In this lecture we will solve the final-value problem derived in the previous lecture 4,

\[
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} &= rV \quad (t < T) \\
V(S, T) &= F(S).
\end{aligned}
\]

For the special case of a call, for which \(F(S) = \max(S - K, 0)\), we will thereby find the celebrated Black and Scholes formula. As we will see during the course of this Pricing course, there are several ways to solve the Black and Scholes PDE and/or prove the Black and Scholes formula (some of which by-pass the PDE entirely - cf. Pricing II). Here we will use the classical approach, was followed by Black and Scholes themselves, of transforming their PDE into the heat equation, and use the long-known classical solution for the initial value problem of the latter. Another way of solving (1) is to use the Feynman - Kac theorem, cf. Math. Methods.

1. **Transforming to the heat equation**

To motivate the transformation below, the key-observation is that if we put \(x = \log S\), then \(S \frac{\partial}{\partial S} = \frac{\partial}{\partial x}\); similarly, \(S^2 \frac{\partial^2}{\partial S^2}\) can be expressed in terms of \(\frac{\partial^2}{\partial x^2}\) and \(\frac{\partial}{\partial x}\). The further trick of replacing \(x\) by \(x + at\) for suitable constant \(a\) will get rid of the first order derivatives in \(x\). Finally, replacing \(t\) by \(T - t\) will transform the final boundary condition at \(t = T\) into an initial one at \(\tau = 0\).

Instead of carrying these transformations out step by step, we will do them in one stroke. We define new coordinates \((x, \tau)\) by

\[
\begin{aligned}
\tau &:= T - \tau, \\
x &:= \log S - \alpha \tau, \text{or} \quad S = e^{\alpha \tau + x}
\end{aligned}
\]

with \(\alpha = -(r - \frac{1}{2} \sigma^2)\) (alternatively, you may for the moment consider \(\alpha\) as a free parameter whose value will be conveniently fixed later on),
and put
\[ u(x, \tau) := V(e^{\alpha \tau + x}, T - \tau) : \]
\( u(x, \tau) \) is simply our value-function \( V \) in terms of these new variables. Observe that \( \tau \) is simply the time-to-maturity.

We then re-write the PDE for \( V(S, t) \) in terms of this new function \( u \): this is an exercise in applying the chain-rule of derivation. First, using that \( S = e^{\alpha \tau + x} \), we find that
\[
\frac{\partial u}{\partial x} = \frac{\partial V}{\partial S} \cdot \frac{\partial S}{\partial x} = e^{\alpha \tau + x} \frac{\partial V}{\partial S}(e^{\alpha \tau + x}, T - \tau),
\]
so that
\[
(4) \quad S \frac{\partial V}{\partial S} = \frac{\partial u}{\partial x},
\]
with the left hand side evaluated in \( S = e^{\alpha \tau + x} \) and \( t = T - \tau \). Next, differentiating once more with respect to \( x \) (and using the Leibnitz rule for deriving a product) we find that
\[
\frac{\partial^2 u}{\partial x^2} = e^{\alpha \tau + x} \frac{\partial V}{\partial S}(e^{\alpha \tau + x}, T - \tau) + e^{2(\alpha \tau + x)} \frac{\partial^2 V}{\partial S^2}(e^{\alpha \tau + x}, T - \tau),
\]
so that, remembering (4),
\[
(5) \quad S^2 \frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}.
\]
Finally, realizing that in (3) the \( \tau \)-variable occurs both in the \( S \)-sloth and the \( t \)-sloth of \( V(S, t) \), we find that
\[
\frac{\partial u}{\partial \tau} = -\frac{\partial V}{\partial t}(e^{\alpha \tau + x}, T - \tau) + \alpha e^{\alpha \tau + x} \frac{\partial V}{\partial S}(e^{\alpha \tau + x}, T - \tau),
\]
so that, using (4) once more,
\[
(6) \quad \frac{\partial V}{\partial t} = -\frac{\partial u}{\partial \tau} + \alpha \frac{\partial u}{\partial x}.
\]
Next substitute (4), (5) and (6) into the Black and Scholes equation. Since
\[
\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S}
= -\frac{\partial u}{\partial \tau} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + (r - \frac{1}{2} \sigma^2 + \alpha) \frac{\partial u}{\partial x},
\]
by the choice of \( \alpha \), we find that \( u \) satisfies
\[
(7) \quad \frac{\partial u}{\partial \tau} - \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} + ru = 0,
\]
together with the (now) initial-value condition
\begin{equation}
  u(x, 0) = F(e^x),
\end{equation}

since \( u(x, 0) = V(e^x, T) = F(e^x) \), by (3). To simplify notations, we put
\begin{equation}
  f(x) := F(e^x).
\end{equation}

If \( r = 0 \), the equation (7) is known as the heat equation. In fact, one simple further transformation of the dependent variable \( u \) allows us to get rid of the \( ru \)-term altogether: if we put
\begin{equation}
  w(x, \tau) := e^{r\tau}u(x, \tau),
\end{equation}
or \( u = e^{-r\tau}w \), then
\[
  \frac{\partial u}{\partial \tau} + ru = e^{-r\tau} \frac{\partial w}{\partial \tau}.
\]
Since \( \frac{\partial^2 u}{\partial x^2} = e^{-r\tau} \frac{\partial^2 w}{\partial x^2} \), we finally find the following initial value problem for \( w(x, \tau) \):
\begin{equation}
  \begin{cases}
    \frac{\partial u}{\partial \tau} - \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (\tau > 0), \\
    w(x, 0) = f(x)
  \end{cases}
\end{equation}

(observe that the initial value condition doesn’t change when going over to \( w(x, \tau) \), since \( w(x, 0) = u(x, 0) \)). As already mentioned, the partial differential equation (11) is called the heat equation: it was proposed by the French mathematician J. Fourier (who also invented the Fourier transform) at the beginning of the 19-th century to model the flow of heat along a rod as function of time and place.

For later reference we record the final pair of transformations for going from \( V \) to \( w \) and vice-versa:
\begin{equation}
  \begin{align*}
    w(x, \tau) &= e^{r\tau}V \left( e^{-(r-\frac{1}{2}\sigma^2)\tau + x}, \tau \right), \\
    V(S, t) &= e^{-r(T-t)}w \left( \log S + (r - \frac{1}{2}\sigma^2)(T - t), T - t \right).
  \end{align*}
\end{equation}

2. Solving the heat equation

The initial-value problem for the heat equation has an explicit solution in the form of an integral:

**Theorem 2.1. (Fourier, \( \sim 1800 \))** The initial-value problem (11) has as solution
\begin{equation}
  w(x, \tau) = \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/2\sigma^2\tau} \frac{dy}{\sigma \sqrt{2\pi\tau}}.
\end{equation}
Proof. We verify that (13) is a solution of (11). First of all, \( w(x, \tau) \) solves the PDE: this follows by differentiation under the integral sign and the fact that
\[
\left( \frac{\partial}{\partial \tau} - \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} \right) \left( \frac{e^{-(x-y)^2/2\sigma^2\tau}}{\sigma \sqrt{2\pi \tau}} \right) = 0,
\]
which can be verified by a direct (if somewhat lengthy) computation.

Next, to show that (13) satisfies the initial condition \( w(x, 0) = f(x) \), we can’t simply insert \( \tau = 0 \) in the formula, since the integrand is not well-defined. We first change variables in the integral: if we first replace \( y \) by \( x - y' \) and then put \( y' = \sigma \sqrt{\tau} z \), we find that
\[
\begin{align*}
(14) \quad w(x, \tau) &= \int_{-\infty}^{\infty} f(x - y') e^{-y'^2/2\sigma^2\tau} \frac{dy'}{\sigma \sqrt{2\pi \tau}} \\
(15) &= \int_{-\infty}^{\infty} f(x - \sigma \sqrt{\tau} z) e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}.
\end{align*}
\]
We can now put \( \tau = 0 \) without any problems, leading to:
\[
\begin{align*}
w(x, 0) &= \int_{-\infty}^{\infty} f(x) e^{z^2/2} \frac{dz}{\sqrt{2\pi}} \\
&= f(x) \int_{-\infty}^{\infty} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} \\
&= f(x),
\end{align*}
\]
since the integral is 1 (total mass of the standard normal distribution).

*Remark.* We did not actually derive the solution (13), but simply verified that it is one. It is legitimate to ask how anyone (e.g. Fourier) found it. One answer is by using Fourier-transform with respect to the \( x \)-variable, which will transform the PDE into a family of easily-solved ODEs. Alternatively, one can use the (one-sided) Laplace transform with respect to \( \tau \). A third method is to start by looking for so-called similarity solutions of the PDE of the form
\[
\frac{1}{\sqrt{\tau}} \psi \left( \frac{x}{\sqrt{\tau}} \right),
\]
motivated by the observation that the PDE itself is invariant under transformations of the form \((x, \tau) \rightarrow (\sqrt{\lambda} x, \lambda \tau) \ (\lambda > 0)\). Since translates \((x, t) \rightarrow u(x - y, t)\) of solutions \( u(x, t) \) are also solutions, one then tries to synthesize more general solutions from translates of the

---

1 This transformation can of course be done in one single step from (13), by putting \( y = x - \sigma \sqrt{\tau} z \).

2 This is called translation invariance of the heat equation.
similarity solution by taking sums, or, more generally, integrals over $y$:

$$
\int_{\mathbb{R}} g(y) \frac{1}{\sqrt{\tau}} \psi \left( \frac{x - y}{\sqrt{\tau}} \right) \, dy,
$$

with $g$ an arbitrary function. Such an integral will be a solution, and will have $g(x)\big|_{x\to x}$ as initial value, provided that $\psi$ integrates to 1 (same proof as above). One can check that for the heat equation, $\psi(x) = C e^{-x^2/2}$, by substituting the similarity solution into the PDE, deriving from this an ODE for $\psi$ and solving the latter.

3. Solution of the Black and Scholes PDE

It is convenient to write the solution of the heat equation in the form

$$
w(x, \tau) = \int_{-\infty}^{\infty} f(x - \sigma \sqrt{\tau} z) e^{-z^2/2} \frac{dz}{\sqrt{2\pi}},
$$

cf. (15). Using the second transformation formula of (12), we then find that

$$
V(S, t) = e^{-r(T-t)} \int_{-\infty}^{\infty} f \left( \log S + (r - \frac{1}{2} \sigma^2)(T - t) + \sigma \sqrt{T-t} z \right) e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}.
$$

Remembering that $f(x) = F(e^x)$, we finally have proved:

**Theorem 3.1.** (Solution of the Black and Scholes PDE for general pay-off $F(S)$). The value at time $t$, when $S_t = S$, of a European derivative paying off $F(S_T)$ is equal to

(16) \[ V(S, t) = e^{-r(T-t)} \int_{-\infty}^{\infty} F \left( Se^{(r - \frac{1}{2} \sigma^2)(T-t) - \sigma \sqrt{T-t} z} \right) e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}. \]

We recall that this formula assumes that the stock-price evolves according to a geometric Brownian motion.

4. Application: the Black and Scholes formula for a European call

We next specialize to a European call, with pay-off function

$$
F(S) = \max(S - K, 0).
$$

We plug this into our general formula (16). We note that the integration will then only extend over those $z$ such that (writing $\tau = T - t$, the time-to-maturity

$$
Se^{(r - \frac{1}{2} \sigma^2) \tau - \sigma \sqrt{\tau} z} \geq K
\implies z \leq \frac{\log S - \log K + (r - \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{\tau}} =: d,
$$
where \( d_- \) is defined by the last expression. The call’s value is then given as the sum of two terms:

\[
e^{-r(T-t)} \left\{ \int_{-\infty}^{d_-} S e^{(r-\frac{1}{2}\sigma^2)\tau-\sigma\sqrt{\tau}z-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}} - x \int_{-\infty}^{d_-} Ke^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}} \right\}.
\]

We recognize the second integral as \( e^{-r\tau}K\Phi(d_-) \), where \( \Phi \) is the cumulative normal distribution function:

\[
(17) \quad \Phi(x) := \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}}.
\]

As to the first term, it equals

\[
S \int_{-\infty}^{d_-} e^{\frac{1}{2}z^2 (\frac{1}{2} - 2\sigma\sqrt{\tau}z + \sigma^2\tau)} \frac{dz}{\sqrt{2\pi}} = \left( z = z + \sigma\sqrt{\tau} \right) S \int_{-\infty}^{d_- + \sigma\sqrt{\tau}} e^{-\frac{1}{2}z'^2} \frac{dz'}{\sqrt{2\pi}} = S\Phi(d_- + \sigma\sqrt{\tau}).
\]

Putting

\[
d_+ := d_- + \sigma\sqrt{\tau}.
\]

We note that \( d_\pm \) can be conveniently expressed as:

\[
(18) \quad d_\pm = \log\left(\frac{S}{Ke^{-r\tau}}\right) \pm \frac{1}{2}\sigma^2\tau.
\]

Summarizing, we have proved the famous Black and Scholes formula for a European call:

**Theorem 4.1.** The value at time \( t \) of a European call with strike \( K \) and exercise time \( T \) is given by

\[
(19) \quad C(S,t) = S\Phi(d_+) - Ke^{-r\tau}\Phi(d_-).
\]

Here \( S \) is the price of the underlying stock at time \( t \), \( \tau = T - t \) is the time-to-maturity, and \( d_\pm \) is given by (18).

**Remark.** If we note that \( \sigma^2\tau \) is simply the total variance, or volatility squared, of the log-return \( \log(S_T/S_t) \) over the remaining life-time \( [t,T] \) of the option\(^3\), we see that \( d_\pm \) can be interpreted in words as:

\[
d_\pm = \log\left(\frac{\text{stock price}}{\text{discounted strike price}}\right) \pm \frac{1}{2}(\text{stock-return variance over } [t,T]) \pm \frac{1}{2}(\text{stock-return volatility over } [t,T]),
\]

which may be a useful for remembering it.

\(^3\)as follows from \( \log(S_T/S_t) = r\tau + \sigma(W_T - W_t) \), together with \( \text{var}(W_T - W_t) = (T-t) = \tau \)
5. Black and Scholes for a European put

We can in principle redo the same computation for a European put, with pay-off function \( \max(K - S, 0) \). Alternatively (and a bit quicker), we can use the put-call parity relation from lecture 1, according to which the time-\( t \) price of the put is given by

\[
P(S, t) = C(S, t) - S + Ke^{-\tau t} = S(\Phi(d_+) - 1) + Ke^{-\tau t}(1 - \Phi(d_-)).
\]

Observing that, using the definition of the cumulative normal distribution \( \Phi(x) \),

\[
1 - \Phi(x) = \int_{x}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} = \Phi(-x),
\]

we conclude that

\[
P(S, t) = Ke^{-\tau t} \Phi(-d_-) - S\Phi(-d_+).
\]

6. The Greeks

6.1. The Delta of an option. As is clear from the derivation of the Black and Scholes equation, an option’s Delta, defined by

\[
\Delta_t = \Delta(S_t, t) := \frac{\partial V}{\partial S}(S_t, t),
\]

is the number of underlying which an investor who has sold the option (is one unit of the option short) has to buy in order to hedge his position. (Concretely, he will at any time \( t \) hold this amount of underlying, and holding the remaining \( V(S_t, t) - \Delta_t S \) in a bank account - borrowing this amount if its negative.) It is therefore an important quantity to know for a trader, arguably as important as the price itself. The \( \Delta \) of a call can be computed explicitly as:

\[
\Delta_c := \frac{\partial C}{\partial S}(S, t) = \Phi(d_+).
\]

Formula (22) can be proved through straightforward brutal computation, by deriving (19) with respect to \( S \): this is do-able, though slightly messy, since the \( d_\pm \) in the Black and Scholes formula also depend on \( S \). Alternatively (and this is a useful trick), we can derive under the integral sign in (16), giving the following integral expression for the \( \Delta \) of any European option:

\[
\frac{\partial V}{\partial S}(S, t) = \int_{-\infty}^{\infty} F'(\text{Se}^{(r-\frac{1}{2}\sigma^2)T-\sigma\sqrt{\tau}z}) e^{-\frac{1}{2}(z+\sigma\sqrt{\tau})^2} \frac{dz}{\sqrt{2\pi}}:
\]
the verification of which is left as an exercise. In the special case of a call,

\[ F'_{\text{call}}(S) = \frac{\partial}{\partial S} \max(S - K, 0) = \begin{cases} 1, & S > K \\ 0, & S < K \end{cases} \]

(which can be summarized as \( F'_{\text{call}}(S) = H(S - K) \), where \( H(x) \) is the Heaviside function). To compute the call’s \( \Delta \) we have to replace \( S \) inhere by \( S_e(r - \frac{1}{2}\sigma^2)\tau - \sigma\sqrt{\tau}z \). Since this is greater than \( K \) iff \( z < d_- \) (cf. the derivation of the Black and Scholes formula in 5.4 above), we find that

\[
\Delta_C = \int_{-\infty}^{d_-} e^{-\frac{1}{2}(z+\sigma\sqrt{\tau})^2} \frac{dz}{\sqrt{2\pi}} = \Phi(d_- + \sigma\sqrt{\tau}) = \Phi(d_+),
\]

as was to be shown.

6.2. **The Gamma of an option.** The second important Greek of an option is its \( \Gamma \), defined as the second derivative of the option price with respect to the underlying:

\[
(24) \quad \Gamma_t = \Gamma(S_t, t) = \frac{\partial^2 V}{\partial S^2}(S_t, t).
\]

Gamma is important because it is a measure of the so-called ”\( \Delta \)-slippage” from not being able to hedge continuously. Continuous hedging is not possible in practice. Suppose we would enter into a \( \Delta \)-edged position at time \( t \) by being one option short (i.e. having one option sold) and \( \Delta_t = \frac{\partial V}{\partial S}(S_t, t) \) stock long. If we would keep the amount \( \Delta_t \) of underlying fixed over a short finite (that is, non-infinitesimal) time interval \([t, t + \delta t]\), then the value of this portfolio would to first approximation change by

\[
\phi = -V(S_{t+\delta t}, t + \delta t) + \Delta_t S_{t+\delta t} - (-V(S_t, t) + \Delta_t S_t)
\]

\[
\simeq -\frac{1}{2} \frac{\partial^2 V}{\partial S^2}(S_t, t) (\delta S_t)^2 - \frac{\partial V}{\partial t}(S_t, t) \delta t
\]

\[
(25) \quad \Delta\text{-slippage} = \frac{\Gamma_t (\delta S_t)^2}{\Theta_t} - \frac{\partial V}{\partial t}(S_t, t) \delta t,
\]

where \( \delta S_t = (S_{t+\delta t} - S_t) \). The point is that for non-infinitesimal \( \delta t \), \((\delta S_t)^2 \) will not be a constant anymore, but a genuine random variable.

---

4When differentiating (16), the chain rule will yield an extra factor of \( \exp(-(r - \frac{1}{2}\sigma^2)\tau - \sigma\sqrt{\tau}z) \) under the integral, part of which cancels out with the discount factor in front of the integral.

\[
H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}
\]
One computes from (22) that the \( \Gamma \) of a European call is given by:

\[
\Gamma_C = \frac{\partial}{\partial S} \Phi(d_+)
\]

\[
e^{-\frac{1}{2}d_+^2} \frac{\partial d_+}{\partial S}
\]

\[
e^{-\frac{1}{2}d_+^2} \frac{\partial d_+}{\partial S} = e^{-\frac{1}{2}d_+^2} \frac{S\sigma}{\sqrt{2\pi\tau}}.
\]

The second (deterministic) term in (25) quantifies the change in time-value of the option. It has been given its own Greek letter.

6.3. The other Greeks (and one non-Greek): Theta, rho and vega. The other Greeks associated to an option are, respectively, its \( \Theta \):

\[
\Theta_t := \Theta(S_t, t) := \frac{\partial V}{\partial t}(S_t, t),
\]

its \( \rho \),

\[
\rho_t := \rho(S_t, t) := \frac{\partial V}{\partial \rho}(S_t, t),
\]

and, finally, its \( \nu \) (which is not a Greek letter at all, but the name of a star):

\[
\nu_t = \nu(S_t, t) = \frac{\partial V}{\partial \sigma}(S_t, t).
\]

(The Greek letter \( \nu \) commonly used to designate it is the upsilon). The last two quantify what one might call parameter risk: the effect of small changes in either the interest rate or the stock’s volatility. Of the two, the vega is the most important, since volatility is both not constant in practice, and the most important factor affecting the option price, after the underlying. All these can be explicitly computed for European calls and puts, most efficiently by differentiating under the integral sign as we did for the \( \Delta \) above - cf. exercises at the end of this lecture.
Exercises to Lecture 5.

*Exercise 5.1. (Similarity solutions to the heat equation). We consider the heat equation for $\sigma = 1$ (for convenience only: the case of arbitrary $\sigma > 0$ can be reduced to this one by transforming $z$ to $\sigma z$):

$$\frac{\partial u}{\partial \tau} - \frac{1}{2} \frac{\partial^2 u}{\partial x^2} = 0.$$

(a) Show that if $u = u(x, t)$ is a solution and if $\lambda > 0$, then the function $u_\lambda$ defined by

$$u_\lambda(x, \tau) := u(\lambda x, \sqrt{\lambda} \tau),$$

is also a solution.

(b) Let us (for the purposes of this exercise only) call a function $u = u(x, \tau)$ homogeneous of order $k$ if, for all $\lambda > 0$,

$$u(\sqrt{\lambda} x, \lambda \tau, \lambda \sigma) = \lambda^k u(x, \tau).$$

(This is different from the usual type of homogeneous function, but a kind of anisotropic homogeneity, since the $x$ and $\tau$ directions are dilated differently.) Show that

$$u(x, \tau) = \tau^k \psi \left( \frac{x}{\sqrt{\tau}} \right).$$

Such a homogeneous function is therefore completely determined by $\psi(x) := u(x, 1)$, its restriction to $\tau = 1$. Conversely, verify that for any function $\psi = \psi(y)$, the function

$$(31) \quad u(x, \tau) := \tau^k \psi \left( \frac{x}{\sqrt{\tau}} \right),$$

is homogeneous of order $k$.

(c) Suppose the function $u$ defined by (31) satisfies the heat equation. Derive an ODE for the function $\psi = \psi(y)$.

(d) Show that $\psi(y) := \Phi(y)$ solves the ODE when $k = 0$. It follows that

$$u(x, \tau) := \Phi \left( \frac{x}{\sqrt{\tau}} \right),$$

is a solution of the heat equation which is homogeneous of order 0.

(e) Show that if $u(x, \tau)$ is a solution of the heat equation, then so is any of its derivatives. Hence, from (d), construct solutions which are homogeneous of degree $-1/2$, $-1$, $-3/2$, etc.

(f) The answer to (e) implies that $e^{-y/2}$ will solve the ODE of (c) for $k = -1/2$. Verify this also directly.

Exercise 5.2. Verify formula (23) for the $\Delta$ of an option.
Exercise 5.3. Compute the \( \Delta \) of a European put.

Exercise 5.4. A *European digital call* with strike \( K \) has a pay-off of £1 if \( S_T \geq K \) and 0 otherwise.

(a) Derive an explicit pricing formula for a digital call.

(b) A (European) *digital put* pays off 1 if \( S < K \), and 0 otherwise. Derive a put-call parity relation for digital calls and puts.

(*hint: holding a digital call and a digital put will guarantee you a pay-off of £1 at maturity \( T \).*)

(c) Hence, or otherwise, price a digital put.

Exercise 5.5. (This exercise requires knowing about the Fourier transform - cf. Math. Methods I). Let

\[ \hat{u}(\xi, \tau) := \int_{-\infty}^{\infty} u(x, \tau) e^{-ix\xi} \, dx, \]

be the Fourier-transform, with respect to \( x \), of \( u(x, \tau) \).

(a) Show that if \( u(x, \tau) \) solves the initial value problem (11), then \( \hat{u} \) satisfies

\[ \begin{cases} \frac{d}{d\tau} \hat{u}(\xi, \tau) = -\frac{1}{2} \sigma^2 \xi^2 \hat{u}(\xi, \tau) \\ \hat{u}(\xi, 0) = \hat{f}(\xi). \end{cases} \]

(b) From this, show that

\[ \hat{u}(\xi, \tau) = \hat{f}(\xi) e^{-\frac{1}{2} \sigma^2 \xi^2 \tau}. \]

(c) By taking the inverse Fourier transform, deduce (13).

Exercise 5.6. Compute the mean and variance of \( (\delta S_t)^2 \) for small but non-infinitesimal \( \delta t \).

Exercise 5.7. In practice, \( \Delta \)-hedging is not done for single options, but for a whole option book, consisting of a portfolio of options on a single asset, but with different pay-offs. Explain how by a single trade in the underlying asset one can hedge the whole book (at each moment in time).

Exercise 5.8. Show that the \( \Gamma \) of a European put is the same as the \( \Gamma \) of the corresponding European call with identical strike and maturity.

Exercise 5.9. Show that the Theta of a European call is given by

\[ \Theta_C = -\frac{\sigma S e^{-\frac{1}{2} d_1^2}}{2\sqrt{2\pi \tau}} - rK e^{-r\tau} \Phi(d_-). \]

Traders sometimes say that "a call is a wasting asset": comment on this. Compute also the Theta of a European put. Is a put always wasting?
Exercise 5.10 Show that the rho of a European call is given by
\[ \rho_C = \tau K e^{-\tau r} \Phi(d_-), \]
and from this derive the rho of a European put.

Exercise 5.11 Show that the vega of a European call and put is equal to
\[ \nu_C = \nu_P = S \sqrt{\tau} e^{-\frac{1}{2} \frac{d_0^2}{\tau}}. \]

Solution of 5.1(c). If \( u(x, \tau) = a(\tau) \psi(x/\sqrt{\tau}) \), with \( a(\tau) \) an arbitrary differentiable function of \( \tau \), then
\[ \partial_x u = \frac{a(\tau)}{\tau} \psi'' \left( \frac{x}{\sqrt{\tau}} \right), \]
while
\[ \partial_{\tau} u = a'(\tau) \psi \left( \frac{x}{\sqrt{\tau}} \right) - \frac{1}{2} a(\tau) x \psi' \left( \frac{x}{\sqrt{\tau}} \right). \]
Substitute into the heat equation, multiply out by \( \tau \) and put \( y := x/\sqrt{\tau} \). We find:
\[ \tau a'(\tau) \psi(y) - \frac{1}{2} a(\tau) y \psi(y) = \frac{1}{2} a(\tau) \psi''(y). \]
Specializing to \( a(\tau) = \tau^k \) and dividing by \( \tau^k \) yields the following ODE for \( y \):
\[ 2k \psi(y) - y \psi'(y) - \psi''(y) = 0. \]