Rudimentary recursion, provident sets and forcing

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This is the corrected and slightly expanded text of the talk given by me on July 6th 2009 at the European Set Theory Meeting held at Bedlewo in honour of Ronald Jensen.

Pages i - iii give the definition of rudimentary recursion, with numerous examples; pages iii and iv establish some fundamental properties of rud rec functions; pages v and vi give the definitions of provident set and canonical progress; pages vii - xi sketch a proof that a set-generic extension of a provident set is provident; finally pages xi-xiii comment on the problem of forcing over a model of Zermelo or of Mac Lane set theory, and culminate in a commuting diagram. More detailed definitions and proofs will be found in my papers [M3, 4, 5, 6]

Rudimentary recursion

The $\Sigma_1$ recursion theorem of Kripke-Platek set theory $\mathsf{KP}$ proves for $G$ a $\Sigma_1$ function that if $G$ is total, so is the function $F$ given by the recursion

$$F(x) = G(F \upharpoonright x),$$

and further $F$ is provably equal to a $\Sigma_1$ function.

If the defining function $G$ is rudimentary in the sense of Jensen, we shall speak of $F$ as given by a rudimentary recursion, or, more briefly, that $F$ is rud rec.

In favourable cases we may also use this terminology when $F$ is intended to be a function defined on $\mathcal{O}$ rather than on $\mathcal{V}$, or defined by recursion on other well-founded relations related to the epsilon relation.

A set of generators for the class of rudimentary functions

We define rudimentary functions $R_0, \ldots, R_8$ and certain auxiliary functions which we show to be generated by $R_0, \ldots, R_7$ under composition.

$$R_0(x, y) = \{x, y\} \quad \quad A_0(x) = \{x\} \quad \quad R_4(x, y) = x \times y \quad \quad R_3(x) = x \cap \{(a, b)_2 \mid a, b \in b\}$$

$$R_1(x, y) = x \setminus y \quad \quad A_4(x, y) = x \cap \{y\} \quad \quad R_5(x) = x \cap \{(a, b, c)_3 \mid (a, b, c)_3 \in x\}$$

$$A_1(x) = \emptyset \quad \quad R_6(x) = \{(b, a, c)_3 \mid (a, b, c)_3 \in x\}$$

$$R_7(x) = \{(b, a, c)_3 \mid (a, b, c)_3 \in x\}$$

$$R_2(x) = \bigcup x \quad \quad A_{14}(x, y) = x^w\{y\} \quad \quad R_8(x, y) = (x^w\{w\} \mid w \in y)$$

$$R_3(x) = \text{Dom}(x)$$

Some rudimentary recursions

EXAMPLE The definition of rank: $$\varrho(x) = \bigcup \{\varrho(y) + 1 \mid y \in x\}$$

EXAMPLE The definition of transitive closure: $$\text{tcl}(x) = x \cup \bigcup \{\text{tcl}(y) \mid y \in x\}$$

EXAMPLE Let $S(x)$ be the set of finite subsets of $x$. Restricted to ordinals, this has a rudimentarily recursive definition:

$$S(0) = \emptyset; \quad S(\zeta + 1) = S(\zeta) \cup \{x \cup \{\zeta\} \mid x \in S(\zeta)\}; \quad S(\lambda) = \bigcup_{\nu < \lambda} S(\nu).$$

Restricted to the hereditarily finite sets, it hasn’t, by an application of an observation of Gandy [G, 1.5.9].
Rudimentary recursion from parameters

Let \( p \) be a set. We call a unary function \( F \) a \( p \)-rud rec if there is a binary rudimentary \( G \) such that for all \( x \),
\[
F(x) = G(p, F\upharpoonright x).
\]

EXAMPLE Ordinal addition is given by the recursion
\[
A(\alpha, 0) = \alpha; \quad A(\alpha, \beta + 1) = A(\alpha, \beta) + 1; \quad A(\alpha, \lambda) = \bigcup_{\nu < \lambda} A(\alpha, \nu)
\]
For each \( \alpha \) that is an \( \alpha \)-rud recursion on the second variable \( \beta \).

REMARK If \( F \) is rud rec (in a parameter), so is \( x \mapsto F\upharpoonright x \) (in the same parameter).

Rudimentary recursion in the theory of forcing

The theory of forcing in the style of Shoenfield furnishes numerous examples of rudimentary recursion from parameters.

In our discussion of forcing we shall suppose that the collection of conditions is a set, \( P \). Experience suggests that we make the following five assumptions about \( P \):

- \( P \) is partially ordered by a relation \( \leq \); if \( p \leq q \) we think that \( p \) contains more information than \( q \), and say that \( p \) is stronger than \( q \).
- To get something interesting we allow the possibility of two conditions being incompatible: we say that \( p \) is compatible with \( q \) if there is some \( r \) stronger than both: \( r \leq p \& r \leq q \); and we say that \( p \) is incompatible with \( q \), in symbols \( p \perp q \), if no such \( r \) exists.
- We assume that any condition can be strengthened in two incompatible ways:

\[
\forall p \exists q : p \leq q \quad \exists r : q \leq r
\]

- We suppose that \( P \) has a greatest element \( 1 = 1_P \), where this condition is the one that gives us no information at all. Thus \( 1 \) is compatible with every condition.
- Finally we suppose that \( P \) is separative: that is,

\[
\forall p \forall q (p \not\leq q \Rightarrow \exists r : p \perp q).
\]

Such a triple \( \mathbb{P} = \langle P, 1_P, \leq \rangle \) is called a notion of forcing.

REMARK The reader of Shoenfield’s paper on forcing or of Kunen’s book on set theory will see that we have reversed the ordered pair in the initial definition: where we shall define \( p \parallel (a, b) \) to mean \( p \in b \), those authors would have defined it to mean \( (a, p) \in b \).

EXAMPLE Suppose we are making a forcing extension using a notion of forcing \( \mathbb{P} \) that is a set of the ground model, assumed transitive. In the theory of forcing, a member \( y \) of the ground model is represented by the term \( \hat{y} \) of the language of forcing, given by the recursion
\[
\hat{y} = \lambda x (\{ (\hat{a}, x) \mid x \in y \})
\]
This is a rudimentary recursion in a parameter, being of the form
\[
F(a) = G(\hat{f}, F\upharpoonright a)
\]
where \( G \) is the rudimentary function \( (\hat{f}, a) \mapsto \{ \hat{f} \} \times \text{Im}(a) \).

If, as is convenient, we specify that \( \hat{f} \) is to be the ordinal 1, \( G \) may rewritten as a pure rud function.

8-iii-2010 Rudimentary recursion, provident sets and forcing Bedlewo ii
EXAMPLE If \( G \) is a generic filter on a notion of forcing \( P \) in a transitive model \( M \), and we follow Shoenfield in treating all members of \( M \) as \( P \)-names, the function \( \text{val}_G(\cdot) \) defined for \( a \in M \) is given by a rudimentary recursion with \( G \) as a parameter:

\[
\text{val}_G(b) =_{df} \{ \text{val}_G(a) \mid \exists p \in G \ (p, a) \in b \}
\]

The generic extension \( M^G[G] \) will then be defined as

\[
\{ \text{val}_G(a) \mid a \in M \}.
\]

REMARK Note that the definition of the forcing relation \( |P^2| \) has not been invoked in making these definitions, but its properties would be needed to show that \( M^G[G] \) has interesting properties.

The rudimentary function \( T \)

The following extremely useful function was introduced and studied in [M3]:

\[
T(u) =_{df} u \cup \{ u \} \cup [u]^1 \cup [u]^2
\]

\[
\cup \{ x \setminus y \mid x, y \in u \}
\]

\[
\cup \{ \bigcup x \mid x \in u \}
\]

\[
\cup \{ \text{Dom}(x) \mid x \in u \}
\]

\[
\cup \{ u \cap (x \times y) \mid x, y \in u \}
\]

\[
\cup \{ x \cap \{(a, b) \mid a, b \in b \} \mid x \in u \}
\]

\[
\cup \{ u \cap \{(b, a, c) \mid a, b, c \in x \} \mid x \in u \}
\]

\[
\cup \{ x \cap \{(b, c, a) \mid a, b, c \in x \} \mid x \in u \}
\]

\[
\cup \{ x^{w} \mid x, w \in u \}
\]

\[
\cup \{ u \cap \{ x^{w} \mid w, w \in y \} \mid y, x \in u \}
\].

**Proposition** \( T \) is rudimentary, \( u \subseteq T(u) \) and \( u \in T(u) \). If \( u \) is transitive, then \( T(u) \) is a set of subsets of \( u \), \( T(u) \) is transitive, \( \text{rank}(T(u)) = \text{rank}(u) + 1 \), and \( \bigcup_{n \in \omega} T^n(u) \) equals \( \text{rud}(u) \), the rud closure of \( u \cup \{ u \} \).

**Remark** Recursively define \( T(x) = \bigcup_{y \in x} T(T(y)) \); then \( T(x) \) always equals \( T_{\| x \|} \), where

\[
T_0 = \emptyset; \quad T_{\nu+1} = T(T_\nu); \quad T_\lambda = \bigcup_{\nu < \lambda} T_\nu
\]

which can be said in one breath as

\[
T_\zeta = \bigcup_{\nu < \zeta} T_\nu.
\]

Then \( L = \bigcup_{n \in \omega} T_n \), and \( J_\nu = T_{\omega \nu} \); but \( \nu \mapsto \omega \nu \) is not rud rec.

**Proposition** If \( F(x) \) is a rudimentary function of several variables, there is an \( \ell \in \omega \) such that for all transitive \( u \), if each argument in \( x \) is in \( u \), then \( F(x) \in T^\ell(u) \).

**Proof** The stated property holds of the nine generating functions and is preserved under composition. 

**Corollary** (Gandy; Jensen) If \( F \) is rudimentary, then there is a finite \( \ell \) such that the rank of the value is at most the maximum of the ranks of the arguments, plus \( \ell \).

**Proof** the function \( T \) increases rank by exactly 1.
Bounding rudimentary functions in a finite progress

**Definition** A \( \xi \)-progress is a sequence \( \langle P_\nu \mid \nu \leq \xi \rangle \) of transitive sets such that for each \( \nu < \xi \), \( T(P_\nu) \subseteq P_{\nu+1} \) and for each limit ordinal \( \lambda \leq \xi \), \( \bigcup_{\nu < \lambda} P_\nu \subseteq P_\lambda \).

The progress is **strict** if for each \( \nu < \xi \), \( P_{\nu+1} \subseteq \mathcal{P}(P_\nu) \); and **continuous** if for each limit \( \lambda \leq \xi \), \( P_\lambda = \bigcup_{\nu < \lambda} P_\nu \).

**Theorem** Let \( R \) be a rudimentary function of \( n \) variables. There is a \( c_R \in \omega \) such that for every \( c_R \)-progress \( P_0, P_1, \ldots, P_\nu \),
\[
R^n P_0^n \subseteq P_{c_R}.
\]

**Definition** We call \( c_R \) the **rudimentary constant** of \( R \).

The canonical progress towards a given transitive set

Let \( c \) be a transitive set. Let \( c_\xi = c \cap \{ x \mid g(x) < \xi \} \). Since \( c \) is transitive, \( c_{\xi+1} \) will be a set of subsets of \( c_\xi \); in fact \( c_{\xi+1} = c \cap \{ x \mid x \subseteq c_\xi \} \); we shall use this as a direct recursive definition below.

If \( c_{\xi+1} = c_\xi \), then \( c_\xi = c \) and for all \( \xi > \zeta \), \( c_\xi = c_\zeta \); so that that first happens when \( \zeta = g(c) \).

Using \( c \) as a parameter we define a sequence of pairs \( \langle (\nu_\nu, c_{\nu_\nu}) \rangle \) by a rud recursion on \( \nu \). Each \( c_\nu \) will be of rank \( \nu \); we shall use the function \( T \), but we shall also “feed” stages of \( c \) into the process.

**Definition**
\[
\begin{align*}
0 &= \emptyset & c_{\nu+1} &= c \cap \{ x \mid x \subseteq c_\nu \} & c_\lambda &= \bigcup_{\nu < \lambda} c_\nu \\
P_0^c &= \emptyset & P_{\nu+1}^c &= T(P_\nu) \cup \{ c_\nu \} \cup c_{\nu+1} & P_\lambda^c &= \bigcup_{\nu < \lambda} P_\nu^c
\end{align*}
\]

**Lemma** Each \( P_\nu^c \) is transitive. \( P_\nu^c \subseteq P_{\nu+1}^c \). \( P_\nu^c \subseteq P_\nu^c \); and so for \( \nu < \zeta \), \( P_\nu = P_\zeta \) and \( P_\lambda = P_\lambda \).

**Remark** \( c_\nu = c \cap P_\nu^c \).

**Remark** \( P_\nu^c \) may also be defined by a single rud recursion on ordinals:
\[
P_0^c = \emptyset; \quad P_{\nu+1}^c = T(P_\nu) \cup \{ c \cap P_\nu \} \cup \{ c \cap \{ x \mid x \subseteq P_\nu \} \}; \quad P_\lambda^c = \bigcup_{\nu < \lambda} P_\nu^c.
\]

**Remark** Each \( P_\nu^c \) is rud closed, for \( \lambda \) a limit ordinal; \( P_\nu^c = V_\omega \).

Two important properties of \( p \)-rud rec functions.

**The Definability Lemma** Let \( F \) be \( p \)-rud recursive, given by \( G \). Then “\( f \) is an \( F \)-attempt” is a \( \Delta_0 \) predicate of \( p \) and \( f \).

**Proof** The predicate in question is
\[ F u(f) \land \bigcup \text{Dom}(f) \subseteq \text{Dom}(f) \land \forall x : x \in \text{Dom}(f) \ f(x) = G(p, f \mid x). \]

**The Propagation Lemma** Let \( G \) be a binary rudimentary function. Then there is a ternary rudimentary function \( H_G \), obtainable uniformly from \( G \), such that for any \( p \), if \( F \) is the \( p \)-rud rec function given by the recursion \( F(x) = G(p, F \mid x) \), and if \( P^+ \) is a transitive set with \( P \subseteq P^+ \subseteq \mathcal{P}(P) \), then
\[
F \upharpoonright P^+ = H_G(p, F \upharpoonright P, P^+).
\]

**Proof** If \( x \in P^+ \), then \( x \subseteq P \), so \( F(x) = (F \upharpoonright P) \mid x \) so \( F(x) = G(p, (F \upharpoonright P) \mid x) \). Hence
\[
F \upharpoonright P^+ = \{ (G(p, (F \upharpoonright P) \mid x), x) \mid x \in P^+ \}.
\]

We take \( H_G(p, f, q) \equiv \{ (G(p, f \mid x), x) \mid x \in q \} \).

\[ \text{Bedlewo iv} \quad 8^\text{viii} 2010 \quad \text{Rudimentary recursion, provident sets and forcing} \quad \text{Bedlewo iv} \]
Provident sets

**Definition** A set $A$ is $p$-provident, where $p$ is a set, if it is non-empty, transitive, closed under pairing and for all $p$-rud rec $F$ and all $x$ in $A$, $F(x) \in A$.

**Remark** If $A$ is $p$-provident, $p \in A$.

**Example** The Jensen fragment $J_\nu$ is $\emptyset$-provident for all $\nu \geq 1$.

**Definition** $A$ is provident if it is $p$-provident for every $p \in A$.

**Example** Each $J_{\omega^\nu}$ is provident.

**Remark** For provident sets, it is unnecessary to demand that they be closed under pairing, for if $x \in A$, the function $y \mapsto \{x, y\}$ is $x$-rud rec, being given by the recursion $F(y) = \{x, \text{Dom } F \upharpoonright y\}$.

A new symbol

**Definition** $F \upharpoonright u = \{ F[x] \mid x \in u \}$

Bounding rudimentarily recursive functions in a single canonical progress

**Theorem** Let $F$ be $p$-rud rec, given by $G$. Then there exist $s_F$ and $q_F$ in $\omega$ such that for any transitive $c$ and any ordinal $\nu_0$ with $p \in P^c_{\nu_0}$, any non-successor ordinal $\lambda$ and any $k \in \omega$,

(i) $F \upharpoonright P^c_{\lambda} \subseteq P^c_{\nu_0 + \lambda}$

(ii) $F \upharpoonright P^c_{\lambda + k} \subseteq P^c_{\nu_0 + \lambda + s_F + k}$

**Theorem** Let $\theta$ be indecomposable and $c$ a transitive set. Then $P^c_\theta$ is provident.

**Proof:** Let $p \in P^c_\theta$; choose $\nu_0 < \theta$ with $p \in P^c_{\nu_0}$. Let $F$ be $p$-rud rec. Then for each limit $\eta < \theta$, $F \upharpoonright P^c_\eta \subseteq P^c_{\nu_0 + \eta} \subseteq P^c_\nu$. So $F \upharpoonright P^c_\theta \subseteq P^c_\theta$, as required.

**Proposition** Let $c$ be a transitive set and $\theta$ an indecomposable ordinal. Then

$$P^c_\theta = P^c_{\theta^c} = \bigcup_{\lambda < \theta} P^c_{\lambda}.$$  

**Proposition** If $\theta$ is an indecomposable ordinal and $C$ is a set of transitive sets such that any two members of $C$ are members of a third, then $B = \bigcup_{c \in C} P^c_\theta$ is provident. More generally, the union of a directed system of provident sets is provident.

**Proof:** Given a parameter $p$ in $B$ and an argument $x$ in $B$, choose $c \in C$ with both $p$ and $x$ in $P^c_\theta$. We know that $P^c_\theta$ is provident, and so if $F$ is $p$-rud rec, $F(x)$ is in $P^c_\theta$ and therefore in $B$.

**Proposition** Let $A$ be a provident set, and let $\theta(A)$ be the least ordinal not in $A$. Then

- $A$ is rud closed;
- $A$ contains the rank $\rho(x)$ of each member $x$ of $A$;
- $A$ contains the transitive closure of each of its members;
- $\theta(A) = \rho(A)$ and is indecomposable;
- and $A = \bigcup \{ P^c_{\theta(d)} \mid d \subseteq d \subseteq A \}$.

**Remark** Let PROV be the finitely axiomatisable theory summarised as:

+ extensionality + the empty set exists
+ all rudimentary functions are defined everywhere
+ every set has a rank + every set has a transitive closure
+ for every transitive $c$ and ordinal $\nu$ the set $P^c_\nu$ exists.

This theory is weaker than Kripke-Platek set theory KP, but stronger than Gandy-Jensen set theory, GJ; all three theories are true in $\text{HF} = V_\omega = J_1 = L_\omega$; if an axiom of infinity be added to each theory, giving the theories KPI, PROVI and GJI, the minimal transitive models are then respectively the Jensen fragments $J_{\omega^\omega}$, $J_\omega$ and $J_2$; and the provident sets are precisely $\text{HF}$ and the transitive models of PROVI.
PROPOSITION Let $\theta$ be an indecomposable ordinal, and let $(Q_\nu)_{\nu \leq \theta}$ be a $\theta$-progress with $Q_\theta = \bigcup_{\nu < \theta} Q_\nu$. Then $Q_\theta$ is provident.

Provident levels of the Jensen and Gödel hierarchies

PROPOSITION If $u$ is transitive and $\emptyset$-provident then so is $\text{rud}(u)$.

Proof: We take $P_\alpha = \mathcal{T}^u(\alpha)$, and $P_\omega = \bigcup_\alpha P_\alpha$. $(P_\nu \mid \nu \leq \omega)$ is then a strict continuous $\omega$-progress, so we may apply a previous proposition with $p = \emptyset$.

COROLLARY Each non-empty $J_\nu$ is $\emptyset$-provident.

THEOREM $J_\nu$ is provident iff $\omega^\nu$ is indecomposable. More generally, if $c$ is a transitive set, $J_\nu(c)$ will be provident iff $\omega^\nu$ is indecomposable and strictly greater than the rank of $c$.

REMARK We need $\omega^\nu$ to exceed the rank of $c$, as provident sets contain the ranks of their members.

REMARK So although for a given $p$ in $L$ we must go to the first indecomposable ordinal above the moment of construction of $p$ to find a $J_\nu$ which is $p$-provident, every subsequent $J_\xi$ will also be $p$-provident.

PROPOSITION $J_1$ is provident. The next one will be $J_\omega$.

PROPOSITION Each $L_\lambda$ is $\emptyset$-provident for limit $\lambda$.

PROPOSITION $L_\lambda$ is provident iff $\lambda$ is indecomposable.

$S$-logic in provident sets

PROPOSITION Let $A$ be provident; let $a \in A$. Then $S(a) \in A$.

REMARK The consequences for certain weak systems of adding the axiom $S(x) \in V$ and extending the syntax by admitting limited quantifiers $\forall y \in S(x), \exists y \in S(x)$, are studied in detail in section 8 of [M3].

Some weak systems of set theory

We rehearse some definitions from [M2].

Denote by $S_0$ the system with axioms of Extensionality, Null Set,Pairing, Union, Difference (“$x \setminus y \in V$”); by $S_1$ the system $S_0$ plus Power Set (“$\{y \mid y \subseteq x \in V\}$”; by $M_0$ the system $S_1$ plus the scheme of Restricted Separation (“$x \cap A \in V$, for each $\Delta_0$ class $A$), known also as $\Delta_0$ Separation and as $\Sigma_0$ Separation; by $M_1$ the system $M_0 + \text{Foundation}$ (“Every non-empty set $x$ has a member $y$ with $x \cap y$ empty”); $\text{Transitive Containment}$ (“Every set is a member of a transitive set”): using the Axiom of Pairing, that may be seen to be equivalent to saying that each set is a subset of a transitive set; by $M$ the system $M_1 + \text{Infinity}$, the latter taken in the form $\omega \in V$ asserting that there exists an infinite von Neumann ordinal; and by $\text{MAC}$ the system $M$ plus the Axiom of Choice, which we may take either as the assertion $\text{AC}$ of the existence of selectors for sets of non-empty sets, or as the assertion $\text{WO}$ that every set has a well-ordering, since $M_0$ suffices for Zermelo’s 1904 proof of their equivalence.

We abbreviate Transitive Containment as $\text{TCo}$. We write $P(x)$ for the power set $\{y \mid y \subseteq x\}$ of $x$.

Zermelo set theory, $Z$, is the result of dropping $\text{TCo}$ from $M$ and adding the full unrestricted Separation scheme. We write $ZC$ for $Z + \text{AC}$. In $Z + \text{TCo}$ the Axiom of Foundation is self-improving to the scheme of Class Foundation, in that for each class $A$ it is provable that $\exists y(y \in A) \implies \exists y(y \in A \cap y \cap A = \emptyset)$; but Jensen and Schröder [C2] and Boffa [C3], [C4] have shown that there are instances of the scheme of Class Foundation which are not theorems of $ZC$, and consequently that $\text{TCo}$ is not provable in $ZC$. A alternative proof of that last fact is to be found in [M3, section 12].

Provident closures

DEFINITION For $M$ a transitive non-empty model of $\text{AxPair}$ and of $\text{TCo}$, we define $\text{Prov}(M) =_{df} \bigcup\{P_\theta^c \mid \emptyset \subseteq c \subseteq M\}$, where $\theta$ is the least indecomposable ordinal not less than $g(M)$.

We call $\text{Prov}(M)$ the provident closure of $M$ as it is a provident set including $M$ and included in all other such. The following perhaps surprising result is proved in [M6].

THEOREM Suppose $M$ is a transitive model of $Z + \text{TCo}$ or of $M$; then so is its provident closure.
Set forcing over provident sets

HISTORICAL NOTE. Since Cohen's creation of forcing as a construction of extensions of models of full ZF, many people have examined the possibility of forcing over models of weaker systems of set theory, to say nothing of those who have transplanted Cohen's ideas to other areas of enquiry outside set theory. Forcing over admissible sets was studied briefly by Barwise in his 1967 Stanford thesis, at greater length by Jensen in an unpublished treatise on admissibility that contained a proof of his celebrated "sequence-of-admissibles" theorem, in Steel's 1978 paper [St1], in Sacks' study [Sa], and in numerous writings of Sy Friedman such as his papers [F1] and [F2], which latter expounds inter alia that result of Jensen.

The paper of Hauser [Ha] and the as yet unpublished notes of Steel [St2] contain explorations of forcing over transitive sets which, whilst not required to be admissible, are nevertheless assumed to possess certain fine-structural properties.

Our purpose here is to outline a development of set forcing in the natural, more general and, the author believes, optimal context of provident sets. The definition of the forcing relation begun below may readily be extended to the class of $\Delta_0$ sentences, and the Forcing Theorem, that what is true in the generic extension is what is forced by a member of the generic filter, proved for that class, as is done in [M5], where details of the definition of the forcing language will be found; but most of the development is concerned with the construction, in the ground model, and evaluation of, in the extension, names for new objects.

The forcing relation for atomic sentences

DEFINITION $p \parallel_0 a \in b \iff \exists\alpha (p, a) \in b$.
LEMMA If $p \parallel_0 a \in b$ then $a \in \bigcup b$.
DEFINITION In future we shall write $\bigcup x$ for $\bigcup x$.
DEFINITION $p \parallel_1 a \in b \iff \exists q \forall \beta \forall \gamma (q \geq p \& (q, a) \in b)$.
LEMMA If $p \parallel_1 a \in b$ and $r \leq p$ then $r \parallel_1 a \in b$.

This last statement shows that $\parallel_1$ improves $\parallel_0$ and starts to resemble a forcing relation.

DEFINITION $p \parallel_0^b = \chi \iff \exists\beta \left( \forall \gamma \left( \exists t \parallel_1^b \beta = \gamma \& t \parallel_1 \chi = \beta \right) \& \exists t \parallel_1^b \beta = \gamma \& t \parallel_1 \chi = \beta \right)$

DEFINITION Let $\chi_p(p, b, c)$ be the characteristic function of the relation $p \parallel^b = \chi$, so that it takes the value 1 if $p \parallel^b = \chi$ and 0 otherwise.

The graph of $\chi_p$ on transitive sets is given by a P-rudimentary recursion.

THE DEFINABILITY LEMMA "f is a $\chi_p$ attempt" is $\Delta_0(P, f)$.

THE PROPAGATION LEMMA Let $F(u) = \chi_p \downarrow((P \times u \times u))$. There is a rudimentary function $H_p$ such that for any transitive $P$, if $P \subseteq P^+ \subseteq P(P)$,

$$F(P^+) = H_p(P, F(P), P^+)$$

Proof of the Propagation Lemma: Let $\Psi(x, f, p, b, c)$ be the $\Delta_0$ formula

$$\forall \beta \in \bigcup^b \exists r \in ((x)_{1+}^2 \{p\}) \left[r \parallel_1^b \beta = \gamma \iff \exists t \in ((x)_{2+}^2 \{r\}) \exists \gamma : \gamma : \bigcup^c \left(f(t, \beta, \gamma) = 1 \& t \parallel_1 \chi = \beta \right) \right]$$

$\& \forall \gamma : \gamma : \bigcup^c \exists r \in ((x)_{1+}^2 \{p\}) \left[r \parallel_1 \gamma \leq \chi \iff \exists t : \exists \beta : \bigcup^b \left(f(t, \gamma, \beta) = 1 \& t \parallel_1 \beta \leq b \right) \right]$

Define $H_p(x, f, v)$ to be

$$\left\{ (0, 1) \times ((x)_0^2 \times (v \times v)) \right\} \cap \left\{ (1, p, b, c)_4 \mid p, b, c \Psi(x, f, p, b, c) \right\} \cup \left\{ (0, p, b, c)_4 \mid p, b, c \neg \Psi(x, f, p, b, c) \right\}$$

8-vii-2010 Rudimentary recursion, provident sets and forcing Bedlewo vii
Propagation of $\chi_\nu$

We have defined the progress $P^c_\nu$ for $c$ a transitive set. We could continue to work with progresses of the above kind, but a problem would then arise at the end of the paper, in the proof that a set-generic extension of a provident set is provident.

Here it is better to work with other progresses, which might be called construction from $c$ as a set and $\chi_\nu$ as a predicate, with the definition of $\chi_\nu$ evolving during the construction.

DEFINITION Let $c$ be a transitive set of which $\mathbb{P}$ is a member; let $\eta = \rho(\mathbb{P})$. We define by a $p$-rudimentary recursion a sequence $((e_\nu, P^{c^+}_\nu, \chi^c_\nu)_\nu)$ of triples, thus obtaining a new progress $(P^{c^+}_\nu)_\nu$.

For every $\nu$, $e_\nu$ will be defined as before; for $\nu \leq \eta$ we set $P^{c^+}_\nu = P^c_\nu$; for $\nu < \eta$, we set $\chi^c_\nu = \emptyset$ but at $\eta$, we set $\chi^c_\eta = \chi_c | P^c_\eta$, which will be a set by the last Corollary. Thereafter we set

$$e_{\nu+1} = e \cap \{x \mid x \subseteq e_\nu\} \quad e_\lambda = \bigcup_{\nu < \lambda} e_\nu$$

$$P^{c^+}_\nu = \mathbb{T}(P^c_\nu) \cup \{e_\nu\} \cup \{\chi^c_\nu \cap P^c_\nu\} \quad P^{c^+}_\lambda = \bigcup_{\nu < \lambda} P^{c^+}_\nu$$

$$\chi^{c^+}_\nu = H^c_\nu(P^c_\nu, \chi^c_\nu, P^{c^+}_\nu) \quad \chi^{c^+}_\lambda = \bigcup_{\nu < \lambda} \chi^c_\nu$$

PROPOSITION Let $e$ be a transitive set, with $\mathbb{P} \in e$, and let $\theta$ be indecomposable and strictly greater than $\rho(\mathbb{P})$. Then $P^c_\theta = P^c_\theta$.

This reconstruction of $P^c_\theta$ shortens the delay for most $\chi^c_\nu$.

PROPOSITION For any ordinal $\nu \geq \eta$, any limit ordinal $\lambda > \eta$ and $k \in \omega$,

$$\chi^c_\nu = \chi_c | P^{c^+}_\nu$$

$$\chi^c_\nu \subseteq P^{c^+}_\nu$$

$$\chi^c_\lambda \subseteq P^{c^+}_\lambda$$

$$\chi^c_\nu \cap P^{c^+}_\nu \subseteq P^{c^+}_\nu$$

Propagating $\chi_c$

We may now define $p \models \exists \beta \in \mathbb{B}$.

DEFINITION $p \models \exists \beta \in \mathbb{B}$ $\iff$ $\exists \beta \in \Omega : \exists t : \exists \beta : t \models \exists \beta \in \mathbb{B}$.

REMARK This is not a definition by recursion: indeed it is visibly rudimentary in $p \models \exists \beta \in \mathbb{B}$.

DEFINITION Let $\chi_c(p,a,b)$ be the characteristic function of the relation $p \models \exists \beta \in \mathbb{B}$.

PROPOSITION There is a natural number $s_1$ such that for each ordinal $\nu \geq \eta$, $\chi_c | P^{c^+}_\nu \subseteq P^{c^+}_{\nu+s_1}$.

Construction of Cohen terms of affine delay for rudimentary functions

THEOREM Let $R$ be a rudimentary function of some number of arguments. Then there is a function $R^p$, of the same number of arguments, with the property that if $A$ is a provident set and $\mathbb{P} \in A$ a notion of forcing, then $A$ is closed under $R^p$ and, further, if $\mathcal{G}$ is an $(A, \mathbb{P})$-generic, then (to take the case of a function of two variables) for all $x$ and $y$ in $A$, $\text{val}_A(R^p(x, y)) = R(\text{val}_A(x), \text{val}_A(y))$.

COROLLARY Let $A$ be provident, $\mathbb{P} \in A$ and $\mathcal{G} (A, \mathbb{P})$-generic. Then $\mathcal{A}^p[\mathcal{G}]$ is rud closed and so a model of $\mathcal{G}_0$.

Propagating of Cohen terms for rud functions

PROPOSITION Let $R$ be a rudimentary function of some number of arguments, and let $R^p$ be the corresponding function of names as given in the above theorem. There is a natural number $s_R$ such that whenever $e$ is a transitive set with $\mathbb{P} \in e$, and $\nu$ is an ordinal not less than $\rho(\mathbb{P})$,

$$R^p | P^{c^+}_\nu \subseteq P^{c^+}_{\nu+s_R}.$$
No new ordinals!

REMARK In section 6 of *The strength of Mac Lane set theory*, a forcing construction is done over a non-standard model \( \mathcal{R} \), and it was there blithely stated without proof that the generic extension would bring no new “ordinals”. Fortunately the model \( \mathcal{R} \) was power-admissible, and therefore certainly a model of PROVI, which is a sub-theory of KP, so that that blithe statement was correct, as forcing over provident sets indeed adds no new ordinals. The reader should be warned, though, that forcing over certain, admittedly pathological, transitive models presented in [M6] leads to generic extensions with, in some cases, more and, in other cases, fewer ordinals than in the ground model.

**Construction of rudimentarily recursive Cohen terms for rank and transitive closure**

Rank and transitive closure are pure rud rec; we show here that P-rud rec Cohen terms exist for them.

Let \( S(\cdot) \) be the basic function \( z \mapsto z \cup \{z\} \).

**Lemma** There is a rud function \( S^p(\cdot) \) such that \( \text{val}_G(S^p(x)) = S(\text{val}_G(x)) \).

**Definition** \( g^p(x) = \{ p, S^p(g^p(y)) \mid (p, y) \in x \land \ p \in \mathbb{P} \} \)

**Remark** \( g^p \) is rud rec in the parameter \( \mathbb{P} \).

**Lemma** Let \( A \) be provident, and \( \mathbb{P} \in A. \) For all \( x \in A \), \( \text{val}_G(g^p(x)) = g(\text{val}_G(x)) \).

**Remark** That all makes sense: if \( x \) is in \( A \), the name \( g^p(x) \) is in \( A \). Note that \( g(\text{val}_G(x)) \) is evaluated in the universe. At present we do not know that the evaluation can be carried out in \( A^\mathbb{P}[G] \).

**Proof**:

\[
g(\text{val}_G(x)) = \bigcup \{ g(y) + 1 \mid y \in \text{val}_G(x) \} \quad \text{definition of } g
\]

\[
= \bigcup \{ g(\text{val}_G(w)) + 1 \mid w \ni \exists p \in G \ (p, w) \in x \} \quad \text{definition of } \text{val}_G(x)
\]

\[
= \bigcup \{ \text{val}_G(\text{val}_G(g^p(w))) + 1 \mid w \ni \exists p \in G \ (p, w) \in x \} \quad \text{induction hypothesis}
\]

\[
= \bigcup \{ \{ (p, S^p(g^p(w))) \mid (p, w) \in x \} \} \quad \text{property of } S^p
\]

\[
= \text{val}_G(\bigcup \{ (p, S^p(g^p(w))) \mid (p, w) \in x \}) \quad \text{by Lemma 6.0}
\]

\[
= \text{val}_G(g^p(x)), \quad \text{by the definition of } g^p.
\]

**Definition** \( tcl^p(x) = \{ (p, z) \mid (p, z) \in x \} \).

**Remark** \( tcl^p \) is rud rec in the parameter \( \mathbb{P} \).

**Lemma** Let \( A \) be provident, and \( \mathbb{P} \in A. \) For all \( x \in A \), \( \text{val}_G(tcl^p(x)) = tcl(\text{val}_G(x)) \).

**Proof**: by similar reasoning.

**Forcing over primitive-recursively closed sets**

Jensen and Karp give, following Gandy, this definition of the class of primitive recursive set functions: there are some initial functions, which are all rudimentary; two versions of substitution: \( F(\bar{x}, \bar{y}) = G(\bar{x}, H(\bar{x}), \bar{y}) \) and \( F(\bar{x}, \bar{y}) = G(H(\bar{x}), \bar{y}) \); and this recursion schema:

\[
F(z, \bar{x}) = G(\bigcup \{ F(u, \bar{x}) \mid u \in z \}, z, \bar{x}).
\]

**Lemma** Let \( A \) be transitive and primitive recursively closed, and let \( F \) be primitive recursive. Then

\[
\text{val}_G(\{ (p, F(y)) \mid p, y (p, y) \in x \}) = \{ \text{val}_G(F(y)) \} \quad \text{by } \exists p \in G \ (p, y) \in x.
\]

**Proof**: as before. \( \vdash \)

For notational simplicity there is only one \( x \) in the following, but it could easily be replaced by a finite sequence.

8·iii·2010 Rudimentary recursion, provident sets and forcing Bedlewo ix
PROP \ (1) Suppose that $G(f, z, x)$ is primitive recursive in the parameter $F$, and that there is a primitive recursive $G^P$ such that for all $f, z, x$ in $A$,

$$\text{val}_G(G^P(f, z, x)) = G(\text{val}_G(f), \text{val}_G(z), \text{val}_G(x)).$$

Suppose that $F(z, x) = G(\cup \{ F(u, x) \mid u \in \mathcal{G}(z), z, x \})$. Define $F^P$ by

$$F^P(z, x) = G^P(\cup \{ (p, F^P(u, x)) \mid p, u \in \mathcal{G}(z), z, x \}).$$

Then $F^P$ is primitive recursive in the parameter $F$, and for all $z, x$ in $A$,

$$\text{val}_G(F^P(z, x)) = F(\text{val}_G(z), \text{val}_G(x)).$$

Proof: for fixed $x$ by recursion on $z$:

$$F(\text{val}_G(z), \text{val}_G(x)) = G(\cup \{ F(w, \text{val}_G(x)) \mid w \in \text{val}_G(z) \}, \text{val}_G(z), \text{val}_G(x))$$

$$= G(\cup \{ F(\text{val}_G(u), \text{val}_G(x)) \mid u \in \text{val}_G(z) \}, \text{val}_G(z), \text{val}_G(x))$$

$$= G(\cup \{ G^P(\cup \{ (p, F^P(u, x)) \mid p, u \in \mathcal{G}(z), z, x \}) \}, \text{val}_G(z), \text{val}_G(x))$$

$$= G^P(\cup \{ (p, F^P(u, x)) \mid p, u \in \mathcal{G}(z), z, x \}), \text{val}_G(z), \text{val}_G(x))$$

$$= \text{val}_G(F^P(z, x))$$

The above confirms an observation made some years ago by Jensen:

COROLLARY A set-generic extension of a primitive recursively closed set is primitive recursively closed.

Construction of Cohen terms for the stages of a progress.

Let $e$ be a transitive set in the ground model of which $P$ is a member, and let $\theta$ be indecomposable, exceeding the rank of $e$. $P^e_\theta$ is provident. Let $d$ be the Cohen term $e \cup \{ \theta \}^P$, so that $\text{val}_G(d)$ will be the transitive set $d = e \cup \{ \theta \}$.

REMARK $d$ will be a member of $P^e_{\theta(P)+k}$ for some (small) $k$, given the definition of $\theta$, our convention that $\theta = 1$ and the fact that $\theta$ is $\mathfrak{d}$-rud rec.

Our task is to build for each $\nu < \theta$ a name $N(\nu)$ for the stage $P^d_\nu$ of the progress towards $d$.

A simplified progress

Now $\mathcal{G}(\theta) \leq \mathcal{G}(P) < \mathcal{G}(P)$, so that for $\nu \geq \eta, d_\nu = e_\nu \cup \{ \mathcal{G} \}$. It might be that $\mathcal{G}(\theta) < \mathcal{G}(P)$; to avoid building names which make allowance for that uncertainty, we shall build names for the terms of a slightly different progress $(Q^d_\nu)_{\nu}$.

DEFINITION\n
For $\nu < \eta$, $Q^d_\nu = P^e_\nu$; $Q^d_\eta = P^e_\eta \cup \{ \mathcal{G} \}$; for $\nu \geq \eta$, $Q^d_{\nu+1} = \mathcal{T}(Q^d_\nu) \cup \{ d_\nu \} \cup d_{\nu+1}$; $Q^d_\lambda = \bigcup_{\nu < \lambda} Q^d_\nu$ if $\lambda = \lambda \geq \eta$.

PROPOSITION If $\theta$ is indecomposable, then $Q^d_\eta$ is provident and equals $P^d_\theta$.

Generic extensions of provident sets and of Jensen fragments

THEOREM Let $\theta$ be an indecomposable ordinal strictly greater than the rank of a transitive set $e$ which contains the notion of forcing, $P$. Let $\mathcal{G}$ be $(P^e_\theta, \mathcal{P})$-generic. Then $(P^e_\theta)^{\mathcal{G}} = \mathcal{P}^{\mathcal{G}}$ and hence is provident.
Proof : \((P^c_\theta)^P[G]\) contains \(P^c_\theta(G)\), as we have for each \(\nu < \theta\) built a name in \(P^c_\theta\) that evaluates under \(G\) to \(Q^c_\nu(G)\), and we know by the previous Proposition that \(Q^c_\nu(G) = P^c_\theta(G)\).

For the converse direction, we know that \(P^c_\theta(G)\) is provident, and has \(G\) as a member and hence can support the \(G\)-rudimentary recursion defining \(val_G(\cdot)\). Further \(P^c_\theta(G)\) includes \((P^c_\theta)_\nu\), which is defined by an \(e\)-rudimentary recursion, and so includes \((P^c_\theta)^P[G]\). ⊣

REMARK Thus, in this special case, a generic extension of a model of \(\text{PROVI}\) is a model of \(\text{PROVI}\). We shall use this result to establish it more generally.

REMARK The theorem remains true if the hypothesis on \(\theta\) is weakened to requiring that \(\theta > \rho(P)\).

Proof that a generic extension of a provident set is provident.

THEOREM Let \(A\) be provident, \(P \in A\) and \(G(A, P)\)-generic. Then \(A^P[G]\) is provident.

Proof : Let \(\theta = \text{df} \cap A\) and let

\[ T = \{c \mid c \in A \& c \text{ is transitive } \& P \in c\}. \]

Then

\[ A = \bigcup \{P^c_\theta \mid c \in T\}, \]

since the union on the right contains each element of \(A\) and is contained in \(A\). It follows that

\[ A^P[G] = \bigcup_{c \in T} (P^c_\theta)^P[G] \]

By the previous theorem, as each \(P^c_\theta\) is provident and contains \(P\),

\[ A^P[G] = \bigcup_{c \in T} P^c_\theta(G) \]

and each \(P^c_\theta(G)\) is provident. Now in [M4, Proposition 5.52] we proved the

LEMMA If \(\theta\) is indecomposable and \(D\) is a collection of transitive sets each of rank less than \(\theta\) and such that the pair of any two is a member of a third, then \(\bigcup_{c \in D} P^c_\theta\) is provident.

Take \(D = \{c \cup \{G\} \mid c \in T\}\) to complete the proof. ⊣

Forcing over models of Mac Lane set or Zermelo set theory

The present treatment of set forcing does not work satisfactorily over a transitive but improvident model of, say, Zermelo set theory, as is shown in [M6]; the difficulties would be removed by passing first to the provident closure of the given model.

The main technical tool of an earlier paper, [M2], was a construction that starting from any model \(M\) of \(M_0\) yields a model, which in [M6] we call Lune(\(M\)), of \(M_1 + \) every well founded extensional relation is isomorphic to a transitive set; it follows that if \(M\) models the well-ordering principle, then Lune(\(M\)) will model KP.

Further, \(M\), if it models \(M_1\), will be interpretable as a submodel of Lune(\(M\)); and it is shown in [M6] that if \(M\) models \(M\), Lune(\(M\)) will model PROVI; so that we then have a second way of extending \(M\) to a model supporting our presentation of forcing.

The passage from \(M\) to Lune(\(M\)) can be considered in two ways. If \(M\) is a transitive set and a \(\beta\)-model in the sense that every binary relation \((a, r) \in M\) that is considered by \(M\) to be extensional and well-founded is genuinely well-founded, we can form the larger model by taking every such \((a, r)\), transitising it, and taking the union of the resulting transitisations.

EXAMPLE Provably in \(ZF\), Lune(\(V_{\omega+\omega}\)) = \(H_{\omega_1}\).
If \( M \) is not a \( \beta \)-model, the construction is still possible: we summarise the method in [M2].

\[ \text{HISTORICAL NOTE} \quad \text{A similar construction, though differing in various details, is presented in Hinnion’s thesis [H] which starting from models of NF yields models of subsystems of ZF.} \]

**Construction of the lune**

We note first that the existence of cartesian products is provable in \( M_0 \):

**PROPOSITION** \( \vdash_{M_0} \) the Cartesian product of two sets is a set.

**DEnFInItIoN** Put

\[
F_1 =_{df} \{ (a, r) \mid r \text{ is an extensional well-founded relation on } a \}
\]

and

\[
W_1 =_{df} \{ (\alpha, a, r) \mid (a, r) \in F_1 \text{ and } \alpha \in a \}.
\]

The elements of \( W_1 \) will be the ingredients of our model: we think of \( (a, r) \) as denoting, in the model we are building, a transitive set of which \( \alpha \) names a member. We must say when two names denote the same element. To assist our intuition we write \( \beta \in_\alpha \alpha \) for \( \beta \in r^{-1}\{\alpha\} \), i.e. \( (\beta, \alpha) \in r \), when \( (a, r) \in F_1 \) and \( \beta \) and \( \alpha \) are in \( a \).

**DEnFInItIoN** Given \( (a, r), (b, s) \in F_1 \), a map \( \phi \) from a subset of \( a \) to a subset of \( b \) will be called a partial isomorphism from \( (a, r) \) to \( (b, s) \) if

(i) \( r^\alpha \text{dom } \phi \subseteq \text{dom } \phi \); 
(ii) for all \( \alpha \in \text{dom } \phi \), \( \phi(\alpha) = =^{_{\text{r}}}(\phi(\beta) \mid \beta \in r^{-1}\{\alpha\}) \); i.e., for all \( \beta \in_\alpha \alpha \), \( \phi(\beta) \in_\alpha \phi(\alpha) \), and for all \( \delta \in_\alpha \phi(\alpha) \) there is a \( \beta \in_\alpha \alpha \) with \( \phi(\beta) = \delta \).

Thus the condition (ii) on \( \phi \) is that \( \{ \delta \mid \delta \in_\alpha \phi(\alpha) \} = \{ \phi(\beta) \mid \beta \in_\alpha \{\alpha\} \} \); or, more succinctly, that \( s^{\alpha}(\phi(\alpha)) = \phi^\alpha r^{-1}(\alpha) \). Note that, provably in \( M_0 \), the property of being a partial isomorphism is \( \Delta_0 \).

**LEMMA** The class of partial isomorphisms from \( (a, r) \) to \( (b, s) \) is a \( \Delta_0(a, r, b, s) \) subset of \( \mathcal{P}(a \times b) \).

**LEMMA** A partial isomorphism is 1-1 as far as it goes.

**LEMMA** Given \( (a, r) \) and \( (b, s) \) in \( F_1 \), any two partial isomorphisms from \( (a, r) \) to \( (b, s) \) agree on their common domain.

**LEMMA** There is a largest partial isomorphism from \( (a, r) \) to \( (b, s) \).

**Proof**: The union of all partial isomorphisms from \( a \) to \( b \) is a set by Lemma 2.9; its domain is closed under \( r^\alpha \), and, by the last lemma, that union is also a partial isomorphism — hence maximal and unique. \( \dashv \)

We write \( \Psi_{arb\alpha} \) for the maximal partial isomorphism from \( (a, r) \) to \( (b, s) \). The following properties are easily checked.

**LEMMA** \( \Psi_{arb\alpha} = id \vert a; \quad \Psi_{arb\alpha}^{-1} = \Psi_{bsar}; \quad \Psi_{arb\alpha}\Psi_{bsect} \subseteq \Psi_{arcT} \).

Now define two relations on \( W_1 \):

\[
(\alpha, a, r) \equiv_1 (\beta, b, s) \iff_{df} \alpha \in \text{dom } \Psi_{arb\alpha} \& \Psi_{arb\alpha}(\alpha) = \beta,
\]

and

\[
(\alpha, a, r)E_1(\beta, b, s) \iff_{df} \alpha \in \text{dom } \Psi_{arb\alpha} \& \Psi_{arb\alpha}(\alpha) \subseteq \beta.
\]

**REmARK** If \( (a, r) \) and \( (c, t) \) are two members of \( F_1 \) in a set \( A \), then any partial isomorphism \( f \) between them is a member of \( \mathcal{P}(\bigcup^0 A \times \bigcup^1 A) \), and hence locally the relations \( \equiv_1 \) and \( E_1 \) are sets.

The above Lemma may be applied to prove the

**PROPOSITION** \( \equiv_1 \) is an equivalence relation, and a congruence with respect to \( E_1 \).

**PROPOSITION** \( E_1 \) is set-like in the sense that given any \( x \) in \( W_1 \), there is a set \( y \) with \( x \in y \) such that

\[
\forall z(z E_1 x \iff \exists w : \in y \ z \equiv_1 w).
\]
Proof: Given $x = (\alpha_0, a_0, r_0)$, let $y = \{(\alpha, a_0, r_0) \mid \alpha \in a_0\}$, which is a set, as it equals $a_0 \times \{(a_0, r_0)\}$. If $(\beta, b, s)E^1(\alpha, a_0, r_0)$, let $\gamma = \Psi_{\Delta_1^0}(\beta)$: then $(\gamma, a_0, r_0)$ is in $y$ and $(\beta, b, s) \equiv^1 (\gamma, a, r)$.

REMARK The equivalence classes under $\equiv^1$ will be proper classes and not sets; in our weak set theory, Scott’s device for choosing subsets of these proper classes is not available to us. Hence all the details of our construction have to be treated locally.

If $M$ is a set, and the ambient set theory is sufficiently strong, then we may form, externally to $M$, the set of equivalence classes under $\equiv^1$, and define on that set the relation induced by $E^1$.

**Definition** $\text{Lune}(M) = \text{def}$ the resulting model.

The commutativity of the operators $\text{Lune}(-)$ and $\text{val}_G(-)$

**Proposition** ([M6]) Let $M$ be a transitive model of $\mathcal{M}$, and let $H = \text{Lune}(M)$, which might be ill-founded. Then the (transitised) standard part of $H$ is provident and thus includes $\text{Prov}(M)$.

**Theorem** ([M6]) Let $I$ and $H$ be transitive sets such that $I$ is a provident model of $\mathcal{Z}$ or at least $M$, and $H = \text{Lune}(I)$. Let $P \in I$ and let $G$ be $(I, P)$ generic. Then $G$ is $(H, P)$ generic, and further $H^P[G] = \text{Lune}(I^P[G])$.

Thus in favourable circumstances the following diagram commutes:

\[
\begin{array}{ccc}
H & \xrightarrow{\text{val}_G} & H^P[G] \\
\uparrow \text{Lune} & & \uparrow \text{Lune} \\
P \in I & \xrightarrow{\text{val}_G} & I^P[G]
\end{array}
\]

**References**


[Hi] R. Hinnion, Sur la théorie des ensembles de Quine, thèse, Université Libre de Bruxelles, 1974/5.


8–iii–2010 ................. Rudimentary recursion, provident sets and forcing ................. Bedlewo xiii