In this section we look at some easy extensions of the Black and Scholes option pricing model:

- options on dividend-paying assets;
- options on forwards and futures, Black’s formula;
- options on assets with time-varying, but deterministic, volatilities, interest rates and dividend rates.

Before doing so, we briefly re-examine the solution of the Black and Scholes equation we established in Lecture 5 from a probabilistic viewpoint.

1. Probabilistic interpretation of the Black and Scholes solution

Recall that we have shown that the solution of the B& S PDE with boundary value $F(S)$ at time $T$ is given by

$$V(S,t) = e^{-rT}E\left(F\left(Se^{-(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}Z}\right)\right),$$

where $Z \sim N(0,1)$ is normally distributed with mean 0 and variance 1. Since $\sqrt{\tau}Z \sim N(0, \tau)$ is equidistributed with (has the same probability distribution as) $W_T - W_t$,

$$V(S,t) = e^{-\tau}E\left(F\left(Se^{-(r-\frac{1}{2}\sigma^2)\tau+\sigma(W_T-W_t)}\right)\right).$$

The interest of this observation is that the the argument of $F$ on the right can be recognized as the solution $\hat{S}_T$ of an initial value problem for a SDE:

$$\left\{ \begin{array}{l}
d\hat{S}_u = r\hat{S}_u + \sigma\hat{S}_u dW_u, \quad u > t. \\
\hat{S}_t = S
\end{array} \right.$$

Hence,

$$V(S,t) = e^{-\tau}E\left(F(\hat{S}_T)|\hat{S}_t = S\right).$$

The same formula also follows from applying the Feynmann-Kac theorem to the Black and Scholes equation, which is covered in the Mathematical Methods module: see exercise 6.1 below for (the proof of) the
particular instance of that theorem relevant for the Black and Scholes PDE.

The SDE for $\hat{S}_t$ is similar to the one for $S_t$ except that the rate of return is now the risk-free rate, $r$, instead of the stock’s own $\mu$. Only risk-neutral investors would invest in a stock which is risky ($\sigma > 0$) but which, on average, would not earn more than the risk-free investment. Apparently investors, in the presence of a continuous-time hedging-technology, should price derivatives by taking discounted expected values of final pay-offs, pretending that the stock only earns the risk-free rate of return, but keeping its risk, as measured by the volatility, unchanged. This is a further instance of the risk-neutral pricing principle, which we already encountered when discussing the binomial model. Here we changed the price-process but, as for the binomial model and as we will see for continuous time-models in part II of the Pricing lecturers, the same thing can be achieved by changing the probability, that is, looking at the price-process through the glasses of a new probability measure called the risk-neutral probability measure.

2. Continuous dividends

Suppose now that the stock pays out a continuous dividend at a rate of $q$: this means that an investor holding one unit of stock at time $t$ will receive a cash\(^1\) payment at $t + dt$ of

$$\begin{equation}
qS_t dt;
\end{equation}$$

$q$ is called the (continuous) dividend yield. In deriving the Black and Scholes equation for an option written on such a dividend-paying stock, we now have to take into account that holding $\Delta_t$ stock at time $t$ will now translate in having $\Delta_t S_{t + dt} + q \Delta_t S_t dt$ at time $t + dt$. Hence

$$\begin{equation}
d(\Delta_t S_t) = \Delta_t dS_t + q \Delta_t S_t dt.
\end{equation}$$

Hedging the option with $\Delta_t$ of the underlying shares shorted, and working through the algebra, we find that $\Delta_t = \frac{\partial V}{\partial S}(S_t, t)$, as before, and that $V$ now has to satisfy the following variant of the Black and Scholes equation:

$$\begin{equation}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} = rV.
\end{equation}$$

Solving this as before with final boundary condition $V(S, T) = F(S)$ given, we find as general solution:

$$\begin{equation}
V(S, t) = e^{-r(t - \tau)} \int_{-\infty}^{\infty} F \left( Se^{(r-q-\frac{1}{2}\sigma^2)\tau - \sigma \sqrt{\tau} z} \right) e^{-z^2/2} \frac{dz}{\sqrt{2\pi}},
\end{equation}$$

where $\tau = T - t$. This can be interpreted as:

$$\begin{equation}
V(S, t) = e^{-r(t - \tau)} \mathbb{E} \left( F(\hat{S}_T) | \hat{S}_t = S \right),
\end{equation}$$

\(^1\)dividends could also be paid differently, e.g. in the form of additional stock
but this time round with $\tilde{S}_t$ evolving according to
\begin{equation}
    d\tilde{S}_t = (r - q)\tilde{S}_t + \sigma \tilde{S}_t dW_t.
\end{equation}
It turns out that to incorporate continuous dividends into the Black and Scholes formula for a European call, we simply have to everywhere replace $S$ by $Se^{-\tau}$, so that
\begin{equation}
    C(S, t) = Se^{-\tau} \Phi(d_+) - Ke^{-r\tau} \Phi(D_-),
\end{equation}
where now
\begin{equation}
    d_\pm = \frac{\log \left( \frac{Se^{-\tau}}{Ke^{-r\tau}} \right) \pm \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}}.
\end{equation}
A natural example of an asset which can be considered to be paying out a continuous dividend would be a foreign currency earning the foreign interest rate.

3. Forwards and futures

A forward contract is an agreement to exchange a non-dividend paying asset $S$ at some future time $T$ for a price of $F = F_t$ which is agreed upon at the time $t < T$ of entering the contract; this amount $F$, which is called the forward price, is known at the time where the deal is struck, but the future market-price $S_T$ of the asset of course isn’t. No money exchanges hands at $t$, only at $T$. To determine $F$, interpret the forward contract as a derivative which pays off $S_T - F$ at $T$, while costing 0 at time $t$. We then must have that
\begin{equation}
    0 = \int_{-\infty}^{\infty} \left( Se^{(r - \frac{1}{2} \sigma^2) \tau - \sigma \sqrt{\tau} z} - F \right) e^{-z^2/2} \frac{dz}{\sqrt{2\pi}},
\end{equation}
where $\tau = T - t$, as before. The integral computes as $S - Fe^{-r\tau}$, from which we find that
\begin{equation}
    F = F_t = Se^{r(T-t)}.
\end{equation}
Parties to a forward contract run credit risk, especially if time to maturity is large: at maturity of the contract one of the parties may not be able to pay $F$, or the other party to may not be able to deliver the asset for $S_T$, especially if the asset is not a purely financial one, but a commodity (e.g. a future harvest)\(^2\). To counter this credit risk, markets have introduced a type of forward contracts with continuous settlement, which are called futures. This works as follows: at some date $t = t_0$, two parties enter into a forward contract with delivery at $T$, agreeing to pay $F_{t_0}$ for $S_T$ at time $T$. At some later time, $t = t_1$ (for example, next day), the market price for such a forward delivery will

\(^2\)For commodities other factors, like storage costs (or cost of carry) and convenience yields, have to be taken into account in the derivation of the forward price.
have changed from \( f_t \) to \( f'_t \). If for example, \( f'_t > f_t \), the party long the forward agreement (that is, the one who would receive \( S \) at \( T \) for \( f_t \)) will now hold a more valuable instrument, and will receive the difference \( f'_t - f_t \) from the party which has to deliver the asset at \( T \). If \( f'_t < f_t \), the party long the contract would receive a negative amount, that is, have to pay that amount to the other party. The day after, at \( t_2 > t_1 \), there will be a new future price \( f_{t_2} \) and the holder of the contract will again receive \( f_{t_2} - f_t \) from the seller, etc.

When modelling this we will idealize to a continuous settlement: at \( t + dt \) the holder receives \( df_t = f_{t+dt} - f_t \) from the seller. It can be shown (cf. exercises) that when interest rates are constant (and in absence of dividends, cost of carry or convenience yields), the the future price equals the forward price

\[
(13) \quad f_t = F_t = e^{r(T-t)}S_t.
\]

This holds more generally for time-dependent, but deterministic interest rates, in the form

\[
(14) \quad f_t = F_t = e^{\int_t^T r_u du}S_t.
\]

When interest rates are stochastic, forward and future prices will in general differ. When pricing derivatives on futures, we will not assume any specific relation between future prices \( f_t \) and spot prices \( S_t \), but simply suppose that the former have their own stochastic evolution\(^3\).

### 4. Options on futures

Suppose that the future price evolves according to the SDE

\[
(14) \quad df_t = \mu f_t dt + \sigma f_t dW_t.
\]

(Such an SDE would follow from (13) and the SDE (??) for \( S_t \), with \( \mu - r \) instead of \( \mu \) but, as stated, we do not assume such a relation, and look at the future as a market-traded asset in its own right). Let \( V(f, t) \) be the price of an option on a futures contract when \( f_t = t \). Suppose you are holding the option at \( t \), and want to hedge your position by entering into \( \Delta_t \) future contracts at a price of \( f_t \). The change in value in your portfolio will be

\[
dV(f_t, t) - \Delta_t df_t.
\]

Using Ito’s lemma on the first term, we see that this change in value becomes deterministic (given \( f_t \)) if we choose

\[
(15) \quad \Delta_t = \frac{\partial V}{\partial f}(f_t, t),
\]

\(^3\)Indeed, for some commodities, like oil, there exists no spot market, at best only future markets with very short maturities.
and therefore has to be equal to $r dt$ times the portfolio value at $t$. However, the value of the portfolio at $t$ will simply be $V(f_t, t)$, since it costs nothing to enter into a futures contract! We therefore find that

$$\left( \frac{\partial V}{\partial f} + \frac{1}{2} \sigma^2 f_t^2 \frac{\partial^2 V}{\partial f^2} \right) dt = r V(f_t, t) dt,$$

all derivatives evaluated in $(f_t, t)$. Conditioning on $f_t = f$ we find that $V(f, t)$ has to satisfy the following variant of the Black and Scholes equation:

$$\frac{\partial V}{\partial f} + \frac{1}{2} \sigma^2 f^2 \frac{\partial^2 V}{\partial f^2} = r V.$$

Mathematically, this is the same as the Black and Scholes equation for an option on a dividend paying asset with continuous dividend yield $q = r$, and we can read off solution formulas, etc., from the latter. We note in particular Black’s formula for a European call on a futures contract:

$$C(f, t) = e^{-r \tau} (f \Phi(d_+) - K \Phi(d_-)),$$

with

$$d_\pm = \frac{\log(f/K) \pm \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}}.$$

5. **Time-dependent interest rates, volatilities and dividend yields**

We can, in our models, allow $r$, $\sigma$ and $q$ to depend on time, but in a deterministic way (we will study the case of stochastic volatilities and/or interest rates in later lectures, using a different, martingale-based, methodology). The coefficients in the Black and Scholes equation then become also time-dependent:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma(t)^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r(t) - q(t)) S \frac{\partial V}{\partial S} = r(t) V.$$

Perhaps amazingly, the PDE can still be explicitly solved either by the transformation method of lecture 5, or, slightly easier, by applying the Feynmann-Kac theorem. It turns out that the main effect of letting the parameters vary with time is that in our previously established pricing formulas for calls, puts, etc. we simply have to replace our constants $r$, $\sigma^2$ and $q$ by their means over the remaining life-time of the option:

$$\begin{align*}
  r &\rightarrow \frac{1}{\tau} \int_t^T r(u) du \quad (\tau = T - t) \\
  \sigma^2 &\rightarrow \frac{1}{\tau} \int_t^T \sigma^2(u) du \\
  q &\rightarrow \frac{1}{\tau} \int_t^T q(u) du.
\end{align*}$$
Of course, in practice, interest rates and even volatilities will be stochastic, and will need to be modelled by their own SDEs. However, as a first approximation, when pricing an option, we can take for \( r(t) \) the short-rate curve which is implied by the yield curve. Similarly, \( \sigma^2(t) \) can be backed out from the term-structure of implied volatilities - see Lecture 8.

### Exercises to Lecture 6

**Exercise 6.1.** Derive the pricing formula (10), (11) for a European call on an asset paying out a continuous dividend at a rate of \( q \).

**Exercise 6.2.** (a) Price a European call on an asset paying out a discrete, known, dividend of \( D_1 \) at time \( t_1 \).

(b) Generalize to sequence of (known) dividends \( D_1 \) at \( t_1 \), \( D_2 \) at \( t_2 \), etc.

*(c) What if the dividends are stochastic, but independent of the stock-price? Could you model correlation with the stock-price? And stochastic ex-dividend dates \( t_1, t_2 \), etc.?*

**Exercise 6.3.** Write out explicitly the pricing formulas for a European call in case of time-dependent but deterministic interest rates, dividend rates, and volatility.

**Exercise 6.4.** (a) Solve the initial-value problem for the time-dependent heat-equation:

\[
\frac{\partial u}{\partial \tau} = \frac{1}{2} \sigma^2(\tau) \frac{\partial^2 u}{\partial x^2}(x, \tau), \quad \tau > 0,
\]

with initial value \( u(x, 0) = f(x) \).

*(Hint. You can use either the Fourier transform, or, alternatively, transform the equation by introducing a new time coordinate \( s = s(\tau) = \int_0^\tau \sigma^2(s) ds \), noting that if \( v(x, s) := u(x, \tau) \) where \( s = s(\tau) \), then

\[
\frac{\partial u}{\partial \tau} = \frac{ds}{d\tau} \frac{\partial v}{ds},
\]

and \( \frac{\partial^2 v}{\partial x^2}(x, s) = \frac{\partial^2 u}{\partial x^2}(x, \tau) \), etc.)*

(b) Transform the Black and Scholes equation with time-dependent (deterministic) volatility to the time-dependent heat equation, and using (a), derive the price of a call for that case.

**Exercise 6.5.** Show that for constant interest rate \( r \), the time-\( t \) future price of \( S_T \) is given by

\[
f_t = S_t e^{r(T-t)}.
\]

*(Hint. Assume wlog that \( t = 0 \), and compare the following two investment strategies, where \( F_0 \) and \( f_0 \) are the forward, respectively future price.*
• **Investment strategy I**: put the amount of $F_0e^{-rT}$ into the savings account\(^4\) and enter into a forward contract with forward price $F_0$.

• **Investment strategy II**: put $f_0e^{-rT}$ into the savings account, and at time $s$ enter into $e^{-r(T-s)}$ long future positions.

Show that both strategies have the identical pay-off of $S_T$.

Modify this argument if interest rates are time-dependent but deterministic (i.e. known in advance).

**Exercise 6.6** (Feynman-Kac for the Black and Scholes PDE.) This exercise requires some knowledge of Ito-integrals, in particular the property of their expectation being 0.

Let $\hat{S}_t$ be a solution of the SDE
\[
d\hat{S}_t = r\hat{S}_t + \sigma \hat{S}_t dW_t,
\]
and let $V = V(S, t)$ be a differentiable function of $S$ and $t$.

(a) Compute $dV(\hat{S}_t, t)$, using Ito’s lemma.

(b) Show that
\[
d\left(e^{r(T-t)} V(\hat{S}_t, t)\right) = e^{r(T-t)} \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \hat{S}_t^2 \frac{\partial^2 V}{\partial S^2} + r \hat{S}_t \frac{\partial V}{\partial S} - rV\right) dt
\]
\[+ \sigma \hat{S}_t e^{r(T-t)} \frac{\partial V}{\partial S} dW_t,
\]
all partial derivatives evaluated in $(\hat{S}_t, t)$.

(c) Conclude that if $V(S, t)$ solves the Black and Scholes equation, then
\[
V(\hat{S}_t, t) = e^{-r(T-t)} V(\hat{S}_T, T) - \int_t^T \sigma \hat{S}_u e^{-r(u-t)} \frac{\partial V}{\partial S}(\hat{S}_u, u) dW_u,
\]
the integral on the right being a stochastic Ito integral. In particular, fixing $\hat{S}_t = S$, we have that
\[
V(S, t) = e^{-r(T-t)} V(\hat{S}_T, T) - \int_t^T \sigma \hat{S}_u e^{-r(u-t)} \frac{\partial V}{\partial S}(\hat{S}_u, u) dW_u,
\]
where $\hat{S}_u$ now satisfies the initial value problem (3).

(d) Taking expectations, conclude that
\[
V(S, t) = e^{-r(T-t)} \mathbb{E}\left(V(\hat{S}_T, T)\right).
\]

(e) By explicitly solving (3) and imposing the final boundary value $V(S, T) = F(S)$ at time $T$, re-derive formula (2).

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\(^4\)equivalently, buy a 0-coupon bond with maturity $T$ and face-value $F_0$. 