The separation of powers:

Weak subsystems of ZF *in the service of philosophy*

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Weaknesses of Zermelo set theory

André Weil and Jean Dieudonné thought that ZC sufficed as a basis for mathematics; and Saunders Mac Lane favoured its subsystem MAC. But note the following:

ZC cannot prove that every set has a rank

ZC cannot prove that every set has a transitive closure

ZC cannot prove that the class of hereditarily finite sets is a set

Z cannot conveniently handle Gödel's concept of constructibility

ZC cannot do Shoenfield-style forcing

ZC does not support certain natural algebraic constructions

ZC cannot prove the determinacy of Borel games

Fix a limit ordinal λ.

Let $\mathbf{M}_{13,\lambda}$ be the union of all transitive sets u with $u \cap \lambda < \lambda$.

Then $\mathbf{M}_{13,\lambda}$ is a supertransitive model of Z containing sets of all ranks but only the ordinals less than λ.

Three hierarchies

$$V_0 = \varnothing \qquad L_0 = \varnothing \qquad T_0 = \varnothing$$

$$V_{\nu+1} = \mathcal{P}(V_\nu) \qquad L_{\nu+1} = \mathrm{Def}(L_\nu) \qquad T_{\nu+1} = \mathbb{T}(T_\nu)$$

$$V_\lambda = \bigcup_{\nu<\lambda} V_\nu \qquad L_\lambda = \bigcup_{\nu<\lambda} L_\nu \qquad T_\lambda = \bigcup_{\nu<\lambda} T_\nu$$

$$V = \bigcup_{\nu} V_\nu \qquad L = \bigcup_{\nu} L_\nu \qquad L = \bigcup_{\nu} T_\nu$$

$$J_\nu = T_{\omega\nu}$$

$$
\begin{aligned}
\mathbb{T}(u) \;=_{\mathrm{df}}\; & u \cup \{u\} \cup [u]^1 \cup [u]^2 \\
& \cup \{x \smallsetminus y \mid_{x,y} x, y \in u\} \\
& \cup \left\{\bigcup x \,\middle|_x\, x \in u\right\} \\
& \cup \left\{\mathrm{Dom}\,(x) \,\middle|_x\, x \in u\right\} \\
& \cup \{u \cap (x \times y) \mid_{x,y} x, y \in u\} \\
& \cup \left\{x \cap \{(a,b)_2 \mid_{a,b} a \in b\} \,\middle|_x\, x \in u\right\} \\
& \cup \left\{u \cap \{(b,a,c)_3 \mid_{a,b,c} (a,b,c)_3 \in x\} \,\middle|_x\, x \in u\right\} \\
& \cup \left\{u \cap \{(b,c,a)_3 \mid_{a,b,c} (a,b,c)_3 \in x\} \,\middle|_x\, x \in u\right\} \\
& \cup \left\{x``\{w\} \,\middle|_{x,w}\, x \in u, w \in u\right\} \\
& \cup \left\{u \cap \left\{x``\{w\} \,\middle|_w\, w \in y\right\} \,\middle|_{x,y}\, x, y \in u\right\}.
\end{aligned}
$$

PROPOSITION *$u \subseteq \mathbb{T}(u)$ and $u \in \mathbb{T}(u)$. Further, if u is transitive, then $\mathbb{T}(u)$ is a set of subsets of u, and hence $\mathbb{T}(u)$ is transitive.*

REMARK In general $u \subseteq v \not\Longrightarrow \mathbb{T}(u) \subseteq \mathbb{T}(v)$; but always

$$u \subseteq \mathbb{T}(u) \subseteq \mathbb{T}^2(u) \ldots.$$

A slow boot to GJ

We formulate set theory with

two undefined binary relations $\in, =$;

propositional connectives $\neg, \ \mathsf{V}\ , \ \&\ , \Longrightarrow, \Longleftrightarrow$;

unrestricted quantifiers $\forall x, \exists x$;

restricted quantifiers $\forall x_{\in y}\ , \exists x_{\in y}\ ;$

where the quantifier binds x but not y

and the class-forming operator $\{\cdot \mid \ldots\}$

We should (but do not) define the result of substituting one variable for another, indicated informally by such usages as $\mathfrak{A}(x)$ and $\mathfrak{A}(y)$.

We add to the usual axioms of classical predicate first-order logic these three schemes:

$$\exists x_{\in y}\, \mathfrak{A} \Longleftrightarrow \exists x[x \in y \;\&\; \mathfrak{A}]$$
$$\forall x_{\in y}\, \mathfrak{A} \Longleftrightarrow \forall x[x \in y \Longrightarrow \mathfrak{A}]$$
$$x \in \{y \mid \mathfrak{A}(y)\} \Longleftrightarrow \mathfrak{A}(x)$$

We extend our notation by the following rules, in which A is a class and t a class or a variable:

$$\exists x_{\in A}\, \mathfrak{A} \;\Longleftrightarrow_{\mathrm{df}}\; \exists x[x \in A \;\&\; \mathfrak{A}];$$
$$t = A \;\Longleftrightarrow_{\mathrm{df}}\; \forall y_{\in t}\, y \in A \;\&\; \forall y_{\in A}\, y \in t;$$
$$A \in t \;\Longleftrightarrow_{\mathrm{df}}\; \exists z_{\in t}\, z = A.$$

We define as usual the classes $\varnothing$, V, $\{x, y\}$, $x \smallsetminus y$, $\bigcup x$ and then we may give axioms for our first, very weak, system:

S_0 The axiom of extensionality, and axioms of empty set, pair set, difference and sumset (or union):

$$[\forall x_{\in a}\, x \in b \;\&\; \forall x_{\in b}\, x \in a] \Longrightarrow a = b,$$
$$\varnothing \in V, \quad \{x, y\} \in V, \quad x \smallsetminus y \in V, \quad \bigcup x \in V.$$

We then define a Δ_0 formula or a Δ_0 class to be one containing no unrestricted quantifiers; a Π_1 formula is one of the form $\forall x \mathfrak{A}$ where $\mathfrak{A}$ is Δ_0; a Σ_2 formula is one of the form $\exists y \mathfrak{B}$ where $\mathfrak{B}$ is Π_1; a Σ_1 formula is one of the form $\exists x \mathfrak{A}$ where $\mathfrak{A}$ is Δ_0, and so on. Then our next systems will be:

ReS_0 S_0 plus the Δ_0 separation axiom: $x \cap A \in V$ for A a Δ_0 class.

ReS ReS_0 plus the scheme of Π_1 foundation:

$$A \neq \varnothing \Longrightarrow \exists x_{\in A}\, x \cap A = \varnothing \text{ for } A \text{ a } \Pi_1 \text{ class.}$$

We shall call functions of the form $x \mapsto x \cap A$ *separators*.

ReS_0 is definitely stronger than S_0:

> Let α be ω^2, and let K^α be the closure of $\alpha \dot{+} 1$ under the operations of pairing, union and difference. Then the set of limit ordinals less than α is not in K^α and hence Δ_0 separation fails there.

Simpson has arguments for showing that ReS is stronger still; but it still does not prove the existence of cartesian products.

A model of ReS in which some cartesian product does not exist is the union of all transitive sets which contain only finitely many ordered pairs.

The next system, DB_0 for "Devlin Basic", adds the existence of cartesian product to ReS_0; it thereby becomes finitely axiomatisable, by work of Bernays. We give it officially as that finite axiomatisation.

DB_0 The system of which the set-theoretic axioms are Extensionality and the following nine set-existence axioms:

$$\begin{array}{lll} \varnothing \in V & \bigcup x \in V & a \cap \{(x,y)_2 \mid_{x,y} x \in y\} \in V \\ \{x,\, y\} \in V & \mathrm{Dom}\,(x) \in V & \{(y,x,z)_3 \mid_{x,y,z} (x,y,z)_3 \in b\} \in V \\ x \smallsetminus y \in V & x \times y \in V & \{(y,z,x)_3 \mid_{x,y,z} (x,y,z)_3 \in c\} \in V \end{array}$$

DB DB_0 plus Π_1 foundation.

REMARK All those nine are theorems of ReS_0 + cartesian product, without foundation.

REMARK We shall usually take the axiom of infinity in the form $\omega \in V$, ω being defined as the class of all von Neumann ordinals such that they and all their predecessors are either 0 or successor ordinals.

REMARK If we add the axiom of infinity plus the scheme of foundation for all classes to DB we obtain the system BS as formulated on page 36 of Devlin's book *Constructibility*:

BS ReS_0 + Cartesian product + full foundation + $\omega \in V$.

Devlin in his book had the aim of finding a theory that

- would hold in all structures L_λ for λ a limit ordinal,
- and in all structures J_α for α an arbitrary non-zero ordinal;
- support a unified development of both hierarchies;
- and yet make no mention of rudimentary functions.

and proposed BS as such a theory. Alas, it proves to be too weak: for any $\mathfrak{k} \geqslant 3$, BS, if consistent, fails to prove that $[\omega]^{\mathfrak{k}}$ exists.

Fix a natural number $\mathfrak{k} \geqslant 3$.

Let $\mathbf{M}_{6,\mathfrak{k}}$ be the union of all transitive sets which contain only finitely sets of size $\mathfrak{k}$.

Then $\mathbf{M}_{6,\mathfrak{k}}$ is a model of BS in which for no infinite member x of $\mathbf{M}_{6,\mathfrak{k}}$ does $[x]^{\mathfrak{k}}$ exist.

Hence BS does not support a definition of the truth predicate $\models_u$.

The Middle Way

We call the cure MW, for "Middle Way":

MW $\qquad \mathsf{DBI} + \forall a \forall k_{\in \omega}\, [a]^k \in V$

Remark Independently of my work, Aczel and Rathjen have been led to study this axiom in their work on constructive set theory.

The point of this system is that the truth relation $\models_u$ is definable in it; but it is still weaker than GJ:

> Let $\mathbf{A}_7$ be the class of all transitive sets u such that only finitely many transitive sets of limit rank are members of u, and let $\mathbf{M}_7$ be $\bigcup \mathbf{A}_7$.
>
> Then $\mathbf{M}_7$ is a transitive model of MW in which for some set B, $\{\bigcup x \mid x \in B\}$ is not a set.

We first formulate the system GJ_0 as DB_0 plus the single axiom called R_8 in [M3], and shall then change tack and investigate certain functions rather than axiom systems.

GJ_0	$\mathsf{DB}_0 + \{x``\{w\} \mid_w w \in y\} \in V$
GJ	GJ_0 + the scheme of Π_1 foundation.

Here $x``\{w\} = \{v \mid (v, w) \in x\}$.

The relation between MW and GJI

PROPOSITION (GJ_0) *Each* $[\omega]^{\mathfrak{k}}$ *exists; indeed, each* $[a]^{\mathfrak{k}}$ *exists for any set* a.

Proof : $[a]^0 = \{\varnothing\} \in V$. $[a]^1 = A_0``a \in V$.

$$[a]^{\mathfrak{k}+1} = \left\{ s \cup \{x\} \,\middle|_{s,x}\, (s,x)_2 \in ([a]^{\mathfrak{k}} \times a) \cap \{(s,x)_2 \mid x \notin s\} \right\},$$

which is in V, being of the form $h``b$ for some set b and rudimentary function h. $\dashv$

THEOREM (GJ) $\forall a \forall k_{\in\omega}\ [a]^k \in V$.

Rudimentary functions

$$R_0(x,y) = \{x,y\}$$

$$R_1(x,y) = x \setminus y$$

$$R_2(x) = \bigcup x$$

$$R_3(x) = \mathrm{Dom}\,(x)$$

$$R_4(x,y) = x \times y$$

$$R_5(x) = x \cap \{(a,b)_2 \mid a \in b\}$$

$$R_6(x) = \{(b,a,c)_3 \mid (a,b,c)_3 \in x\}$$

$$R_7(x) = \{(b,c,a)_3 \mid (a,b,c)_3 \in x\}$$

$$A_{14}(x,y) = x``\{y\}\,[= \mathrm{Dom}\,((x \cap ([\bigcup\bigcup x] \times \{y\}))^{-1})]$$

$$R_8(x,y) = \{x``\{w\} \mid w \in y\}$$

DEFINITION Let $\mathcal{B}$ be the closure of $R_0 \ldots R_7$ under composition.

PROPOSITION *For each Δ_0 class A the map $x \mapsto x \cap A$ is in $\mathcal{B}$.*

REMARK That corresponds to the derivability of Δ_0 separation in DB_0.

DEFINITION Let $\mathcal{R}$ be the closure of $R_0 \ldots R_8$ under composition.

$\mathcal{R}$ is closed under formation of images: by which is meant that if F is in $\mathcal{R}$ so is $x \mapsto F``x$.

PROPOSITION *If $F(\vec{x})$ is a rudimentary function of several variables, there is an $\ell \in \omega$ such that for all transitive u, if each argument in $\vec{x}$ is in u, then $F(\vec{x}) \in \mathbb{T}^\ell(u)$.*

Proof : The stated property holds of the nine generating functions and is preserved under composition. $\dashv$

COROLLARY (Gandy; Jensen) *If F is rudimentary, then there is a finite ℓ such that the rank of the value is at most the maximum of the ranks of the arguments, plus ℓ.*

Proof : the function $\mathbb{T}$ increases rank by exactly 1. $\dashv$

PROPOSITION *For any transitive u, $\bigcup_{n\in\omega}\mathbb{T}^n(u)$ is the rudimentary closure of $u \cup \{u\}$ and models* TCo.

Bounding rudimentary functions in a finite progress

DEFINITION A ξ-*progress* is a sequence $\langle P_\nu \mid \nu \leqslant \xi \rangle$ of transitive sets such that for each $\nu < \xi$, $\mathbb{T}(P_\nu) \subseteq P_{\nu+1}$ and for each limit ordinal $\lambda \leqslant \xi$, $\bigcup_{\nu<\lambda} P_\nu \subseteq P_\lambda$.

THEOREM *Let R be a rudimentary function of n variables. There is a $c_R \in \omega$ such that for every c_R-progress $P_0, P_1, \ldots, P_{c_R}$,*

$$R``P_0^n \subseteq P_{c_R}.$$

DEFINITION We call c_R the *rudimentary constant* of R.

Rudimentary recursion

The Σ_1 recursion theorem of Kripke-Platek set theory KP proves for G a Σ_1 function that if G is total, so is the function F given by the recursion

$$F(x) = G(F \restriction x),$$

and further F is provably equal to a Σ_1 function.

If the defining function G is rudimentary in the sense of Jensen, we shall speak of F as given by a *rudimentary recursion*, or, more briefly, that F *is rud rec*.

In favourable cases we may also use this terminology when F is intended to be a function defined on On rather than on V, or defined by recursion on other well-founded relations related to the epsilon relation.

Some rudimentary recursions

EXAMPLE The definition of *rank*:

$$\varrho(x) = \bigcup\{\varrho(y) + 1 \mid_y y \in x\}$$

EXAMPLE The definition of *transitive closure*:

$$\mathrm{tcl}(x) = x \cup \bigcup\{\mathrm{tcl}(y) \mid_y y \in x\}$$

EXAMPLE Let $\mathcal{S}(x)$ be the set of finite subsets of x. *Restricted to ordinals*, this has a rudimentarily recursive definition:

$$\mathcal{S}(0) = \{\varnothing\};\ \mathcal{S}(\zeta{+}1) = \mathcal{S}(\zeta) \cup \{x \cup \{\zeta\} \mid_x x \in \mathcal{S}(\zeta)\};\ \mathcal{S}(\lambda) = \bigcup_{\nu<\lambda} \mathcal{S}(\nu$$

Restricted to the hereditarily finite sets, it hasn't.

Remark Recursively define $T(x) = \bigcup_{y \in x} \mathbb{T}(T(y))$; then $T(x)$ always equals $T_{\varrho(x)}$, where

$$T_0 = \varnothing; \qquad T_{\nu+1} = \mathbb{T}(T_\nu); \qquad T_\lambda = \bigcup_{\nu<\lambda} T_\nu$$

which can be said in one breath as

$$T_\zeta = \bigcup_{\nu<\zeta} \mathbb{T}(T_\nu).$$

Then $L = \bigcup_{\nu \in ON} T_\nu$, and $J_\nu = T_{\omega\nu}$; but $\nu \mapsto \omega\nu$ is not rud rec.

Rudimentary recursion from parameters

Let p be a set. We call a unary function F *p-rud rec* if there is a binary rudimentary G such that for all x,

$$F(x) = G(p, F \restriction x).$$

EXAMPLE Ordinal addition is given by the recursion

$$A(\alpha, 0) = \alpha;\ A(\alpha, \beta + 1) = A(\alpha, \beta) + 1;\ A(\alpha, \lambda) = \bigcup_{\nu < \lambda} A(\alpha, \nu)$$

For each α that is an α-rud recursion on the second variable β.

REMARK If F is rud rec (in a parameter), so is $x \mapsto F \restriction x$ (in the same parameter).

Two important properties of p-rud rec functions.

THE DEFINABILITY LEMMA *Let F be p-rud recursive, given by G. Then "f is an F-attempt" is a Δ_0 predicate of p and f.*
Proof : The predicate in question is

$$Fn(f) \;\&\; \bigcup \mathrm{Dom}\,(f) \subseteq \mathrm{Dom}\,(f) \;\&\; \forall x_{\in \mathrm{Dom}\,(f)} \;\; f(x) = G(p, f \restriction x)$$

THE PROPAGATION LEMMA *Let G be a binary rudimentary function. Then there is a ternary rudimentary function H_G, obtainable uniformly from G, such that for any set p, if F be the p-rud rec function given by the recursion $F(x) = G(p, F \restriction x)$, and if P^+ and P be transitive sets with $P \subseteq P^+ \subseteq \mathcal{P}(P)$, then*

$$F \restriction P^+ = H_G(p, F \restriction P, P^+).$$

Proof : If $x \in P^+$, then $x \subseteq P$, so $F \restriction x = (F \restriction P) \restriction x$ so $F(x) = G(p, (F \restriction P) \restriction x)$. Hence

$$F \restriction P^+ = \{(G(p, (F \restriction P) \restriction x), x)_2 \mid_x x \in P^+\}.$$

We take $H_G(p, f, q) \equiv \{(G(p, f \restriction x), x)_2 \mid_x x \in q\}. \dashv$

The canonical progress towards a given transitive set

Let c be a transitive set. Let $c_\zeta = c \cap \{x \mid \varrho(x) < \zeta\}$. Since c is transitive, $c_{\zeta+1}$ will be a set of subsets of c_ζ; in fact $c_{\zeta+1} = c \cap \{x \mid x \subseteq c_\zeta\}$; we shall use this as a direct recursive definition below.

If $c_{\zeta+1} = c_\zeta$, then $c_\zeta = c$ and for all $\xi > \zeta$, $c_\xi = c_\zeta$; so that that first happens when $\zeta = \varrho(c)$.

Using c as a parameter we define a sequence of pairs $((c_\nu, P^c_\nu))_\nu$ by a rud recursion on ν. Each P^c_ν will be of rank ν; we shall use the function $\mathbb{T}$, but we shall also "feed" stages of c into the process.

DEFINITION

$$c_0 = \varnothing \quad c_{\nu+1} = c \cap \{x \mid x \subseteq c_\nu\} \qquad c_\lambda = \bigcup_{\nu<\lambda} c_\nu$$

$$P^c_0 = \varnothing \quad P^c_{\nu+1} = \mathbb{T}(P^c_\nu) \cup \{c_\nu\} \cup c_{\nu+1} \quad P^c_\lambda = \bigcup_{\nu<\lambda} P^c_\nu$$

LEMMA *Each P^c_ν is transitive. $P^c_\nu \subseteq P^c_{\nu+1}$. $P^c_\nu \in P^c_{\nu+1}$; and so for $\nu < \zeta$, $P^c_\nu \subseteq P^c_\zeta$ and $P^c_\nu \in P^c_\zeta$.*

REMARK $c_\nu = c \cap P^c_\nu$; $\varrho(P^c_\nu) = \nu$; $L(c) = \bigcup_{\nu \in \mathbf{ON}} P^c_\nu$.

REMARK P^c_ν may be defined by a single rud recursion on ordinals:

$$P^c_0 = \varnothing;\ P^c_{\nu+1} = \mathbb{T}(P^c_\nu) \cup \{c \cap P^c_\nu\} \cup (c \cap \{x \mid x \subseteq P^c_\nu\});\ P^c_\lambda = \bigcup_{\nu<\lambda} P^c_\nu.$$

REMARK Each P^c_λ is rud closed, for λ a limit ordinal; $P^c_\omega = V_\omega$.

A new symbol

DEFINITION $F \restriction u =_{\mathrm{df}} \{F \restriction x \mid x \in u\}$

Bounding rudimentarily recursive functions in a single canonical progress

THEOREM *Let F be p-rud rec, given by G. Then there exist s_F and g_F in ω such that for any transitive c and any ordinal ν_0 with $p \in P^c_{\nu_0}$, any non-successor ordinal λ and any $k \in \omega$,*

$$\text{(i) } F \restriction P^c_\lambda \subseteq P^c_{\nu_0+\lambda}; \qquad \text{(ii) } F \restriction P^c_{\lambda+k} \in P^c_{\nu_0+\lambda+s_F+k\cdot g_F}.$$

Provident sets

DEFINITION A set A is *provident* if it is non-empty, transitive, and for all $p \in A$, all p-rud rec F and all x in A, $F(x) \in A$.

THEOREM *Let θ be indecomposable and c a transitive set. Then P^c_θ is provident.*

Proof : Let $p \in P^c_\theta$; choose $\nu_0 < \theta$ with $p \in P^c_{\nu_0}$. Let F be p-rud rec. Then for each limit $\eta < \theta$, $F \restriction P^c_\eta \subseteq P^c_{\nu_0+\eta} \subseteq P^c_\theta$. So $F \restriction P^c_\theta \subseteq P^c_\theta$, as required. $\dashv$

PROPOSITION *Let c be a transitive set and θ an indecomposable ordinal. Then*

$$P^c_\theta = P^{c_\theta}_\theta = \bigcup_{\lambda<\theta} P^{c_\lambda}_\theta .$$

PROPOSITION *If θ is an indecomposable ordinal and C is a set of transitive sets such that any two members of C are members of a third, then $B =_{\mathrm{df}} \bigcup_{c \in C} P_\theta^c$ is provident. More generally, the union of a directed system of provident sets is provident.*

Proof : Given a parameter p in B and an argument x in B, choose $c \in C$ with both p and x in P_θ^c. We know that P_θ^c is provident, and so if F is p-rud rec, $F(x)$ is in P_θ^c and therefore in B. $\dashv$

PROPOSITION *Let A be a provident set, and write $\theta(A)$ for the least ordinal not in A.*

A is rud closed;

A contains the rank $\varrho(x)$ of each member x of A;

A contains the transitive closure of each of its members;

$\theta(A)$ is indecomposable;

$\theta(A) = \varrho(A)$;

$A = \bigcup\{P^d_{\theta(A)} \mid_d \bigcup d \subseteq d \in A\}$.

PROPOSITION *Let θ be an indecomposable ordinal, and let $(Q_\nu)_{\nu\leqslant\theta}$ be a θ-progress with $Q_\theta = \bigcup_{\nu<\theta} Q_\nu$. Then Q_θ is provident.*

Provident levels of the Jensen and Gödel hierarchies

THEOREM *J_ν is provident iff $\omega\nu$ is indecomposable. More generally, if c is a transitive set, $J_\nu(c)$ will be provident iff $\omega\nu$ is indecomposable and strictly greater than the rank of c.*

We need $\omega\nu$ to exceed the rank of c, as provident sets contain the ranks of their members.

COROLLARY *J_ω is provident. The next one will be J_{ω^2}.*

PROPOSITION *L_λ is provident iff λ is indecomposable.*

Nathan Bowler's observations

REMARK (Bowler) The family of rud rec functions is NOT closed under composition.

PROPOSITION (Bowler) *The family of functions of the form $H \circ F$ where H is rudimentary and F is rudimentarily recursive IS closed under composition.*

Rudimentary recursion and forcing

EXAMPLE Suppose we are making a forcing extension using a notion of forcing $\mathbb{P}$ that is a set of the ground model, assumed transitive. In the theory of forcing, a member y of the ground model is represented by the term $\hat{y}$ of the language of forcing, given by the recursion

$$\hat{y} =_{\mathrm{df}} \{(\mathbb{1}^{\mathbb{P}}, \hat{x}) \mid_x x \in y\}$$

This is a rudimentary recursion in a parameter, being of the form

$$F(a) = G(\mathbb{1}^{\mathbb{P}}, F \restriction a)$$

where G is the rudimentary function $(\mathbb{1}^{\mathbb{P}}, a) \mapsto \{\mathbb{1}^{\mathbb{P}}\} \times \mathrm{Im}(a)$.

EXAMPLE If $\mathcal{G}$ is a generic filter on a notion of forcing $\mathbb{P}$ in a transitive model M, and we follow Shoenfield in treating all members of M as $\mathbb{P}$-names, the function $\mathrm{val}_{\mathcal{G}}(\cdot)$ defined for $a \in M$ is given by a rudimentary recursion with $\mathcal{G}$ as a parameter.

$$\mathrm{val}_{\mathcal{G}}(b) \ =_{\mathrm{df}} \ \{\mathrm{val}_{\mathcal{G}}(a) \mid_a \exists p_{\in \mathcal{G}} \ (p,\, a) \in b\}$$

The generic extension $M[\mathcal{G}]$ will then be defined as

$$\{\mathrm{val}_{\mathcal{G}}(a) \mid_a a \in M\}.$$

REMARK Note that the definition of the forcing relation $\Vdash$ has not been invoked in making these definitions, but its properties would be needed to show that $M[\mathcal{G}]$ has interesting properties.

Forcing in provident sets

DEFINITION $p \Vdash_0 \underline{a} \in \underline{b} \iff_{\mathrm{df}} (p, a) \in b$.

$\Vdash_0$ is our first approximation to the relation $\Vdash$.

LEMMA *If* $p \Vdash_0 \underline{a} \in \underline{b}$ *then* $a \in \bigcup\bigcup b$.

DEFINITION In future we shall write $\bigcup^2 x$ for $\bigcup\bigcup x$.

DEFINITION $p \Vdash_1 \underline{a} \in \underline{b} \iff_{\mathrm{df}} \exists q{:}\in\bigcup^2 b\ [q \geq p \ \&\ (q, a) \in b]$.

LEMMA *If* $p \Vdash_1 \underline{a} \in \underline{b}$ *and* $r \leq p$ *then* $r \Vdash_1 \underline{a} \in \underline{b}$.

This last statement shows that $\Vdash_1$ improves $\Vdash_0$ and starts to resemble a forcing relation.

DEFINITION $p \Vdash \underline{b} = \underline{c} \iff_{\mathrm{df}}$

$$\forall \beta :\in \bigcup^2 b \; \forall r :\leq p$$
$$\big[r \Vdash_1 \underline{\beta} \in \underline{b} \Rightarrow \exists t :\leq r \; \exists \gamma :\in \bigcup^2 c \, (t \Vdash \underline{\beta} = \underline{\gamma} \;\&\; t \Vdash_1 \underline{\gamma} \in \underline{c}) \big] \;\&$$
$$\forall \gamma :\in \bigcup^2 c \; \forall r :\leq p$$
$$\big[r \Vdash_1 \underline{\gamma} \in \underline{c} \Rightarrow \exists t :\leq r \; \exists \beta :\in \bigcup^2 b \, (t \Vdash \underline{\gamma} = \underline{\beta} \;\&\; t \Vdash_1 \underline{\beta} \in \underline{b}) \big]$$

DEFINITION Let $\chi_{=}(p, b, c)$ be the characteristic function of the relation $p \Vdash^{\mathbb{P}} \underline{b} = \underline{c}$, so that it takes the value 1 if $p \Vdash^{\mathbb{P}} \underline{b} = \underline{c}$ and 0 otherwise.

The graph of $\chi_{=}$ on transitive sets is given by a $\mathbb{P}$-rudimentary recursion.

THE DEFINABILITY LEMMA "f is a $\chi_{=}$ attempt" is $\Delta_0(\mathbb{P}, f)$.

THE PROPAGATION LEMMA *Let* $F(u) = \chi_{=} \restriction (\mathbb{P} \times u \times u)$. *There is a rudimentary function* $H_{=}$ *such that for any transitive* P, *if* $P \subseteq P^+ \subseteq \mathcal{P}(P)$,

$$F(P^+) = H_{=}(\mathbb{P}, F(P), P^+).$$

Propagation of $\chi_{=}$

We have defined the progress P^c_ν for c a transitive set. We could continue to work with progresses of the above kind, but a problem would then arise at the end of the paper, in the proof that a set-generic extension of a provident set is provident.

Here it is better to work with other progresses, which might be called construction from e as a set and $\chi_{=}$ as a predicate, with the definition of $\chi_{=}$ evolving during the construction.

DEFINITION Let e be a transitive set of which $\mathbb{P}$ is a member; let $\eta = \varrho(\mathbb{P})$. We define by a p-rudimentary recursion a sequence $((e_\nu, P^{e;\,=}_\nu, \chi^e_\nu)_3)_\nu$ of triples, thus obtaining a new progress $(P^{e;\,=}_\nu)_\nu$.

For every ν, e_ν will be defined as before; for $\nu \leqslant \eta$ we set $P^{e;\,\mathsf{=}}_\nu = P^e_\nu$; for $\nu < \eta$, we set $\chi^e_\nu = \varnothing$ but at η, we set $\chi^e_\eta = \chi_{\mathsf{=}} \restriction P^e_\eta$, which will be a set by the last Corollary. Thereafter we set

$$
\begin{aligned}
e_{\nu+1} &= e \cap \{x \mid x \subseteq e_\nu\} & e_\lambda &= \textstyle\bigcup_{\nu<\lambda} e_\nu \\
P^{e;\,\mathsf{=}}_{\nu+1} &= \mathbb{T}(P^{e;\,\mathsf{=}}_\nu) \cup \{e_\nu\} \cup e_{\nu+1} \cup \{\chi^e_\nu \cap P^{e;\,\mathsf{=}}_\nu\} & P^{e;\,\mathsf{=}}_\lambda &= \textstyle\bigcup_{\nu<\lambda} P^{e;\,\mathsf{=}}_\nu \\
\chi^e_{\nu+1} &= H_{\mathsf{=}}(\mathbb{P}, \chi^e_\nu, P^{e;\,\mathsf{=}}_{\nu+1}) & \chi^e_\lambda &= \textstyle\bigcup_{\nu<\lambda} \chi^e_\nu
\end{aligned}
$$

Proposition *Let e be transitive, with $\mathbb{P} \in e$, and let θ be indecomposable and strictly greater than $\varrho(\mathbb{P})$. Then $P^{e;\,\mathsf{=}}_\theta = P^e_\theta$.*

This reconstruction of P^e_θ shortens the delay for most χ^e_ν:

PROPOSITION *For any ordinal $\nu \geqslant \eta$, any limit ordinal $\lambda > \eta$ and $k \in \omega$,*

$$\chi^e_\nu = \chi_{=} \restriction P^{e;=}_\nu;$$
$$\chi^e_\nu \subseteq P^{e;=}_{\nu+6};$$
$$\chi^e_\lambda \subseteq P^{e;=}_\lambda;$$
$$\chi_{=} \restriction P^{e;=}_\nu \in P^{e;=}_{\nu+12}.$$

Propagation of χ_ϵ

We may now define $p \Vdash \underline{a} \in \underline{b}$.

DEFINITION $p \Vdash \underline{a} \in \underline{b} \iff_{\mathrm{df}} \forall s :\leq p\ \exists t :\leq s\ \exists \beta :\in \bigcup^2 b\ \big[t \Vdash \underline{\beta} = \underline{a}\ \&\ t \Vdash_1 \underline{\beta} \in \underline{b}\big]$.

REMARK This is not a definition by recursion: indeed it is visibly rudimentary in $p \Vdash \underline{b} = \underline{c}$.

DEFINITION Let $\chi_\epsilon(p, a, b)$ be the characteristic function of the relation $p \Vdash^{\mathbb{P}} \underline{a} \in \underline{b}$.

PROPOSITION *There is a natural number s_ϵ such that for each ordinal $\nu \geqslant \eta$, $\chi_\epsilon \restriction P_\nu^{e;=} \in P_{\nu+s_\epsilon}^{e;=}$.*

Construction of Cohen terms of affine delay for rudimentary functions

THEOREM *Let R be a rudimentary function of some number of arguments. Then there is a function $R^{\mathbb{P}}$, of the same number of arguments, with the property that if A is a provident set and $\mathbb{P} \in A$ a notion of forcing, then A is closed under $R^{\mathbb{P}}$ and, further, if $\mathcal{G}$ is an $(A, \mathbb{P})$-generic, then (to take the case of a function of two variables) for all x and y in A, $\mathrm{val}_{\mathcal{G}}(R^{\mathbb{P}}(x, y)) = R(\mathrm{val}_{\mathcal{G}}(x), \mathrm{val}_{\mathcal{G}}(y))$.*

COROLLARY *Let A be provident, $\mathbb{P} \in A$ and $\mathcal{G}$ $(A, \mathbb{P})$-generic. Then $A[\mathcal{G}]$ is rud closed and so a model of GJ_0.*

Propagation of Cohen terms for rud functions

PROPOSITION *Let R be a rudimentary function of some number of arguments, and let $R^{\mathbb{P}}$ be the corresponding function of names that we have defined. There is a natural number s_R such that whenever e is a transitive set with $\mathbb{P} \in e$, and ν is an ordinal not less than $\varrho(\mathbb{P})$,*

$$R^{\mathbb{P}} \restriction P_{\nu}^{e;\,=} \in P_{\nu+s_R}^{e;\,=}.$$

Construction of rudimentarily recursive Cohen terms for rank and transitive closure

Rank and transitive closure are pure rud rec; we show here that $\mathbb{P}$-rud rec Cohen terms exist for them.

Let $S(\cdot)$ be the function $z \mapsto z \cup \{z\}$.

LEMMA *There is a rud function* $S^{\mathbb{P}}(\cdot)$ *such that* $\mathrm{val}_{\mathcal{G}}(S^{\mathbb{P}}(x)) = S(\mathrm{val}_{\mathcal{G}}(x))$.

DEFINITION $\varrho^{\mathbb{P}}(x) =_{\mathrm{df}} \bigcup^{\mathbb{P}}\{(p, S^{\mathbb{P}}(\varrho^{\mathbb{P}}(y)) \mid (p, y) \in x \;\&\; p \in \mathbb{P}\}$

REMARK $\varrho^{\mathbb{P}}$ is rud rec in the parameter $\mathbb{P}$.

LEMMA *Let A be provident, and $\mathbb{P} \in A$. For all $x \in A$,*

$$\mathrm{val}_{\mathcal{G}}(\varrho^{\mathbb{P}}(x)) = \varrho(\mathrm{val}_{\mathcal{G}}(x))$$

DEFINITION $\mathrm{tcl}^{\mathbb{P}}(x) =_{\mathrm{df}} x \cup^{\mathbb{P}} \bigcup^{\mathbb{P}}(\{(p, \mathrm{tcl}^{\mathbb{P}}(z)) \mid (p,\, z) \in x\})$.

REMARK $\mathrm{tcl}^{\mathbb{P}}$ is rud rec in the parameter $\mathbb{P}$.

LEMMA *Let A be provident, and $\mathbb{P} \in A$. For all $x \in A$,*

$$\mathrm{val}_{\mathcal{G}}(\mathrm{tcl}^{\mathbb{P}}(x)) = \mathrm{tcl}(\mathrm{val}_{\mathcal{G}}(x)).$$

Construction of Cohen terms for the stages of a progress.

Let e be a transitive set in the ground model of which $\mathbb{P}$ is a member, and let θ be indecomposable, exceeding the rank of e. P^e_θ is provident. Let $\dot{d}$ be the Cohen term $\hat{e} \cup \{\dot{\mathcal{G}}\}^{\mathbb{P}}$, so that $\mathrm{val}_{\mathcal{G}}(\dot{d})$ will be the transitive set $d = e \cup \{\mathcal{G}\}$.

REMARK $\dot{d}$ will be a member of $P^e_{\varrho(\mathbb{P})+k}$ for some (small) k, given the definition of $\dot{\mathcal{G}}$, our convention that $\mathbb{1} = 1$ and the fact that $\hat{\cdot}$ is $\mathbb{1}$-rud rec.

Our task is to build for each $\nu < \theta$ a name $N(\nu)$ for the stage P^d_ν of the progress towards d.

A simplified progress

Now $\varrho(\mathcal{G}) \leqslant \varrho(\mathbb{P}) < \varrho(\mathbb{P})$, so that for $\nu \geqslant \eta$, $d_\nu = e_\nu \cup \{\mathcal{G}\}$. It might be that $\varrho(\mathcal{G}) < \varrho(\mathbb{P})$; to avoid building names which make allowance for that uncertainty, we shall build names for the terms of a slightly different progress $(Q^d_\nu)_\nu$.

DEFINITION

for $\nu < \eta$, $\quad Q^d_\nu = P^e_\nu$; $\qquad Q^d_\eta = P^e_\eta \cup \{\mathcal{G}\}$;

for $\nu \geqslant \eta$, $Q^d_{\nu+1} = \mathbb{T}(Q^e_\nu) \cup \{d_\nu\} \cup d_{\nu+1}$; $Q^d_\lambda = \bigcup_{\nu<\lambda} Q^d_\nu$ if $\lambda = \bigcup \lambda$

PROPOSITION *If θ is indecomposable, then Q^d_θ is provident and equals P^d_θ.*

Generic extensions of provident sets

THEOREM *Let θ be an indecomposable ordinal strictly greater than the rank of a transitive set e which contains the notion of forcing, $\mathbb{P}$. Let $\mathcal{G}$ be $(P^e_\theta, \mathbb{P})$- generic. Then $(P^e_\theta)^{\mathbb{P}}[\mathcal{G}] = P^{e\cup\{\mathcal{G}\}}_\theta$ and hence is provident.*

Proof : $(P^e_\theta)^{\mathbb{P}}[\mathcal{G}]$ contains $P^{e\cup\{\mathcal{G}\}}_\theta$, as we have for each $\nu < \theta$ built a name in P^e_θ that evaluates under $\mathcal{G}$ to $Q^{e\cup\{\mathcal{G}\}}_\nu$, and we know by the previous Proposition that $Q^{e\cup\{\mathcal{G}\}}_\theta$ equals $P^{e\cup\{\mathcal{G}\}}_\theta$. For the converse direction, we know that $P^{e\cup\{\mathcal{G}\}}_\theta$ is provident, and has $\mathcal{G}$ as a member and hence can support the $\mathcal{G}$-rudimentary recursion defining $\mathrm{val}_{\mathcal{G}}(\cdot)$. Further $P^{e\cup\{\mathcal{G}\}}_\theta$ includes $(P^e_\nu)_\nu$, which is defined by an e-rudimentary recursion, and so includes $(P^e_\theta)^{\mathbb{P}}[\mathcal{G}]$. $\dashv$

REMARK Thus, in this special case, a generic extension of a model of PROVI is a model of PROVI. We shall use this result to establish it more generally.

REMARK The theorem remains true if the hypothesis on θ is weakened to requiring that $\theta > \varrho(\mathbb{P})$.

Proof that a generic extension of a provident set is provident.

THEOREM *Let A be provident, $\mathbb{P} \in A$ and $\mathcal{G}$ $(A, \mathbb{P})$-generic. Then $A^{\mathbb{P}}[\mathcal{G}]$ is provident.*

Proof : Let $\theta =_{\mathrm{df}} On \cap A$ and let

$$T = \{c \mid c \in A \ \& \ c \text{ is transitive } \ \& \ \mathbb{P} \in c\}.$$

Then

$$A = \bigcup \{P_\theta^c \mid c \in T\},$$

since the union on the right contains each element of A and is contained in A. It follows that

$$A^{\mathbb{P}}[\mathcal{G}] = \bigcup_{c \in T} (P_\theta^c)^{\mathbb{P}}[\mathcal{G}]$$

By the previous theorem, as each P_θ^c is provident and contains $\mathbb{P}$,

$$A^{\mathbb{P}}[\mathcal{G}] = \bigcup_{c \in T} P_\theta^{c \cup \{\mathcal{G}\}}$$

and each $P_\theta^{c \cup \{\mathcal{G}\}}$ is provident. Now in [M4, Proposition 5·52] we proved the

LEMMA *If θ is indecomposable and D is a collection of transitive sets each of rank less than θ and such that the pair of any two is a member of a third, then $\bigcup_{d \in D} P_\theta^d$ is provident.*

Take $D = \{c \cup \{\mathcal{G}\} \mid c \in T\}$ to complete the proof. $\dashv$

Forcing over models of Zermelo set theory

No new ordinals ?

REMARK In section 6 of *The strength of Mac Lane set theory*, a forcing construction is done over a non-standard model $\mathfrak{N}$, and it was there blithely stated without proof that the generic extension would bring no new "ordinals". Fortunately the model $\mathfrak{N}$ was power-admissible, and therefore certainly a model of PROVI, which is a sub-theory of KP, so that that blithe statement was correct, as forcing over provident sets indeed adds no new ordinals.

BUT BEWARE ! Forcing over certain admittedly pathological transitive models of Z leads to generic extensions with, in some cases, more and, in other cases, fewer ordinals than in the ground model.

The provident closure

If M is a transitive non-empty model of AxPair and of TCo, the *provident closure* of M is defined thus:

DEFINITION $\mathrm{Prov}(M) =_{\mathrm{df}} \bigcup\{P^c_\theta \mid \bigcup c \subseteq c \in M\}$, where θ is the least indecomposable ordinal not less than $\varrho(M)$.

REMARK $\varrho(M)$ will be a limit ordinal. If $\varrho(M)$ is indecomposable, $\theta = \varrho(M)$, otherwise θ will be the ordinal product $\varrho(M)\omega$, and will equal the supremum of finite sums of ordinals less than $\varrho(M)$.

THEOREM *Suppose* $\mathbf{M}$ *is a transitive model of* $\mathsf{Z} + \mathsf{TCo}$ *or of* M*; then so is its provident closure* I*.*

The lune

The lune of a model $\mathbf{M}$ of Zermelo or of Mac Lane set theory is a canonical extension of it which models the same theory plus Kripke-Platek. For example, $\mathrm{Lune}(V_{\omega+\omega}) = H_{\beth_\omega}$.

A similar construction, though differing in various details, is presented in Hinnion's thesis which starting from models of NF *yields models of subsystems of* ZF.

The passage from $\mathbf{M}$ to $\mathrm{Lune}(\mathbf{M})$ can be considered in two ways. If $\mathbf{M}$ is (a transitive set and) a β-model in the sense that every binary relation $(a, r) \in \mathbf{M}$ that is considered by $\mathbf{M}$ to be extensional and well-founded is genuinely well-founded, we can form the larger model by taking every such (a, r), transitising it, and taking the union of the transitisations of all such well-founded

extensional relations in $\mathbf{M}$.

But if $\mathbf{M}$ is not a β-model, the construction still is useful; we take all extensional well-founded relations in the sense of $\mathbf{M}$, and working within $\mathbf{M}$, define an interpretation $(\cdot)^1$ as in *The Strength of Mac Lane set theory*; externally to $\mathbf{M}$ we can define an equivalence relation and form the set of equivalence classes, and define a "membership" relation on that set.

DEFINITION $\mathrm{Lune}(\mathbf{M}) =_{\mathrm{df}}$ the resulting model.

A commuting diagram

THEOREM *Let I and H be transitive sets such that I is a provident model of* Z *or at least* M, *and* $H = \mathrm{Lune}(I)$. *Let* $\mathbb{P} \in I$ *and let* $\mathcal{G}$ *be* $(I, \mathbb{P})$ *generic. Then* $\mathcal{G}$ *is* $(H, \mathbb{P})$ *generic, and further* $H[\mathcal{G}] = \mathrm{Lune}(I[\mathcal{G}])$.

... or to put it in a form which would please Mac Lane, ...

$$
\begin{array}{ccccc}
 & & H & \xrightarrow[\mathrm{val}_{\mathcal{G}}]{} & H[\mathcal{G}] \\
 & & \uparrow \text{Lune} & & \uparrow \text{Lune} \\
\mathbb{P} & \in & I & \xrightarrow[\mathrm{val}_{\mathcal{G}}]{} & I[\mathcal{G}]
\end{array}
$$

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