Determinacy and Identification with Optimal Rules *

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Abstract

This paper contributes to the McCallum-Cochrane debate on the determinacy of Taylor rules in two ways. Firstly, it shows that Cochrane’s critique of rules does not necessarily generalize to all simple rules. Secondly, it shows that when fully optimal or timeless rules are formulated in a transparent manner, then Cochrane’s critique always applies. The implication is that fully optimal or timeless rules will always be subject to local instability.

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1 Introduction and Literature Review

This paper reviews the McCallum-Cochrane debate on the determinacy of simple rules - both generalizing the Cochrane (2011) results to any linear rational expectations model, and then extending the ideas to optimal commitment rules.

McCallum (2009a), Cochrane (2009), and Cochrane (2011) start from the result that the New Keynesian (hereafter NK) model, where monetary policy is governed by a simple interest rate rule of the Taylor form, has multiple solutions. There will be explosive or non-local solutions with explosive inflation, and these cannot be ruled out by transversality conditions that are concerned with real variables only. The issue is then whether it is it possible to rule out all solutions other than the locally bounded one. McCallum (2009a) suggests that ”learnability” may be sufficient where the rule satisfies the Taylor principle. Cochrane (2009) and Cochrane (2011) counters that McCallum’s argument assumes that the monetary shock is observable, which it is not. Taking away that assumption, he shows that the Taylor rule is not identified and, moreover, that McCallum’s determinacy results with learning are reversed; under the Taylor principle, the locally bounded equilibrium is the only one that is not learnable. On the other hand, all non local equilibria appear to be learnable.

We firstly analyse more general linear models than that discussed by the two previous authors, obtaining a generalization of their results for simple rules.

As is detailed in the next section, both authors largely confine themselves to a simple NK model comprising a Fisher equation and a Taylor rule for interest rates dependent on inflation and a monetary policy shock where is latter is governed by an AR(1) process. The present paper generalises the results reported so far for the case of simple rules and extends the analysis used by the two authors to more general linear models. Next, it analyses the conduct of optimal monetary policy for commitment rules using the taxonomy of decision frameworks for monetary policy described in Svensson and Woodford (2004) in three categories. First, a general targeting rule that specifies target variables, desired levels of such variables a set of instruments and a loss function defined in terms of these three elements. The second, a specific targeting rule, specifies a criterion involving endogenous variables that must be satisfied along the path of the optimal equilibrium. The third, an explicit instrument rule, specifies the exact setting of the policymakers’ instruments as a function of endogenous or exogenous variables along the equilibrium path.

From these the examples we concentrate on are the explicit instrument rules (although, in our judgement, the analysis is applicable to the other two ways of implementing the optimal equilibrium) For this, we propose four desirable properties for assessing instrument rules. These are transparency (rules expressed purely in terms of macroeconomic variables that are part of the private sectors assumed information set), implementability (rules that are both transparent and saddle-path stable), timelessness (invariance of the form of rule for $t \geq 0$), and finally robustness (rules that are independent of the exogenous shock processes).
An illustration of their use is provided in what is a new result for the literature when we show that the unique general form of the timeless and transparent rule as presented by Currie and Levine (1993) is not necessarily implementable. We go on to show that there are non-standard, multiple forms of the timeless, transparent rule that are saddle-path stable and therefore implementable. The multiplicity of such equilibria then creates an identification problem for the econometrician setting out to estimate such a rule, and is therefore subject to the Cochrane critique; we also note that these results also apply to the universal optimal rule as defined by Damjanovic et al. (2008). We then show that the ‘robustness criterion’ first introduced by Woodford (2003) only applies to the transparent timeless rule in set-ups such as the standard New Keynesian model without backward-looking variables such as capital stock.

This paper is by no means the first to criticise the timeless approach to optimal policy largely associated with Woodford (2003). Most notably Dennis (2010) and Blake and Kirsanova (2004) have shown that the timeless rule may in some circumstances yield a higher welfare loss than the non-precommitment time-consistent policy.

More broadly, central bank transparency is a vital requirement in the New Keynesian Taylor rule model as it is clearly essential in ensuring central bank credibility and thereby its management of agents expectations (see Geraats (2005) and Eijffinger and Geraats (2006)). Thus, in the highly influential account of the transmission mechanism by two leading New Keynesian exponents, Gali and Gertler (2007), it is noted that, since the structural equations of the model depend on forward-looking expectations of households and firms, then current values of output and inflation depend not on the current value of the policy rate of interest but on its expected future path.

The present extension contributes further important considerations to this. One is to argue that it would be inappropriate for policy makers to use the timeless form of the policy rule as this appears subject to the Cochrane critique in general. Furthermore, for any system other than that of a purely forward-looking one, our analysis makes clear that if the rule is announced by the authorities, it needs to be sufficiently simple in the sense we describe. More complicated rules lead to local instability and thus do not lend themselves to ensuring transparency.

Section 2 outlines the McCallum-Cochrane debate, and then extends the results of Cochrane (2011) to more general rational expectations (RE) models. Section 3 lays the ground for our more general results on optimal policy by showing how the optimal rule can be presented in an infinite number of equivalent structural forms even for the simplest New Keynesian model. Section 4 extends the Cochrane critique to fully optimal (and timeless) policy for the general linear RE case with quadratic objective function. Section 5 concludes.

1In a model that incorporates a wealth effect from government debt, and therefore both forward and backward-looking variables, they utilize the timeless form of the optimal conditions to obtain a representation of the rule.
For simple rules Cochrane (2009) notes that the Blanchard and Kahn (1980) conditions are insufficient to pin down the solutions to rational expectations models. While McCallum (2009a) agrees with this, he claims that learnability in the sense of Evans and Honkapohja (2000) gets round this problem. Cochrane (2011) disputes whether learnability can be applied in the basic New Keynesian model because of identifiability problems for the simple rule. Cochrane (2009) uses the following to illustrate his point:

Cochrane uses the following simplified NK model to illustrate his arguments. Assuming full price flexibility so that \( y_t = \bar{y}_t \) for all \( t \), where \( y_t, \bar{y}_t \) represent output and the flex-price natural rate of output, respectively. To further simplify, the natural rate is assumed to be constant. This means that in the standard three equation NK model (comprising an AD based on the first order for households, a Calvo form of the AS equation and an interest rate rule depending on the output and inflation gap), the Calvo equation disappears and, where the real interest rate is constant, the AD equation becomes the Fisher equation,

\[
R_t = E_t \pi_{t+1} \tag{1}
\]

where \( R_t \) is the nominal interest rate, \( \pi_t \) is inflation and \( E_t \) represents expectations given the measurement set at time \( t \). In turn, as the output gap is assumed zero, the central bank policy rule can be written as a Taylor rule, including serially correlated shocks \( e_t \):

\[
R_t = \frac{1}{a} \pi_t + e_t \quad e_t = \rho e_{t-1} + \varepsilon_t \tag{2}
\]

where \( \{\varepsilon_t\} \) is a sequence of white noise and \( 0 < \rho < 1 \). It is easy to show that if \( a < 1 \) then the saddlepath solution yields

\[
\pi_t = \frac{1}{1 - \rho a} e_t \tag{3}
\]

However the latter is only one of an infinite set of possible solutions that are given by

\[
\pi_{t+1} = \frac{1}{a} \pi_t + e_t + \delta_{t+1} \tag{4}
\]

and \( \delta_{t+1} \) is any random variable with mean 0.

For (3) to be a valid solution, McCallum (2009a) argues that the system should be learnable. By this it is meant that the expectation for \( \pi_{t+1} \) is formed using a recursive discounted least squares estimate of (3) recursively, i.e. in period \( t \) estimate

\[
\hat{\pi}_t = \hat{\phi}_t e_t \quad \text{where} \quad \hat{\phi}_t = \frac{\sum^{t-1}_{\tau=1} \lambda^{t-1-\tau} e_{t-1} \hat{\pi}_t}{\sum^{t-1}_{\tau=1} \lambda^{t-1-\tau} e^2_{t-1}} \tag{5}
\]

and using this estimate form the expectation under learning:

\[
\hat{\pi}_{t+1} = \hat{\phi}_t \rho e_t \tag{6}
\]
This is known as learnable if $\phi_t$ converges to the parameter associated with the rational expectations solution $\pi_t = \frac{1}{1-ae_t}$, i.e., if $\phi_t \to \frac{1}{1-pa}$. The conditions for learnability of the stationary solution turn out to be identical to those for saddlepath stability, so in his response to Cochrane (2009), McCallum (2009b) argues that in the simple model the explosive equilibria are not learnable (as claimed by Cochrane (2009), although not central to the debate) and the unique bounded equilibrium is the only learnable one.

Cochrane’s critique is then two related but conceptually distinct issues, and are to do with information sets and identification. Thus, on the first point he argues that McCallum is in error about the information sets available to the public as he (McCallum) assumes they can directly observe the monetary policy shock. Making the opposite assumption, that the policy disturbance is not directly observable, so that agents must run regressions to measure it, gives the opposite result; explosive equilibria are learnable (although he is wrong on this score - see below), but more importantly, the unique local equilibrium is not learnable. On the second, he notes that the result above ”is closely tied to identification”.

At this point it is useful to distinguish between the two issues of identifiability and learnability. Identifiability specifically refers to whether all of the parameters of a particular set of equations of the system (or alternatively the system as a whole) can be estimated consistently; for our purposes, and in general for Cochrane, this is narrowed to identifiability of the rule. Learnability is also in part about estimation, but only applies to estimation of the relationship between the forward-looking variables and the backward-looking variables. McCallum (2007) proves that determinacy is a sufficient condition for learnability; Ellison and Pearlman (2011) narrow this down to the same sufficient condition when agents learn about the saddlepath (and also extend this to any information set)\(^2\). Technically the latter is a weak e-stability result, whereas the former is a strong e-stability result. The reason why agents are interested in an estimated relationship for the forward-looking variables e.g. $\pi_t$, is because they can then project forward to form expectations of next period’s $\pi_{t+1}$.

The obvious implication of Cochrane (2009) and Cochrane (2011) is that a necessary condition for a system to exhibit saddlepath stability is that

- the system is learnable
- the rule is identifiable

The reason for the second condition is that if the rule is not identifiable, then it is only directly relevant off the saddlepath equilibrium; on the saddlepath equilibrium it is observationally equivalent to an infinite number of other structurally equivalent rules - in the case considered above, for any value $a > 1$. If the policymaker either makes a mistake, or deliberately chooses $a < 1$, but agents believe $a > 1$, then there is a unique equilibrium; if agents always wish to check that $a > 1$, then they must on occasion form expectations that are inconsistent with the unique equilibrium. Once this (aggregate) expectational bubble

\(^2\)Ellison and Pearlman (2011) show that in the case of indeterminacy the sunspot solution is also learnable, so that one cannot use the learnability criterion to isolate any particular equilibrium in this case.
is formed, the system then evolves with dynamics that are dependent on $a$, and it can then be estimated. Not surprisingly, it can be shown that if $a > 1$, then subsequent expectations formed using discounted OLS do not converge to rational expectations, but the essence of Cochrane’s argument would appear to be that once the estimate of $a$ is seen to be converging to a number greater than 1, then agents will prick the bubble, and the system will return to saddlepath stability.

Note that Cochrane’s critique is still valid if the interest rate rule is expressed as

$$R_t = \alpha R_{t-1} + \frac{1}{a} \pi_t + e_t$$  \hspace{1cm} (7)

where $\alpha < 1$. In this case, the saddlepath relationship links $\pi_t$ to $R_{t-1}$ and $e_t$. Any multiple of this added to (7) retains the structure of (7), so that the identification problem remains at the heart of learnability if $e_t$ is unobserved.

2.1 A General Result for Simple Rules

Suppose that the system is given by

$$\begin{bmatrix} z_{t+1} \\ E_t x_{t+1} \end{bmatrix} = A \begin{bmatrix} z_t \\ x_t \end{bmatrix} + B w_t + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$ \hspace{1cm} (8)

where $z$ represents backward-looking variables, $x$ represents forward-looking variables, and $w$ are the policy variables.

Suppose for simplicity that there is only one policy variable (although the result below trivially extends to any number), and that there is now a simple rule in place, of the form $w_t = a^T z_t + b^T x_t$, where $z_t$ includes $w_{t-1}$. Also assume that the rule ensures that the system is saddlepath stable i.e. the number of unstable eigenvalues of the system exactly matches the number of forward-looking variables. We then have the following:

**Theorem 1:**

The system is learnable, and the rule is identified provided that the number of non-zero elements of $[a^T \ b^T] \leq$ the number of backward-looking variables (apart from a set $\{a^T, \ b^T\}$ of measure zero).

**Proof:** Using McCallum (2007) and Ellison and Pearlman (2011), we have the result that saddlepath stability imples learnability. The saddlepath condition can be written as

$$x_t + N z_t = 0$$ \hspace{1cm} (9)

where $[I \ N]$ is a linear combination of the eigenvectors corresponding to the set of unstable eigenvalues of the system; we note that a characteristic of eigenvectors is that apart from a set of measure zero, any submatrix of $[I \ N]$ has the property that it is of full rank i.e. its rank is equal to the minimum of the number of rows or columns, and in particular this
implies that all elements of \( N \) are non-zero. Firstly then, if \( b^T = 0 \), there is no linear combination of \( x_t + N z_t \) that can be added to the rule such that the rule still omits \( x_t \). Similarly, if there is one non-zero element of \( b^T \), and one zero element of \( a^T \), then the same argument applies. Simple logic then extends this argument to any number of non-zero elements of \( b^T \).

**Corollary 1**

When the rule being used is the time-consistent optimal rule and the resultant system is saddlepath stable, then the Cochrane critique does not apply.

**Proof:** This is because the time-consistent optimal rule (see Currie and Levine (1993)) is Markov perfect and depends only on the variables \( z_t \).

**Corollary 2**

If \( z_t^T = [z_{U,t}^T z_{O,t}^T] \) where \( z_U \) represents unobservable shocks, and \( z_O \) represents observable economic variables, then the system is learnable, and the rule \( w_t = a^T z_{O,t} + b^T x_t \) is identified provided that the number of non-zero elements of \( [a^T b^T] \) is less than the number of variables \( z_{O,t} \) (apart from a set \( \{a^T b^T\} \) of measure zero).

**Proof:** Substantially the same as for the theorem, except that one would use discounted instrumental variables for learning, rather than discounted OLS. If all shocks are AR(1) processes, then one would use one-period lagged variables as the instruments.

Thus we can conclude that if a rule is sufficiently simple, then it satisfies the necessary conditions for local stability.

**Example:**

Consider the NK model with habit:

\[
\begin{align*}
y_t - h y_{t-1} &= E_t(y_{t+1} - h y_t - \sigma(i_t - E_t \pi_{t+1})) \\
\pi_t &= E_t \pi_{t+1} + \gamma y_t \\
i_t &= \phi \pi_t + \varepsilon_t
\end{align*}
\]

where the last equation is a simple rule with \( \phi > 1 \) and \( \varepsilon_t \) is white noise.

Substituting (12) into (10), one can easily write this in state space form. It is well known that this system has exactly one stable root, which means that the jump variables \( y_t \) and \( \pi_t \) can be written in terms of the pre-determined variables \( y_{t-1} \) and \( \varepsilon_t \):

\[
\begin{align*}
y_t &= \alpha_1 y_{t-1} + \alpha_2 \varepsilon_t \\
\pi_t &= \beta_1 y_{t-1} + \beta_2 \varepsilon_t
\end{align*}
\]

Can we now estimate \( \phi \) in (12)? Clearly not by OLS, because from (14) \( \pi_t \) is correlated with \( \varepsilon_t \). However \( y_{t-1} \) is uncorrelated with the latter, so we can use it as an instrumental
variable, so that
\[ \hat{\phi}_{IV} = \frac{\sum_t t y_t - 1/N}{\sum_t \pi_t y_t - 1/N} \] (15)
and we note that \( \sum_t \pi_t y_t - 1/N \) is well-defined because the OLS estimator of \( \beta_1(\neq 0) \) in (14) is given by
\[ \hat{\beta}_{OLS} = \frac{\sum_t \pi_t y_t - 1/N}{\sum_t y_t^2 - 1/N} \] (16)
Note that had the rule (12) been a Taylor rule \( i_t = \phi \pi_t + \theta y_t + \varepsilon_t \), \( \phi \) and \( \theta \) would not have been identifiable. This is because the saddlepath solution would have been qualitatively the same as (13)-(14), from which it follows that \( \beta_1 y_t - \alpha_1 \pi_t = (\beta_1 - \alpha_1) \varepsilon_t \); adding any multiple of this to the Taylor rule leaves the latter structurally unchanged.

A further elaboration of this makes the need for simplicity of rules even starker. If agents know the parameters and structure of the rest of the economy, then it turns out that a Taylor rule feeding back on both inflation and output is sufficient to pass the identifiability test. This is because the parameter \( \alpha_1 \) is then a function \( \alpha_1 = \alpha_1(\theta, \phi) \) of the system parameters and of the Taylor rule parameters. There is no further information that can be gleaned about \( (\theta, \phi) \) from estimation of \( \beta_1 \) from (14) because from (11) one can see that in the limit \( \beta_1 = \gamma \alpha_1 / (1 - \alpha_1) \). However estimation of the interest rate rule in the reduced form \( i_t = \lambda y_{t-1} + \varepsilon_t \) provides information on the Taylor rule parameters because \( \lambda = \lambda(\theta, \phi) \). On the other hand, if the Taylor rule includes \( y_{t-1} \) as well, then there are three Taylor rule parameters; since the reduced form of the rule is still in terms of \( y_{t-1} \) it follows that these three parameters are not identifiable.

Thus as mentioned earlier, if the rule is simple enough then it avoids the Cochrane critique, although the degree of simplicity needed may depend on agents’ knowledge of the other parameters.

### 3 Optimal Rules for the Basic New Keynesian Model and the Cochrane Critique

Consider the basic New Keynesian model, with a Phillips curve that arises from firms choosing not to re-optimize prices at each period and an aggregate demand curve given by an Euler equation, with \( \pi_t, y_t \) and \( r_t \) representing inflation, the output gap, and the interest rate respectively.

\[ \pi_t = E_t \pi_{t+1} + y_t + \nu_t \] (17)
\[ y_t = E_t y_{t+1} - \sigma (r_t - E_t \pi_{t+1}) \] (18)
where \( \nu_t \) is associated with a taste shock, and \( E_t \) denotes expectations based on full information at time \( t \). For convenience we have assumed that the consumers discount rate is 0, and we have also assumed for expositional purposes that the coefficient on the output gap
in (17) is unity.

Now consider a policymaker whose objective is to minimize the welfare loss function (without discounting, for convenience)

\[ L = \sum \frac{1}{2} (q\pi_t^2 + y_t^2) \]  

(19)

subject to (17) and (18). Because the interest rate is not costed in the welfare loss, it turns out that we can ignore the constraint (18), and just consider the Lagrangian problem

\[ L = \sum \frac{1}{2} (q\pi_t^2 + y_t^2) - \lambda_{t+1} (\pi_{t+1,t} - \pi_t + y_t - v_t) \]  

(20)

where we assume for convenience that the output gap is the instrument, although it is merely a proxy, and the interest rate may be obtained by substituting into the Euler condition (18).

First-order conditions are

\[ q\pi_t - \lambda_t + \lambda_{t+1} = 0 \quad y_t - \lambda_{t+1} = 0 \]  

(21)

with a first-order condition \( q\pi_0 + \lambda_1 = 0 \) in the initial period \( t = 0 \), which represents the time inconsistency problem. We note that the two focs provide the dynamics of \( y_t \) as a representative solution to the timeless problem:

\[ y_t = y_{t-1} - q\pi_t \]  

(22)

and this equation can also be described as the optimal condition for the policy problem.

This representation of the policy rule is transparent, in that it can be expressed in terms of observable macroeconomic variables i.e they are part of the private sector’s assumed information set. It is also implementable in that it is also saddlepath stable. This particular representation of the timeless solution is also robustly optimal in that it is independent of the shock process, although this issue is irrelevant to the main results of the paper.

In order to connect to Cochrane’s critique, let us first consider the case with no shock, \( v_t = 0 \) for all \( t \). It is useful at this stage to define the value function for the case of perfect foresight (\( \pi_{t+1,t} = \pi_{t+1} \), which corresponds to a backward-looking system), which is given by \( \frac{1}{2} s\pi_t^2 \) where \( s > 0 \) is given by the Riccati equation

\[ s = q + \frac{s}{1 + s} \]  

(23)

\[ ^3 \text{Henceforth we use the notation } x_{t+1,t} \text{ for } E_t x_{t+1}. \]

\[ ^4 \text{Any change to interest rate has a first-round effect on output, followed later by the impact on prices. } \]
It is then easy to show that along the optimal path there is an equilibrium relationship\(^5\)

\[
y_t = \frac{s}{1 + s} \pi_t
\]  

(24)

Note that this cannot represent the \textit{policy rule} under RE because that would imply a relationship \(\pi_{t+1} = \frac{1}{1+s} \pi_t\), which has a stable root and therefore would lead to indeterminacy. However, given this equilibrium relationship we could formally divide up \(y_t = \mu y_t + (1 - \mu) y_t\) for any \(\mu\) and (22) can also be written as

\[
\mu y_t = y_{t-1} - \frac{(1 - \mu)s}{1 + s} \pi_t - q\pi_t
\]

(25)

Thus we get the result that there is a continuum of representations of the optimal rule that all have the same structure, with (22) as a special case of (25). Coupled with the NKPC, it produces the same behaviour of \(\pi_t, y_t\)\(^6\) for all \(\mu\).\(^7\)

Note that the eigenvalues of the system (although not the equilibrium relationship \(y_t = \frac{s}{1 + s} \pi_t\)) depend on \(\mu\), and are given by \(\frac{1}{1+s}\) and \((1 + s)/\mu\). Clearly for \(\mu\) between 0 and \(1 + s\) the system is saddlepath stable, with the stable eigenvalue equal to \(\frac{1}{1+s}\) for all \(\mu\). However for \(\mu > 1 + s\) both eigenvalues will have modulus less than 1, implying that the system under the timeless rule is indeterminate. This immediately connects up with the Cochrane critique, as we have now shown that the system can be saddlepath stable - and therefore learnable - but the rule cannot be identified. Only if we allow for local instability will the optimal (or timeless) rule be identifiable. But if the optimal rule is locally unstable, this means that it cannot be optimal!

Intuitively, we can regard the implication of this in the following way. Since there is a continuum of rules that represent the timeless rule, this means that it is impossible for the econometrician and the private sector to identify this rule, and must therefore take it on trust. But if the private sector is unwilling to trust the policymaker, and is also unable to verify the rule by using data, it will therefore be unable to rule out the possibility of an implementation of a rule that has two stable eigenvalues, which introduces indeterminacy and the generation of sunspots. Thus we can imagine a scenario in which agents test the optimal policy by generating sunspot expectations; if the rule is wrongly implemented, then these sunspots die out, and agents will be tempted to generate profitable sunspots in the future. If the rule is correctly implemented, then these sunspots will produce explosive

\(^5\)The perfect foresight case uses the value function iteration \(s\pi_t^2 = max_{y_t}(y_t^2 + q\pi_t^2 + s\pi_t^2)\) and this generates (24) and (23).

\(^6\)One can write the state space setup as \[
\begin{bmatrix}
y_{t+1,1} \\
\pi_{t+1,1}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\mu} & -\frac{(1-\mu)s+q}{\mu(1+s)} \\
-\frac{1}{\mu} & 1 + \frac{(1-\mu)+q}{\mu(1+s)}
\end{bmatrix} \begin{bmatrix}
y_{t-1,1} \\
\pi_t
\end{bmatrix} \] (after substituting for \(q\) from (23)). The eigenvalues of the matrix are \(\frac{1}{1+s}\) and \((1 + s)/\mu\); assuming the latter is greater than 1, its left eigenvector then implies \(y_{t-1} = s\pi_t\) which, when substituted into (25), yields (24).

\(^7\)Svensson (2003) in footnote 21 briefly mentions the same idea in the context of time-consistent rules, but does not take it any further. Later on he discusses whether optimal rules are verifiable, but in the context of complexity, rather than identifiability as here.
effects which agents will eventually notice and correct. In either case, the welfare effects will be negative.

In Section 4, for the more general case of timeless and/or optimal rules, we shall see that with the introduction of a matrix that plays a similar role to that of \( \mu \) the choice of this is not arbitrary, but essential to ensure that the system under the timeless rule is saddlepath stable and determinate.

There is of course one particular representation of the rule that is identifiable when there are shocks present, namely (22), because the latter incorporates no shocks. This is because when shocks are present, then the saddlepath relationship includes the shocks as well, and therefore so does (25). Woodford terms (22) the robustly optimal rule, and we shall see that the Cochrane critique is not applicable to robustly optimal rules for any system with only jump variables. This is because only in these cases is the robustly optimal rule transparent as well (i.e. is not in part forward-looking).

4 Implementation of Fully Optimal Rules in a General Setting

Consider now a general linear economic model and welfare loss function. The aim of this section is to show that in principle we have exactly the same problem as described earlier, that the fully optimal or timeless rule cannot be implemented in a unique transparent way. This compounds the problem for timeless policy, given the well-documented result that it might not even be as good in welfare terms as the optimal time-consistent (or discretionary) policy (see Blake and Kirsanova (2004)).

Accordingly we consider the general linear-quadratic problem under RE. For convenience we focus on the purely deterministic problem, but it is simple to extend the results to the stochastic case. We start with a general model of the form

\[
\begin{bmatrix}
  z_{t+1} \\
  E_t x_{t+1}
\end{bmatrix} = A \begin{bmatrix}
  z_t \\
  x_t
\end{bmatrix} + B w_t
\]

where \( z_t \) is an \((n - m) \times 1\) vector of predetermined variables including non-stationary processed, \( z_0 \) is given, \( w_t \) is a vector of policy variables, \( x_t \) is an \( m \times 1\) vector of non-predetermined variables and \( E_t x_{t+1} \) denotes rational (model consistent) expectations of \( x_{t+1} \) formed at time \( t \). Then \( E_t x_{t+1} = x_{t+1} \) and letting \( y_t^T = [z_t^T, x_t^T] \) (26) becomes

\[
y_{t+1} = Ay_t + Bw_t
\]

(27)

The policy-maker’s loss function at time \( t \) is given by

\[
\Omega_t = \frac{1}{2} \sum_{i=0}^{\infty} \beta^i [y_{t+i}^T Q y_{t+i} + w_{t+i}^T R w_{t+i}]
\]

(28)
where $Q$ is a symmetric and non-negative definite matrix, $R$ is a positive definite matrix and \( \beta \in (0, 1) \) is discount factor. The slightly more general case involves terms in $y_t^T U w_{t+1}$, but the analysis below is barely changed with the introduction of this extension, so we ignore this term.

Consider the policy-maker’s \textit{ex-ante} optimal policy at $t = 0$. This is found by minimizing $\Omega_0$ given by (28) subject to (27) and and given $z_0$. We proceed by defining the Hamiltonian

$$
H_t(y_t, y_{t+1}, \mu_{t+1}) = \frac{1}{2} \beta^t (y_t^T Q y_t + w_t^T R w_t) + \mu_{t+1} (A y_t + B w_t - y_{t+1})
$$

where $\mu_t$ is a row vector of costate variables. By standard Lagrange multiplier theory we minimize

$$
\mathcal{L}_0(y_0, y_1, \ldots, w_0, w_1, \ldots, \mu_1, \mu_2, \ldots) = \sum_{t=0}^{\infty} H_t
$$

with respect to the arguments of $\mathcal{L}_0$ (except $z_0$ which is given). Then at the optimum, $\mathcal{L}_0 = \Omega_0$.

Redefining a new costate column vector $p_t = \beta^{-t} \mu_t^T$, the first-order conditions lead to

$$
w_t = -R^{-1} \beta B^T p_{t+1}
$$

$$
\beta A^T p_{t+1} - p_t = -Q y_t
$$

Substituting (31) into (27) we arrive at the following system under control

$$
\begin{bmatrix}
I & \beta B R^{-1} B^T \\
0 & \beta A^T
\end{bmatrix}
\begin{bmatrix}
y_{t+1} \\
p_{t+1}
\end{bmatrix}
= \begin{bmatrix}
A & 0 \\
-Q & I
\end{bmatrix}
\begin{bmatrix}
y_t \\
p_t
\end{bmatrix}
$$

To complete the solution we require $2n$ boundary conditions for (33). Specifying $z_0$ gives us $n - m$ of these conditions. The remaining condition is the ‘transversality condition’

$$
\lim_{t \to \infty} \mu_t^T = \lim_{t \to \infty} \beta^t p_t = 0
$$

and for the \textit{fully optimal} rule, the initial condition

$$
p_{20} = 0
$$

where $p_t^T = [p_1^T, p_2^T]$ is partitioned so that $p_1 t$ is of dimension $(n - m) \times 1$. Equations (31), (33) together with the $2n$ boundary conditions constitute the system under optimal control. For the \textit{timeless} rule, the initial value $p_{20}$ is not 0, but is dependent on the value of $y_0$. However this is not the focus of our attention, so we ignore the details of this.

At this point, we briefly note the following proposition, which is an obvious generalization of that for the basic New Keynesian model:

\textbf{Proposition 1:} When all variables are forward-looking, and the discount factor $\beta$ is
sufficiently close to 1, the optimal rule can be written in a unique way so as to satisfy the
robustly optimal criterion. In addition the rule is not subject to the Cochrane critique.

**Proof:** For a linear quadratic optimal control problem it is well known that the eigen-
values of the system (33), coupled with the dynamics of the Lagrange multipliers, has
eigenvalue pairs of the form \( \{ \lambda, 1/(\beta\lambda) \} \). Thus for \( \beta \) close enough to 1, the system together
with Lagrange multipliers is saddlepath stable. Since the optimal policy variable is a lin-
ear function of the Lagrange multipliers, which are governed by equations independent of
shocks, the result follows that the rule is robustly optimal and not subject to the Cochrane
critique.

4.1 **Possible Implementations of the Optimal Rule**

Currie and Levine (1993) and Dennis (2010) describe the optimal rule using the following
unique representation that represents the rule solely in terms of the predetermined variables
and the costate variables \( p_2 \):

\[
 w_t = -F \begin{bmatrix} I & 0 \\ -S_{22}^{-1}S_{21} & S_{22}^{-1} \end{bmatrix} \begin{bmatrix} z_t \\ p_{2t} \end{bmatrix}
\]

(36)

where

\[
 \begin{bmatrix} z_{t+1} \\ p_{2t+1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ S_{21} & S_{22} \end{bmatrix} G \begin{bmatrix} I & 0 \\ -S_{22}^{-1}S_{21} & S_{22}^{-1} \end{bmatrix} \begin{bmatrix} z_t \\ p_{2t} \end{bmatrix}
\]

(37)

\[
 x_t = \begin{bmatrix} -S_{22}^{-1}S_{21} & S_{22}^{-1} \end{bmatrix} \begin{bmatrix} z_t \\ p_{2t} \end{bmatrix}
\]

(38)

where \( F = \beta(R + \beta B^T S B)^{-1} B^T SA \), \( G = A - BF = (I + \beta BR^{-1} B^T S)^{-1} A \) and

\[
 S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}
\]

(39)

is partitioned conformably with \( z, x \) and is the solution to the steady-state Ricatti equation

\[
 S = Q + \beta A^T S(I + \beta BR^{-1} B^T S)^{-1} A
\]

(40)

Note the well-known result that \( G \) has only stable eigenvalues for \( \beta \) sufficiently close to
1, and that the Ricatti equation arises from the relationship \( p = Sy \).

Attractive as this unique representation is, if all variables are forward-looking, then \( z_t \)
does not exist, so from (37) we see that \( p_{2t} \) only feeds back on itself. It therefore follows
that the policy variable \( w_t \) is not dependent on the state variable \( x_t \) at all. Thus if the
underlying system is completely forward-looking and is indeterminate under no policy rule,
then the Currie-Levine implementation of the optimal rule will be indeterminate as well.
Thus it is flawed as a general implementation, and fails in particular for the basic New
Keynesian model.
Dennis (2010) partially gets round this problem by noting from (36) and (38) that \( w_t, x_t \) are linearly dependent on \( z_t, p_t^2 \). By finding the pseudo-inverse of this linear relationship, he writes \( p_t^2 \), (in a non-unique way) as a linear function of \( \{ z_t, x_t, w_t \} \), which generates an ARMA process for \( \{ w_t \} \) in terms of \( z_t, x_t \), but provides no means of ensuring determinacy.

4.2 Transparent Rules that are Saddlepath Stable

Clearly, to ensure that the system is determinate, we must find an expression for the transparent rule that is not of the Currie-Levine form. For the example used in Section 3, we found that we could write the policy instrument \( y \) in terms of its own past value and current \( \pi \), and it is easy to check that this leads to a determinate system. Thus the most obvious possibility is to ensure that the policy feedback rule for the general case involves a feedback on \( x \) as well as on \( z \) and \( p^2 \); there is no point of course in using \( p^1 \) in the feedback rule, because it is a forward-looking variable, and it would be confusing to agents to introduce yet another forward-looking variable into the system.\(^8\) How do we achieve this feedback? The essence is to rewrite the optimal rule \( w_t = -R^{-1}B^T p_{t+1} \) of (31) in terms of \( z_t, x_t \) and \( p^2_t \).

We first note (dropping the \( t \) subscript for the moment) that the \( p_1 = S_{11} z + S_{12} x \) relationship can be expanded to

\[
p_1 = S_{11} z + S_{12} x \quad p_2 = S_{21} z + S_{22} x
\]

so that we can obtain an infinite number of representations of \( p_1 \) in terms of the other variables:

\[
p_1 = S_{11} z + S_{12} x + M(p_2 - S_{21} z - S_{22} x) \quad (42)
\]

where \( M \) is a matrix of appropriate dimensions. The Currie-Levine implementation uses \( M = S_{12} S_{22}^{-1} \) so that \( p_1 \) only depends on \( z \) and \( p_2 \). Thus, given that the policy rule is expressed in terms of \( p_{t+1} \), our objective is to write the policy rule in a way that includes \( p^2_t \), but eliminates the variable \( p^1_t \), and replaces it by (42). The appendix indicates how to obtain a representation of the optimal rule in terms of the variables \( z_t, x_t \) and \( p^2_t \).

We need to choose \( M \) in such a way that the implemented system is saddlepath stable. The following theorem, which is the first main result of this section, not only shows how to do this, but is also the first ever general statement that an optimal rule exists that ensures that an RE system under control is saddlepath stable.

**Theorem 2:**

For non-zero \( M \), the system is determinate provided that \( G_{22} + M^T G_{12} \) has stable eigenvalues, where \( G_{12} \) is the \((n - m) \times m\) top right matrix of \( G \). Furthermore, there exist an infinite number of such matrices with the required property.

**Proof:** See Appendix.

\(^8\)Woodford (2003) however uses this for the robustly optimal rule.
The first main implication of Theorem 2 is:

**Corollary 3:** There are an infinite number of representations of the optimal/timeless policy that have identical structure but with different parameter values, with one exception - the case when $M = S_{12}S_{22}^{-1}$ and $G_{22} + (S_{12}S_{22}^{-1})^T G_{12}$ has stable eigenvalues.

The implication of this is that implementable optimal rules are in general not identifiable; there is just one exception to this, which is when the Currie-Levine representation of the optimal rule, that feeds back only on current and past backward-looking variables, is saddlepath stable. However the latter is not a generic result.

The results above also apply to the ‘universally optimal’ rule of Damjanovic *et al.* (2008). The latter shows that when the optimal choice of instrument must be chosen over all equilibrium realizations of initial conditions, then this is equivalent to solving the problem of maximizing (28) with the discount factor $\beta$ (within (28)) set equal to 1. The effect on the basic New Keynesian model, which is slightly more general than that of (17), as there is a term $\beta$ multiplying $E_t \pi_{t+1}$, is that the timeless rule remains as (22), whereas the universally optimal rule becomes $y_t = \beta y_{t-1} - q \pi_t$. In general then, since the only thing that is changed from Corollary 3 is the discount factor, we have the following result:

**Corollary 4:** There are an infinite number of representations of the universally optimal policy, which yield determinacy, that have identical structure but with differing parameter values.

The final implication of all this is the following result:

**Theorem 3:**

If the model incorporates at least one backward-looking variable, then the optimal (or timeless) policy, when implemented in transparent form, is (in general) subject to the Cochrane critique.

In other words, such an implemented optimal policy will not necessarily evolve along the saddlepath but will be subject to local instability. Note that because the locally unstable solution has eigenvalues that are dependent on the choice of matrix $M$, this means that parameters of the rule are indeed identifiable when the system evolves off the optimal path.

As explained earlier, the intuitive explanation of this result is that since the main impact of the policy is when the path of the economy is off the saddlepath, the only way that the private sector can test the policy is by forming sunspot expectations, and then learning from trajectory that is subsequently followed.

5 Conclusion

We have shown that although the Cochrane critique holds for simple rules when all variables are forward-looking, as in the simplest New Keynesian model, it need not hold for fully
optimal or timeless policy.

On the other hand, for more general models with backward-looking variables, we have found that sufficiently simple rules are not subject to the Cochrane critique. However the fully optimal (and timeless) policy suffers from an identifiability problem which implies that the Cochrane critique holds, implying that it is subject to local instability.

The explicit conclusion of our paper is that for a rule to satisfy the Cochrane criterion (assuming that there are backward-looking variables) it needs to be sufficiently simple. Anything else will lead to local instability.

APPENDIX

A Proof of Theorem 1

First of all write $S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$, $A = [A_1 ~ A_2]$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ conformably with $z$ and $x$. Also note that $G = (I + \beta BR^{-1} B')^{-1} A = A - \beta BR^{-1} B'(I + \beta SBR^{-1} B')^{-1} SA$, so that $G_{22} = A_{22} - \beta B_{22} R^{-1} B'(I + \beta SBR^{-1} B')^{-1} SA_{22}$, $G_{12} = A_{12} - \beta B_{12} R^{-1} B'(I + \beta SBR^{-1} B')^{-1} SA_{12}$. In addition the Ricatti equation implies that $S_2 = Q_2 + \beta A_2^2 (I + \beta SBR^{-1} B')^{-1} SA$.

The system under control is (33), but with $p_1 = S_1 y + M (p_2 - S_2 y)$. It follows that we can then write the system as

$$y_{t+1} + \beta BR^{-1} B_1^t (S_1 y_{t+1} + M (p_2, t+1 - S_2 y_{t+1}) + \beta BR^{-1} B_2^t p_{2,t+1} = Ay_t \quad (A.1)$$

$$\beta A_{22}^' p_{2,t+1} + \beta A_{12}^' (S_1 y_{t+1} + M (p_2, t+1 - S_2 y_{t+1})) = -Q_2 y_t + p_2 A_2 \quad (A.2)$$

For the saddle-path condition to hold, we need this system to have $n$ stable eigenvalues and $m$ unstable eigenvalues. We first note that (A.1) and (A.2) can in turn be rewritten as

$$(I + \beta BR^{-1} B') y_{t+1} + \beta (BR^{-1} B_1^t M + BR^{-1} B_2^t) (p_{2,t+1} - S_2 y_{t+1}) = Ay_t \quad (A.3)$$

$$\beta A_{22}^t S y_{t+1} + \beta (A_{12}^t S_1 + A_{12}^t M) (p_{2,t+1} - S_2 y_{t+1}) = (S_2 - Q_2) y_t + (p_{2,t} - S_2 y_t) \quad (A.4)$$

where (A.4) uses $A_{12}^t S_1 + A_{22}^t S_2 = A_2 S$. After substituting for $y_{t+1}$ from (A.3) it follows that (A.4) can be rewritten as

$$(G_2^t + G_1^t M) (p_{2,t+1} - S_2 y_{t+1}) = (p_{2,t} - S_2 y_t) \quad (A.5)$$

Hence the eigenvalues under this implementation of the optimal rule are the union of the eigenvalues of $G$, all $n$ of which are stable, and the inverse of the eigenvalues of $G_2^t + G_1^t M$; if the latter inverses are stable, then the whole system will be saddle-path stable. We first
show that there is at least one value of the matrix $M$ which ensures that $G'_{22} + G'_{12}M$ is stable.

Consider the set of left eigenvectors of $G$, each of which we write conformably with $z$ and $x$ as $[e'_1 f'_1]$, $i = 1, ..., n$, so that $[e'_1 f'_1]G = \lambda_i[e'_1 f'_1]$ $i = 1, ..., n$. It is a standard result that this set of eigenvectors spans $n$-dimensional space, so this implies that there must be at least one subset of $m$ eigenvectors which has the property that the set $\{f_i\}$ span $m$-dimensional space. Choose one of these subsets and write

$$
E = \begin{bmatrix} e'_{i_1} \\ \vdots \\ e'_{i_m} \end{bmatrix}, \quad F = \begin{bmatrix} f'_{i_1} \\ \vdots \\ f'_{i_m} \end{bmatrix}, \quad L = diag(\lambda_{i_1}, ..., \lambda_{i_m}) \quad (A.6)
$$

It follows that $[E \ F]G = L[E \ F]$. If we multiply this through by $F^{-1}$, we obtain

$$
[F^{-1}E \ I]G = F^{-1}LF[F^{-1}E \ I] \quad (A.7)
$$
or equivalently $[N \ I]G = \Lambda[N \ I]$ where $\Lambda = F^{-1}LF$ is a matrix with stable eigenvalues.

It follows that we may rewrite this as $[N \ I]\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \Lambda[N \ I]$, where $\Lambda$ is a square matrix with stable eigenvalues; but this implies in particular that $NG_{12} + G_{22} = \Lambda$, so $M = N'$ is one particular choice of $M$ that ensures determinacy of the system.

Finally, by continuity, there exists a neighbourhood of $M = N'$ for which $G'_{22} + G'_{12}M$ is stable. This completes the proof.

### B Representations of the Implementable Timeless Rule

Without loss of generality we can include target variables $s_t$ in the state vector so that $U = 0$. The equations governing the optimal equilibrium are then

$$
y_{t+1} = Ay_t + Bw_t \quad w_t = -\beta R^{-1}B'p_{t+1} \quad p_t = Sy_t \quad (B.1)
$$

Substituting $p_{t+1} = Sy_{t+1}$ and for $y_{t+1}$ yields the expression

$$
w_t = -\beta R^{-1}B'S(Ay_t + Bw_t) \quad (B.2)
$$

which can be solved to give the standard expression $w_t = -(I + \beta R^{-1}B'SB)^{-1}\beta R^{-1}B'SAy_t$.

For the case in hand, the relevant equations are:

$$
y_{t+1} = Ay_t + Bw_t \quad w_t = -\beta R^{-1}(B'p_{1,t+1} + B'p_{2,t+1}) \quad p_{1t} = S_1y_t + M(p_{2t} - S_2y_t) \quad (B.3)
$$

$$
\beta A'_2Sy_{t+1} + \beta(A'_{22} + A'_{12}M)(p_{2,t+1} - S_2y_{t+1}) = -Q_2y_t + p_{2t} \quad (B.4)
$$
Hence we have
\[ p_{2t+1} = \frac{1}{\beta} (A'_{22} + A'_{12}M)^{-1}(-Q_S y_t + p_{2t} - \beta A'_{12}(S_1 - MS_2)(Ay_t + Bw_t)) \]  
(B.5)

and that for \(w_t\) is given by
\[ (I + JB)w_t = -JAy_t + R^{-1}(B_2' + B_1'M)(A'_{22} + A'_{12}M)^{-1}(Q_S y_t - p_{2t}) \]  
(B.6)

where \(J = \beta R^{-1}(B_1' - B_2'(A_{22}^{-1})'A_{12}')(I + M(A_{22}^{-1})'A_{12}^{-1})(S_1 - MS_2)\)

This constitutes an implementable representation of the optimal rule provided that \(M\) satisfies the requirements of Theorem 2.

REFERENCES


