{M})$ :

$$\forall x \exists F A(x, F)$$

$$\forall x \bigvee_{i \in \omega} \exists \bar{y}_i A(x, \{u : \phi_i(u, x, \bar{y}_i)\})$$

By recursive saturation, there is  $n$  such that

$$\forall x \bigvee_{i \leq n} \exists \bar{y}_i A(x, \{u : \phi_i(u, x, \bar{y}_i)\})$$

If  $n = 0$ , put  $R(u, x, \bar{y}_0)$  as  $\phi_0(u, x, \bar{y}_0)$ . General case also holds. □

(Barwise-Schlipf:1975), (Ferreira-Wehmeier:2002), (Walsh:ta)

# $\Sigma_1^1$ axiom of choice: a proof-theoretic proof

This is a very rough sketch. We assume that the theory  $T$  has pairing.

Add to the Tait predicative calculus the following rule:

$$\frac{\Gamma, \exists F A(F, a)}{\Gamma, \exists R \forall x \exists y A(R_{x,y}, x)}$$

where  $A$  is arithmetical and  $a$  is an eigenvariable. This extended calculus proves the following strengthening of modified  $\Sigma_1^1$ -choice:  $\forall x \exists F A(F, x) \rightarrow \exists R \forall x \exists y A(R_{x,y}, x)$ . Hence, it proves  $\Delta_1^1$ -comprehension.

By a *partial cut-elimination* theorem, if the conclusion is of the form  $\exists F B(F)$ , with  $B$  arithmetical, then there is a proof in the extended calculus where each formula of the sequents has that form or the form  $\forall F B(F)$ .

In this situation, one can show that the new rule is superfluous.



# The limits of strict predicativity (I)

$$\begin{array}{ll} \text{superexp}(x, 0) = 1 & \text{superexp}(x, y + 1) = x^{\text{superexp}(x, y)} \\ \text{super}^2\text{exp}(x, 0) = 1 & \text{super}^2\text{exp}(x, y + 1) = \text{superexp}(x, \text{super}^2\text{exp}(x, y)) \end{array}$$

The cut-elimination theorem is provable in  $I\Delta_0(\text{superexp})$ .

- $I\Delta_0(\text{superexp}) \vdash \text{Con}_Q$
- $I\Delta_0(\text{superexp}) \vdash \text{Con}_{\text{RCOF}}$

The predicative cut-elimination theorem is provable in  $I\Delta_0(\text{super}^2\text{exp})$ .

- $I\Delta_0(\text{super}^2\text{exp}) \vdash \text{Con}_T \rightarrow \text{Con}_{\text{TP}}$  (perhaps even,  $\text{Con}_{\text{TD}}$ )
- $I\Delta_0(\text{super}^2\text{exp}) \vdash \text{Con}_{\text{RDA}_1}$
- $\text{RDA}_1$  does not interpret  $I\Delta_0(\text{super}^2\text{exp})$

Unclear claims:

- For each  $k$ ,  $I\Delta_0(\text{super}^2\text{exp}) \vdash \text{Con}_{\text{RDA}_k}$
- $\text{RDA}$  does not interpret  $I\Delta_0(\text{super}^2\text{exp})$

(Hájek-Pudlák:1993), (Buss:1998), (Burgess:2005)

# The limits of strict predicativity (II)

Variable-binding term-forming operators:  $\phi(x) \rightsquigarrow \hat{x}\phi(x)$       *versus*

functor operator:  $F \rightsquigarrow \ddagger F$ , with Law V:

$\forall F \forall G (\forall x (Fx \leftrightarrow Gx) \leftrightarrow \ddagger F = \ddagger G)$ .

PV is the counterpart of Heck's predicative set theory.

- $I\Delta_0(\text{super}^2\text{exp}) \vdash \text{Con}_{\text{PV}}$
- PV does not interpret  $I\Delta_0(\text{super}^2\text{exp})$

## Loose ends:

- Clarify the system in which the predicative cut-elimination can be proved. The same for the cut-elimination for ramified systems.
- Can *superexp* (or more) be developed predicatively?
- Prove finitistically the consistency of Heck's predicative set theory.
- Heck's predicative set theory can be ramified and it is a consistent theory (via a model theoretic argument). Prove finitistically the consistency of the ramified versions.
- Consider also  $\Delta_1^1$  comprehension. Open problems in both the variable-binding term-forming operators and functor operators.

(Heck:1996), (Wehmeier:1999), (Burgess:2005)

# Provable reducibility: first example

Axiom scheme of reducibility:  $\forall F^k \exists F^0 \forall x (F^k x \leftrightarrow F^0 x)$ .

Setting of ramified Frege arithmetic.

Hume's principle:  $Nx.\phi(x) = Nx.\psi(x) \leftrightarrow \exists R_0$  ("R<sub>0</sub> witnesses  $\phi \approx_x \psi$ ")

$P(x, y) := \exists F_0 \exists u (y = Nw.F_0 w \wedge F_0 u \wedge x = Nw.(F_0 w \wedge w \neq u))$

$y \leq x := \forall F_0 (F_0 y \wedge \forall z, w (F_0 z \wedge P(z, w) \rightarrow F_0 w) \rightarrow F_0 x)$

## Theorem (a restricted reducibility)

*Ramified Frege arithmetic proves*

$$\mathbb{N}_2(x) \rightarrow \exists F_0 \forall y (F_0 y \leftrightarrow y \leq x)$$

## Corollary

*Ramified Frege arithmetic proves  $\forall x (\mathbb{N}_2 x \rightarrow \exists y (\mathbb{N}_2 y \wedge Pxy))$ .*

## Theorem

*In ramified Frege arithmetic, the 4-numbers form a model of  $\text{I}\Delta_0(\text{exp})$ .*

(Heck: ta)

# In transition: finite reducibility

Primitive quantification over finite sets?

(Feferman-Hellman:1995, 1998), (Ferreira:1999), (Parsons:2008)

## An alternative.

Heck's ramified set theory with the scheme of reducibility collapses to Frege's set theory and, therefore, it is inconsistent. However:

### Theorem

*Heck's ramified set theory with the scheme of finite reducibility is consistent.*

$Dwo(R_0, F_0) := \forall H_0 (\emptyset \neq H_0 \subseteq F_0 \rightarrow \exists x Min(x, R_0|_{H_0}) \wedge \exists y Max(y, R_0|_{H_0}))$

$Fin(F_0) := \exists R_0 Dwo(R_0, F_0)$

*Axiom scheme of finite reducibility:*

$$\forall F_0 \forall H_k (Fin(F_0) \wedge H_k \subseteq F_0 \rightarrow \exists G_0 \forall x (G_0 x \leftrightarrow H_k x))$$

### Theorem

*Heck's ramified set theory with the scheme of finite reducibility interprets Peano arithmetic PA (in a Fregean way).*

(Ferreira:2005)

# Weyl's approach

“(the) house (of analysis) is to a large degree built on sand”

Hermann Weyl in preface to *Das Kontinuum* (1918)

According to Feferman, Weyl's system is essentially  $ACA_0$ . It is a predicative system *given the natural numbers*.

The least upper bound principle does not hold in this system, but every bounded sequence of natural numbers has a least upper bound:

$$\sup\{X_n : n \in \mathbb{N}\} = \{q \in \mathbb{Q} : \exists n \in \mathbb{N} (q \in X_n)\}$$

The following is provable in  $ACA_0$ :

- The Bolzano-Weierstrass theorem: Every bounded sequence of real numbers has a convergent subsequence.
- Every sequence of points in a compact metric space has a convergent subsequence.
- Every countable commutative ring has a maximal ideal.
- König's lemma: Every infinite, finitely branching tree, has an infinite path.

# Weyl's approach continued

- Every continuous real-valued function defined on the  $[0, 1]$  (or in any compact metric space) is uniformly continuous and has a maximum.
- Brouwer's fixed point theorem.
- Gödel's completeness theorem for countable languages.
- The Hahn-Banach theorem for separable normed spaces.
- The Banach-Steinhaus and open mapping theorems of functional analysis (for separable normed spaces).

Feferman's thesis: “all of *applicable classical and modern analysis* can be developed in  $(ACA_0)$ ” or “all scientifically applicable mathematics can be formalizable in  $(ACA_0)$ ”

(Feferman:1998), (Simpson:1999)

# A (tentative) proposal of Kreisel

Given  $\mathcal{S}$  a subset of  $\mathcal{P}(\mathbb{N})$ , we say that  $S \subseteq \mathbb{N}$  is in  $\mathcal{S}^*$  if there is a formula  $\phi(x)$  of the language of second-order arithmetic (possibly with parameters in  $\mathcal{S}$ ) such that  $S = \{n \in \mathbb{N} : (\mathbb{N}, \mathcal{S}) \models \phi(n)\}$ .

$$R_0 = \text{Arithm}, R_{\alpha+1} = (R_\alpha)^*, \text{ and for limit } \lambda, R_\lambda = \bigcup_{\alpha < \lambda} R_\alpha$$

*Kreisel's proposal:*

The predicatively definable sets, given the set of natural numbers, are the members of  $R_{\omega_1^{CK}}$ . ( $\omega_1^{CK}$  is the first non-recursive ordinal.)

Bootstrap condition: an ordinal should be considered predicatively definable if, and only if, it is isomorphic to a well-ordering of  $\omega$  which is in some  $R_\alpha$ , with  $\alpha$  predicatively definable.

## Theorem (Spector)

*Every well-ordering in  $R_{\omega_1^{CK}}$  is isomorphic to a recursive ordinal.*

(Spector:1955), (Kreisel:1958), (Sacks:1990), (Feferman:2007)

# Provable reducibility: second example

The cumulative hierarchy:

$$V_0 = \emptyset, V_{\alpha+1} = \mathcal{P}(V_\alpha), \text{ and for limit } \lambda, V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$$

Given  $S$  a set, we say that a subset  $X$  of  $S$  is in  $\mathcal{P}_{\text{Df}}(S)$  if there is a formula of set theory  $\phi(x)$  (possibly with parameters from  $S$ ) such that  $X = \{a \in S : (S, \in) \models \phi(a)\}$ .

Gödel's constructible hierarchy:

$$L_0 = \emptyset, L_{\alpha+1} = \mathcal{P}_{\text{Df}}(L_\alpha), \text{ and for limit } \lambda, L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha$$

## Theorem

*Zermelo-Fraenkel set theory ZF proves*

$$\forall x \in L (x \subseteq \omega \rightarrow x \in L_{\omega_1^L})$$

(Gödel:1944), (Kunen:1980)

# Some recursion theory

- Given  $X, Y \subseteq \omega$ , we say that  $X$  is many-one reducible to  $Y$  if there is a recursive function  $f$  such that  $x \in X \leftrightarrow f(x) \in Y$ , for all  $x \in \omega$ .
- $T(e, x, z)$ : the Turing machine with Gödel number  $e$  has a halting computation  $z$  when its input is  $x$ . In this case,  $U(z)$  is the output of the halting computation. Both  $T$  and  $U$  are primitive recursive predicates.
- $\{e\}(x) \downarrow$ : the Turing machine with Gödel number  $e$  halts with input  $x$ .
- $W_e := \{x \in \omega : \{e\}(x) \downarrow\}$  ( $e$  is an index for the r.e. set  $W_e$ )
- $R_e := \{(x, y) \in \omega^2 : \{e\}(x, y) \downarrow\}$  ( $e$  is an index for the r.e. relation  $R_e$ )
- $\{e\}(x) = y$ : the Turing machine with Gödel number  $e$  halts with input  $x$  and outputs  $y$ .
- $\{e\}^X(x) \downarrow$  and  $\{e\}^X(x) = y$ : relativizations to an oracle  $X$ .
- The Turing jump of  $X$ :  $X' := \{e \in \omega : \{e_0\}^X(e_1) \downarrow\}$

# A normal form for $\Pi_1^1$ formulas

A set  $S \subseteq \omega$  is  $\Pi_1^1$  if it is definable (in the power model) by a second-order formula of the form  $\forall X A(x, X)$ , where  $A$  is an arithmetical predicate.

By skolemization, every arithmetical predicate  $A(x, X)$  is of the form

$$\forall f \in \omega^\omega \exists y B'(x, y, X, f)$$

for some bounded formula  $B'$ . Only finitely many values of  $X$  and  $f$  are required to decide  $B'(x, y, X, f)$ . In the end (collapse quantifiers), one can find a primitive recursive relation  $B$  such that

$$x \in S \leftrightarrow \forall f \in \omega^\omega \exists y B(x, \bar{f}(y))$$

where  $\bar{f}(y)$  denotes the finite sequence  $\langle f(0), f(1), \dots, f(y-1) \rangle$ . Let

$$T_S(x) := \{ \sigma \in \omega^{<\omega} : \forall \tau (\sigma \preceq \tau \rightarrow \neg B(x, \tau)) \}$$

where  $\sigma \prec \tau$  means that  $\sigma$  strictly extends  $\tau$  as a sequence.

$$x \in S \leftrightarrow T_S(x) \text{ is a well-founded tree}$$

# A $\Pi_1^1$ universal formula

## Theorem

*There is a  $\Pi_1^1$  formula  $U(z, x)$  such that, for every  $\Pi_1^1$  set  $S$ , there is  $e \in \omega$  such that, for all  $x \in \omega$ ,  $x \in S \leftrightarrow U(e, x)$ .*

## Proof.

We know that  $x \in S \leftrightarrow \forall f \in \omega^\omega \exists y B(x, \bar{f}(y)) \leftrightarrow \forall f \in \omega^\omega \exists y \{e\}(x, \bar{f}(y)) \downarrow$ , where  $e$  depends on  $B$  (hence, on  $S$ ). Put

$$U(e, x) := \forall f \in \omega^\omega \exists y \{e\}(x, \bar{f}(y)) \downarrow. \quad \square$$

## Corollary

*There is a  $\Pi_1^1$  set which is not  $\Sigma_1^1$ .*

## Proof.

Consider the set  $V := \{x \in \omega : U(x, x)\}$ . Its complement cannot be  $\Pi_1^1$ . If it were, there would be  $e \in \omega$  such that, for every  $x \in \omega$ ,  $\neg U(x, x) \leftrightarrow U(e, x)$ . This gives a contradiction for  $x = e$ . □

# Kleene's $\mathcal{O}$ and the hyperarithmetic sets (I)

Consider the least subset of  $\omega \times \omega$  which has the following closure properties

- if  $(x, y) \in X$  then  $(x, 2^y) \in X$
- if, for every  $x \in \omega$ ,  $\{e\}(x) \downarrow$  and  $(\{e\}(x), \{e\}(x+1)) \in X$ , then for every  $x \in \omega$   $(\{e\}(x), 3 \cdot 5^e) \in X$
- if  $(x, y), (y, z) \in X$  then  $(x, z) \in X$

and contains the pair  $(1, 2)$ .

We write  $x <_{\mathcal{O}} y$  to say that the pair  $(x, y)$  is in this least set. Note that one expresses that  $X$  is closed under the above three clauses via an arithmetical formula  $A(X)$ . Therefore the above least set (of ordered pairs) is  $\Pi_1^1$ :

$$x <_{\mathcal{O}} y \equiv \forall X (A(X) \wedge (1, 2) \in X \rightarrow (x, y) \in X)$$

The field of  $<_{\mathcal{O}}$  is Kleene's  $\mathcal{O}$ , the set of notations for *constructive ordinals*.  $\mathcal{O}$  is a  $\Pi_1^1$  set. The ordering  $x <_{\mathcal{O}} y$  is well-founded but not linear.

# Kleene's $\mathcal{O}$ and the hyperarithmetical sets (II)

Given  $x \in \mathcal{O}$ , let  $|x|$  the order type of  $x$  in the well-order  $<_{\mathcal{O}}$ . We say that  $x$  is a notation for (the constructive) ordinal  $|x|$ . Note that:

- 1 is the notation for the ordinal 0
- 2 is the notation for the ordinal 1
- if  $x$  is a notation for  $\alpha$  then  $2^x$  is a notation for  $\alpha + 1$
- if, for each  $x \in \omega$ ,  $\{e\}(x)$  is a notation for  $\alpha_x$ , then  $3 \cdot 5^e$  is a notation for  $\sup_x \alpha_x$

## Theorem (Kleene, Markwald)

*Every constructive ordinal is a recursive ordinal.*

## Proof.

Let  $\alpha$  be a constructive ordinal. W.l.o.g.,  $\alpha$  is infinite. Let  $\alpha = |x|$ , for some  $x \in \mathcal{O}$ . It can be proved that the restriction of the ordering  $<_{\mathcal{O}}$  to the set of elements less than  $x$  is a *linear* r.e. relation  $R$ . The field of  $R$  is, of course, an infinite r.e. set  $W$ . Then there is a one-one recursive function  $f$  that maps  $\omega$  onto  $W$ . Define,  $x < y \equiv (f(x), f(y)) \in R$ .

$<$  is a recursive relation of order type  $\alpha$ .



# Kleene's $\mathcal{O}$ and the hyperarithmetic sets (III)

## Theorem (Kleene)

Every  $\Pi_1^1$  set is many-one reducible to  $\mathcal{O}$ .

## Proof.

Let  $S \in \Pi_1^1$ . Then, for all  $x \in \omega$ ,  $x \in S$  if, and only if,  $T_S(x)$  is a well-founded tree. This tree is r.e. (in fact, it is recursive). It is possible to define a total recursive function  $f$  such that, if  $e$  is the index of a r.e. binary relation, then  $R_e$  is well-founded if, and only if,  $f(e) \in \mathcal{O}$ . (Moreover, if  $R_e$  is well-founded,  $|R_e| \leq |f(e)|$ .) Therefore,

$$x \in S \leftrightarrow f(t(x)) \in \mathcal{O}$$

where  $t_S$  is a recursive function such that, for every  $x$ ,  $t_S(x)$  is the index of the relation associated with the tree  $T_S(x)$ .  $\square$

## Corollary

$\mathcal{O} \notin \Sigma_1^1$ .

# Kleene's $\mathcal{O}$ and the hyperarithmetic sets (IV)

## Corollary (Spector)

Suppose  $X \subseteq \mathcal{O}$  and  $X \in \Sigma_1^1$ . Then there is  $b \in \mathcal{O}$  such that, for every  $x \in X$ ,  $|x| < |b|$ . ( $\Sigma_1^1$  boundedness.)

## Proof.

Suppose not. We see that  $\mathcal{O}$  has a  $\Sigma_1^1$ -definition.

$$x \in \mathcal{O} \leftrightarrow T_{\mathcal{O}}(x) \text{ is well-founded} \leftrightarrow R_{T_{\mathcal{O}}(x)} \text{ is well-founded}$$

$$\leftrightarrow \exists b \in X (R_{T_{\mathcal{O}}(x)} \text{ embeds into the r.e. set } \{z : z <_{\mathcal{O}} b\})$$

This is a  $\Sigma_1^1$  predicate. □

The  $H$ -sets are subsets of  $\omega$  defined by recursion on  $<_{\mathcal{O}}$ :

- $H_0 = \emptyset$
- $H_{2^x} = (H_x)'$
- $H_{3 \cdot 5^e} = \{2^k 3^n : k \in H_{\{e\}(n)}\}$

# Kleene's $\mathcal{O}$ and the hyperarithmetical sets (V)

## Definition

A set  $X \subseteq \omega$  is *hyperarithmetical* if it is recursive in some  $H$ -set.

## Theorem (Kleene)

Given  $X \subseteq \omega$ , the following are equivalent:

- $X$  is hyperarithmetical.
- $X$  is  $\Delta_1^1$ .
- $X$  is in  $R_{\omega_1^{CK}}$ .

## Proof.

We only prove that  $\Delta_1^1$ -sets are hyperarithmetical.

Let  $X$  be  $\Delta_1^1$ . There is a recursive  $g$  such that  $x \in X \leftrightarrow g(x) \in \mathcal{O}$ .

$\{z \in \omega : \exists x \in X (g(x) = z)\}$  is a  $\Sigma_1^1$  subset of  $\mathcal{O}$ . Therefore, it is bounded by some  $b \in \mathcal{O}$ . We get,  $x \in X \leftrightarrow |g(x)| < |b|$ .

It can be show that the condition “ $|g(x)| < |b|$ ” is hyperarithmetical.  $\square$

# Another proposal of Kreisel

We can now prove that every well-ordering  $\prec$  in  $R_{\omega_1^{CK}}$  is isomorphic to a recursive ordinal. Take  $\prec$  in  $R_{\omega_1^{CK}}$ . Assume that its order type is not smaller than  $\omega_1^{CK}$ . Then,

$x \in \mathcal{O} \leftrightarrow R_{t_{\mathcal{O}}(x)}$  is well-founded  $\leftrightarrow \exists f (f : R_{t_{\mathcal{O}}(x)} \mapsto \prec \text{ is order preserving})$

Note that this is a  $\Sigma_1^1$  definition (because “membership in  $\prec$ ” is  $\Delta_1^1$ ).

Basic question: are the recursive ordinals indeed predicative?

Note that the notion of well-order is *impredicative*.

Should one not demand, for a recursive ordinal to be counted as predicatively obtained, that it be predicatively recognized as a well-ordering?

(Kreisel:1960)

# Ramified Analysis

Idea: Transfinite progression of semi-formal systems  $RA_\alpha$ , in which an ordinal  $\alpha$  is to be accepted as the index for a system if an well-ordering of that type has been proved in a previous system.

The languages  $L_\alpha$  of the systems  $RA_\alpha$  are based on the first-order language of PA and, for each  $\beta \leq \alpha$ , has denumerable many set variables  $X^\beta, Y^\beta, Z^\beta, \dots$

- Quantifier-free axioms for the primitive recursive equations.
- (Ramified comprehension axioms)

$$\exists X^\beta \forall x (x \in X^\beta \leftrightarrow A(x))$$

where  $A(x)$  is a formula of  $L_\alpha$  with bound variables of level less than  $\beta$  and set parameters of level less than or equal to  $\beta$ .

- ( $\omega$ -rule) From  $A(0), A(1), A(2), \dots$  conclude  $\forall x A(x)$ .
- (Limit generalization) For each limit ordinal  $\lambda < \alpha$  and each formula  $A(X^\lambda)$  with only  $X^\lambda$  free, infer  $A(X^\lambda)$  from  $A(X^0), A(X^1), \dots, A(X^\beta), \dots$  for all  $\beta < \lambda$ .

Note that  $R_\alpha$  is a natural model of  $RA_\alpha$ .

# Autonomy

Let  $\prec$  be a sufficiently long (to be determined) primitive recursive well-ordering of  $\omega$ . Let:

$$WO(\prec, z) := \forall X^0 (\forall y (\forall x \prec y (x \in X^0) \rightarrow y \in X^0) \rightarrow z \in X^0)$$

## Definition

We say that an ordinal  $\alpha$  is autonomous (with respect to ramified analysis) if it is in the smallest class  $A$  of ordinals containing all ordinals less than  $\omega^2$  and such that

- if  $D$  is a (infinitary) derivation of  $WO(\prec, n)$  in  $RA_\alpha$  with  $\alpha, |D| \in A$  then  $|n|_\prec \in A$ .

Here  $|D|$  is the height of the derivation tree  $D$ . The statement of well-ordering is not as special as it seems, because of a “lifting” argument.

## Theorem

*The autonomous ordinals are exactly the ordinals less than the Feferman-Schütte ordinal  $\Gamma_0$ .*

# The Veblen hierarchy

Let  $X \subseteq \omega_1$ .  $X$  is *unbounded* if  $\forall \alpha < \omega_1 \exists \beta \in X (\alpha < \beta)$ .  $X$  is *closed* if for all countable  $S \subseteq X$ ,  $\sup S \in X$ .  $X$  is a *club* if it is closed and unbounded.

## Theorem

Let  $X \subseteq \omega_1$  be a club. There is a unique increasing onto function  $en_X : \omega_1 \rightarrow X$ . The derived set  $X' := \{\alpha < \omega_1 : en_X(\alpha) = \alpha\}$  is a club.

## Proof.

Let  $en_X(0)$  be the first ordinal in  $X$ , let  $en_X(\alpha + 1)$  be the first ordinal in  $X$  after  $en_X(\alpha)$  and, for limit  $\lambda$ , let  $en_X(\lambda) = \sup_{\alpha < \lambda} en_X(\alpha)$ . We only show that the derived set is unbounded.

Take  $\alpha < \omega_1$ . W.l.o.g.,  $\alpha \in X$ . Let  $\beta := \sup\{\alpha, en_X(\alpha), en_X(en_X(\alpha)), \dots\}$ . It is easy to show that  $en_X(\beta) = \beta$ .  $\square$

- $Cr(0) = \{\omega^\beta : \beta < \omega_1\}$
- $Cr(\alpha + 1) = Cr(\alpha)'$
- $Cr(\lambda) = \bigcap_{\alpha < \lambda} Cr(\alpha)$

# The Feferman-Schütte ordinal $\Gamma_0$

## Definition

$$\varphi_\alpha := \text{en}_{Cr(\alpha)}.$$

$$\varphi_0(\alpha) = \omega^\alpha$$

$$\varphi_1(0) = \varepsilon_0, \text{ the least solution to the equation } \omega^\alpha = \alpha: \omega^{\omega^{\omega^{\omega^{\dots}}}}$$

$$\varphi_1(\alpha) = \varepsilon_\alpha$$

$$\varphi_2(0) = \varepsilon_{\varepsilon_{\varepsilon_{\varepsilon_{\dots}}}}, \text{ the least solution to the equation } \varepsilon_\alpha = \alpha.$$

## Theorem

*The set  $\{\alpha < \omega_1 : \alpha \in Cr(\alpha)\}$  is a club.*

## Proof.

Given a family of clubs  $(X_\alpha)_{\alpha < \omega_1}$ , it is a result that its diagonal intersection  $\Delta_{\alpha < \omega_1} X_\alpha := \{\beta < \omega_1 : \beta \in \bigcap_{\alpha < \beta} X_\alpha\}$  is still a club.

The set in the theorem is the diagonal intersection of  $(Cr(\alpha))_{\alpha < \omega_1}$ .  $\square$

The ordinal  $\Gamma_0$  is the least  $\alpha$  such that  $\alpha \in Cr(\alpha)$ . It is the least ordinal closed under the binary Veblen function  $\varphi$ .

(Pohlers:2009)

# (Rough) ideas of proof

In general, cut-elimination theorem does not hold for theories (it holds for pure logic). It is possible to see induction as a “logical principle” if one allows the  $\omega$ -rule.

Suppose that one has  $A(0)$  and  $\forall x(A(x) \rightarrow A(x + 1))$ . For each  $n \in \omega$ ,  $A(n) \rightarrow A(n + 1)$  is a consequence. Using modus ponens (cut) once, we get  $A(1)$ . Using twice, we get  $A(2)$ . Thrice,  $A(3)$ . Etc. By the  $\omega$ -rule, we can conclude  $\forall x A(x)$ .

Note that this proof figure can be seen as a tree of height  $\omega$ .

## Ideas:

- Go to an infinitary calculus where induction can be deduced.
- Prove a cut-elimination theorem for the infinitary calculus.
- Bound the order type of a proof of well-ordering by the height of a cut free proof of it.
- The predicative systems can be embedded in (suitable) infinitary calculi.

# Tait's infinitary propositional calculus: language

In Tait's infinitary propositional calculus there is a denumerable infinite stock of propositional letters,  $p, q, r, \dots$ . These letters came in pairs: opposite propositional letters  $p$  and  $\bar{p}$ .

The propositional formulas are built from propositional letters by means of denumerable (finite or infinite) conjunctions and disjunctions:

$$\bigwedge_{i \in I} A_i \quad \text{and} \quad \bigvee_{i \in I} A_i$$

where  $I$  is denumerable finite or infinite.

Negation is *defined* via negation normal form.

The *rank* of a propositional letter is 0. The rank of a conjunction (disjunction) as above is the least upper bound of the successor of the ranks of the conjuncts (disjuncts).

If we consider the closed atomic formulas of the language of first-order PA as the propositional letters of an infinitary propositional calculus, first-order formulas “translate” naturally into propositional formulas of finite rank.

# Tait's infinitary propositional calculus: proofs

Derivations concern finite sets  $\Gamma, \Delta, \dots$  of propositional formulas (interpreted disjunctively).

A collection  $S$  of finite sets of atoms is called an *axiom system* if it has the following property:

*Intersection property:* If  $\Delta, p$  and  $\Gamma, \bar{p}$  are in  $S$  then so is some finite subset of  $\Delta \cup \Gamma$ .

Usually one requires that unordered pairs sets  $\{p, \bar{p}\}$  are always in the axiom system.

## Definition

We inductively define  $\left| \frac{\alpha}{\rho} \Delta \right.$ , for countable ordinals  $\alpha$  and  $\rho$  and finite sets of propositional formulas  $\Delta$ :

- if  $\Delta \in S$ , then  $\left| \frac{\alpha}{\rho} \Gamma, \Delta \right.$  for any finite set of propositional formulas  $\Gamma$  and any countable ordinals  $\alpha$  and  $\rho$ .
- if for every  $k \in I$ ,  $\left| \frac{\alpha_k}{\rho} \Delta, A_k \right.$  and  $\alpha_k < \alpha$ , then  $\left| \frac{\alpha}{\rho} \Delta, \bigwedge_{i \in I} A_i \right.$ .
- if for some  $k \in I$ ,  $\left| \frac{\alpha_k}{\rho} \Delta, A_k \right.$  and  $\alpha_k < \alpha$ , then  $\left| \frac{\alpha}{\rho} \Delta, \bigvee_{i \in I} A_i \right.$ .
- if  $\left| \frac{\beta}{\rho} \Delta, C \right.$  and  $\left| \frac{\beta}{\rho} \Delta, \neg C \right.$  and  $rk(C) < \rho$  and  $\beta < \alpha$  then  $\left| \frac{\alpha}{\rho} \Delta \right.$ .

# Predicative cut-elimination and the stage theorem

## Theorem (Predicative cut elimination)

If  $\frac{\alpha}{\beta + \omega^\rho} \Delta$  then  $\frac{\varphi_\rho(\alpha)}{\beta} \Delta$ .

Note that when  $\rho = 0$  the conclusion is  $\frac{\omega^\alpha}{\beta} \Delta$ . In particular, if  $\frac{\alpha}{n} \Delta$  and  $\alpha < \varepsilon_0$  then  $\frac{\varepsilon_0}{0} \Delta$ .

Suppose  $p_0, p_1, p_2, \dots, \bar{p}_0, \bar{p}_1, \bar{p}_2, \dots$  are all distinct propositional letters and that, for each  $n \in \omega$ , the only axiom containing either  $p_n$  or  $\bar{p}_n$  is  $\{p_n, \bar{p}_n\}$ . Let  $\prec$  be a well-ordering of  $\omega$ . Define:

$$Prog(\prec) := \bigwedge_n \left( \left( \bigvee_{k \prec n} \bar{p}_k \right) \vee p_n \right) \quad \text{and} \quad WO(\prec, n) := \left\{ \neg Prog(\prec), \bigwedge_{m \prec n} p_m \right\}$$

## Theorem (Stage theorem)

If the empty set is not an axiom and  $\frac{\beta}{0} WO(\prec, n)$  and  $\beta < \varepsilon_\alpha$  then  $|n|_\prec < \varepsilon_\alpha$ .

(Tait:1968), (Schütte:1977), (Pohlers:2009)

# Unramified theories (I)

Language of analysis: based on the language of PA and with only one type of second-order variables.

- The *Hierarchy Axiom*:

$$\forall Z(WO(Z) \rightarrow \forall X\exists Y H(Z, X, Y))$$

where  $WO(Z)$  is the  $\Pi_1^1$  formula expressing that  $Z$  is a well-ordering of the natural numbers, and  $H(X, Y, Z)$  is an arithmetical formula expressing that  $Y$  is the Turing jump hierarchy along  $Z$  starting on  $X$ .

- The *Hierarchy Rule*:

$$\frac{WO(\prec)}{\forall X\exists Y H(\prec, X, Y)}$$

where  $\prec$  is a primitive recursive predicate.

- The *Bar Rule*:

$$\frac{WO(\prec)}{\forall y(\forall x \prec y A(x) \rightarrow A(y)) \rightarrow \forall x A(x)}$$

where  $A(x)$  is an arbitrary formula.

# Unramified theories (II)

The  $\Delta_1^1$ -comprehension Rule:

$$\frac{\forall x(\exists X A(x, X) \leftrightarrow \forall X B(x, X))}{\exists Z \forall x(x \in Z \leftrightarrow \exists X A(x, X))}$$

where  $A$  and  $B$  are arithmetical formulas.

## Definition (Feferman)

The theory  $\text{IR}$  is  $\text{ACA}_0$  together with the  $\Delta_1^1$ -comprehension rule, the hierarchy rule and the bar rule.

$\text{IR}$  has a (somewhat) natural interpretation in  $\text{RA}_{\Gamma_0}$  and, hence, it is (locally) predicatively justified.

## Definition (H. Friedman)

The theory  $\text{ATR}_0$  is  $\text{ACA}_0$  together with the hierarchy axiom.

The theory  $\text{ATR}_0$  proves the same  $\Pi_1^1$ -sentences as  $\text{IR}$  and, hence, can be considered predicatively reducible.  $\text{ATR}_0$  proves the  $\Delta_1^1$ -comprehension *axiom*. Note, however, that  $\text{ATR}_0$  has only restricted induction.

(Buchholz-Feferman-Pohlers-Sieg:1981),

# A predicative logic

Second-order propositional logic.

- Propositional constants and variables are formulas.
- If  $A$  and  $B$  are formulae, then  $(A \rightarrow B)$  is a formula.
- If  $A$  is a formula and  $F$  is a propositional variable, then  $\forall F(A)$  (or  $\forall F.A$ ) is a formula.

Prawitz's definitions:

$$\begin{aligned}\neg A &=_{df} A \rightarrow \forall F.F \\ A \wedge B &=_{df} \forall F((A \rightarrow (B \rightarrow F)) \rightarrow F) \\ A \vee B &=_{df} \forall F((A \rightarrow F) \rightarrow ((B \rightarrow F) \rightarrow F)) \\ \exists G.A &=_{df} \forall F(\forall G(A \rightarrow F) \rightarrow F).\end{aligned}$$

(Russell:1903), (Prawitz:1965)

# A predicative logic, continued

Rules for the natural deduction calculus:

$$\frac{\begin{array}{c} [A] \\ \vdots \\ B \end{array}}{A \rightarrow B} \rightarrow I \qquad \frac{\begin{array}{c} \vdots \\ A \end{array}}{\forall F.A} \forall I$$

where, in the second rule,  $F$  does not occur free in any undischarged hypothesis. The elimination rules are

$$\frac{\begin{array}{c} \vdots \\ A \rightarrow B \end{array} \quad \begin{array}{c} \vdots \\ A \end{array}}{B} \rightarrow E \qquad \frac{\begin{array}{c} \vdots \\ \forall F.A \end{array}}{A[C/F]} \forall E$$

with  $C$  an arbitrary formula, free for  $F$  in  $A$ .

This is an intuitionistic *impredicative* theory. It is the deductive side of Girard's polymorphic  $\lambda$ -calculus.

(Girard-Lafont-Taylor:1989)

# Provable reducibility: third example

Let  $F$  be the second-order propositional system described. Consider  $F_{\text{at}}$  the restriction of  $F$  in which  $C$  is atomic in the  $\forall E$ -rule. Of course,  $F_{\text{at}}$  is predicative.

## Theorem

$F_{\text{at}}$  proves the usual introduction and elimination rules of the natural deduction calculus for the connectives as defined by Prawitz.

## Proof.

It suffices to show that, for *any* formula  $C$  of the second-order propositional language, one has:

- $\neg A \rightarrow (A \rightarrow C)$
- $A \wedge B \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow C)$
- $A \vee B \rightarrow ((A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow C))$
- $\exists G.A \rightarrow (\forall G(A \rightarrow C) \rightarrow C)$

This can be argued by induction on the complexity of  $C$ . □

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