INSPECTING THE MECHANISM: CLOSED-FORM SOLUTIONS FOR ASSET PRICES IN REAL BUSINESS CYCLE MODELS*

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We derive closed-form solutions for asset prices in an RBC economy. The equations are based on a log-linear solution of the RBC model and allow a clearer understanding of the determination of risk premia in models with production. We demonstrate not only why the premium of equity over the risk-free rate is small but also why the premium of equity over a real long-term bond is small and often negative. In particular, risk premia for equity and long real bonds are negative when technology shocks are permanent.

Recently, a growing literature has explored the asset pricing implications of real business cycle (RBC) models. Examples are Rouwenhorst (1995), Jermann (1998) and Boldrin et al. (1995).¹ From a methodological point of view, models with production allow a more realistic modelling of consumption and dividends than do the pure exchange economies of Lucas (1978). However, explaining the behaviour of asset prices in production economies is also more challenging. For example, in an exchange economy, an increase in risk premia can be obtained by increasing risk aversion. This is not necessarily true in production economies, since agents can choose a smoother consumption path by substituting between work and leisure in response to productivity shocks. The papers just cited find that RBC models tend to generate counterfactual asset pricing implications unless some extreme form of rigidities is introduced.

This paper attempts to provide a clearer understanding of the challenges posed by asset pricing behaviour for RBC models. Instead of solving a standard model numerically and simulating it over a large number of parameter permutations, we argue here that this understanding can be better developed by analysing approximate closed-form solutions for prices of a variety of financial assets. Although there are now many accurate solution algorithms available for solving RBC models (e.g., see the overview in Taylor and Uhlig (1990)), the approximate closed-form expressions obtained here make the relationship between asset prices and exogenous technology shocks particularly transparent by providing simple analytical expressions that decompose the effect of technology shocks on asset prices into (i) a direct effect due to the shock itself and (ii) an indirect effect stemming from capital accumulation.

¹ Cochrane (1991) uses the first-order condition of firms instead of consumers to derive asset prices in a production economy.
This capital accumulation channel is absent in exchange economies and is the source of most of the puzzling asset pricing implications of RBC models. For example, Rouwenhorst (1995) reports that, for certain parameter values, the equity premium can be smaller than the long-bond premium. In some cases the equity premium is even negative, so that equity becomes a hedge against technology shocks. The analytical log-linear framework allows us to show that premia on long-term assets are unambiguously negative when technology shocks are permanent. In particular, we show how uncertainty about the cash flow of firms drives a wedge between risk premia of equity and a long real bond (with fixed dividend payments.) This wedge is actually negative for many sensible parameter configurations, causing the long bond to be riskier than equity. Intuitively, technology shocks raise the capital stock of firms, which in turn (ceteris paribus) decreases their marginal product of capital and hence cash flows. This effect can be strong enough to offset the positive direct effect of technology shock, so that long bonds have a higher risk premium than equity. Of course, the specific form of risk premia depend on the particular functional forms chosen in this paper. However, the decomposition of asset prices into a direct effect (due to the technology shock) and an indirect effect (due to capital accumulation) is of a more general nature and will be present in any model with production.

The starting point of this paper is Campbell’s (1994) approach to solving the RBC model. After approximating all relevant equations in log-linear form, he provides analytical solutions for the elasticities of endogenous variables with respect to the state variables. In this paper we extend Campbell’s solution for the real variables to financial asset prices. Given the elasticities of the real variables, we solve for expected risk premia of equity and long real bonds as well as the stochastic process of the risk-free interest rate. The closed-form solutions for risk premia are written as functions of the elasticities of the real variables given by the solution of the growth model. Thus it is straightforward to compute asset premia once the RBC model has been solved using Campbell’s (1994) technique. To illustrate the flexibility of the method, we extend the basic model to allow for adjustment cost in investment and leveraged firms. Another advantage of our approach is that the equations provided here are valid for any RBC model that can be solved using Campbell’s approach.

Risk premia of bonds and equity are determined by the dynamic response of future expected returns and cash flows to technology shocks, as shown in Campbell and Shiller (1988). The analytical solutions for asset premia decompose this response into the sum of a direct effect from the technology shock and the effect due to capital accumulation. For example, a positive technology shock increases cash flows (holding capital constant), since firms are more productive. However, the shock leads also to the accumulation of capital, which decreases cash flows (for a given level of technology). The analytical solution quantifies these two effects and shows how their relative sizes depend on the deep parameters of the model, such as the persistence of the shock and risk aversion. The risk premium of a infinitely lived real bond can be decomposed in a similar fashion.

In addition to the risk premia of long bonds and equity, we also compute the Sharpe-ratio as a general measure of the risk–return trade-off in the model.
Hansen and Jagannathan (1991) compute volatility bounds of asset returns and show that the marginal rate of substitution must be fairly volatile to be consistent with these bounds. Moreover, the volatility of the marginal rate of substitution determines the slope of the capital market line – that is, the Sharpe-ratio. In the RBC model with CRRA preferences, the Sharpe-ratio is determined by risk aversion, the elasticity of consumption with respect to the technology shock and the standard deviation of the technology shock. It is not surprising that the Sharpe-ratio generated by the model is very low, indicating that the marginal rate of substitution is not volatile enough to explain observed asset returns.

The first two versions of the RBC model follow Campbell (1994). We start with a model with fixed-labour input. Second, we allow for variable labour input. We also consider extensions allowing for leveraged firms and adjustment costs in investment. In the model with fixed labour supply, we vary risk aversion and the persistence of the shock.

The major findings can be summarised as follows. Higher risk aversion and persistence lead to a higher Sharpe-ratio. To approach the Sharpe-ratio in the data, shocks must be (almost) permanent and risk aversion must be very high. However, given risk aversion, risk premia of long bonds and equity are decreasing in the persistence of the shock. The reason is that the higher capital accumulation causes capital losses due to higher expected returns. For permanent shocks, bonds and equity have strongly negative risk premia. Another puzzling implication of the model is that the difference between the risk premia for equity and long bonds is very small for all reasonable parameter values. Moreover, the risk premium on long bonds is larger than the equity premium for a wide range of parameters. The reason for this is that the net effect of dividend growth following a positive technology shock can be negative owing to capital accumulation.

If labour input is variable, then the model produces a somewhat higher Sharpe-ratio, since agents work more after a positive shock to take advantage of higher wages. However, the effect is rather small. The levels of risk premia are about the same as those with fixed labour input. Introducing leverage in firms can potentially enlarge the wedge between risk premia for equity and bonds. We show that higher leverage can actually decrease the equity premium but extreme levels of leverage are required to affect the equity premium substantially.

Next, we allow for adjustment costs in investment. Increasing adjustment costs make investments more costly, so agents consume more after a positive shock. This increases the Sharpe-ratio and also leads to higher risk premia. However, risk aversion must still be high to achieve reasonable risk premia. In particular, adjustment costs are not able to drive a wedge between the risk premia of long bonds and stocks. Apart from the generally low level of risk premia and the Sharpe-ratio, this puzzle remains a major shortcoming of asset prices in the RBC model.

The approach taken in this paper is related to that of Jermann (1998). He decomposes risk premia in a similar fashion, although his method still requires a numerical solution to the underlying RBC model. Abel (1999) derives closed-form solutions for equity and term premia in a Lucas-tree type economy. The advantage of the technique presented here is that it combines closed-form solutions of the
production model with closed-form solution for asset prices. Jermann (1998) and Abel (1999) also allow for habit formation. In principle the techniques used in this paper are also applicable to models with habits but the expressions become quite complicated since an additional state variable has to be introduced. Essentially, the equations can be restated in matrix form along the lines of Uhlig (1999). Unfortunately, much of the intuition is lost and I therefore do not pursue this extension.

The method used in this paper is based on a log-linear approximation. It is natural to question the accuracy of the approximation. Taylor and Uhlig (1990) provide an extensive comparison of various solution methods for real business cycle methods and Den Haan and Marcet (1994) present an accuracy test. More recently, Judd (1998) summarises the large literature on numerical methods. Solving real business cycle models using linearisation has a long history; see, e.g., Kydland and Prescott (1982), King et al. (1988), Christiano (1990) and Campbell (1994), to name just a few. Den Haan and Marcet (1994) report that the log-linear approximation, as employed here, is significantly more accurate than approximation in levels. Compared with numerically based parameterised expectations, the log-linear approximation performs well as long as the innovation variance of the technology shock is not too high. Note that the calibrated innovation variance used in this paper falls well within the range for which the log-linear approximation works well.

The paper proceeds as follows. In Section 1, we state some stylised facts from asset markets. Campbell’s (1994) solution method is summarised in Section 2. In Section 3 we derive the closed-form solution for asset prices, and Section 5 presents the results for the standard RBC model. In Section 6 we consider three extensions of the basic model: variable labour input, leveraged firms and, adjustment costs in investments. Section 7 concludes.

1. Some Stylised Asset Market Facts

Table 1 gives a short list of key facts about asset markets using US data from 1948Q1 to 1996Q2. Stocks carry an equity premium of 2% per quarter: explaining that premium continues to be the subject of a large and vibrant literature. The premium for long Government bonds is much smaller, about 0.2% per quarter. However, returns on long bonds are fairly volatile. The average risk-free rate is 0.2% per quarter and does not vary much over time. The slope of the capital market line (i.e., the Sharpe-ratio) is 0.27 using the SP500 as a proxy for the market portfolio. We will use this set of facts to evaluate the performance of the RBC model concerning asset markets.

2. A Simple RBC Model and the Log-linear Approximation

This section summarises the standard RBC model and Campbell’s (1994) solution method. He suggests a log-linear approximation around the steady-state to analyse stochastic fluctuations. The advantage of this approach is that it provides analytical solutions of the dynamic behaviour of the economy once the model has been
The obvious advantage over numerical techniques is that it allows a more intuitive understanding of the mechanisms at work. The first version of the model assumes that labour supply is fixed. This is relaxed in our first extension, so that agents derive utility from consumption and leisure. Another extension allows for adjustment costs in investments. Campbell’s solution technique is generally applicable in any model that can be approximated in log-linear form. Uhlig (1999) extends the method to models that have a larger state space and also analytically computes HP-filtered versions of the model’s variables.

The model consists of many identical utility-maximising agents (or a representative agent) and profit-maximising firms. As is usual in these models, we assume that the exogenous uncertainty affects the productivity of firms. Alternative ways to introduce uncertainty include, for example, shocks to tax rates; see Campbell (1994). Firms convert capital and labour into the consumption good via a Cobb–Douglas production function. Formally, the social planner solves

\[
\max_{C_t, N_t} \mathbb{E}_0 \sum_{i=0}^{\infty} \beta^i U(C_t, 1 - N_t)
\]

subject to

\[
K_{t+1} = Y_t + (1 - \delta)K_t - C_t
\]

where \(C_t\) is consumption, \(1 - N_t\) is leisure (normalised to lie in the unit interval), \(K_t\) is capital, \(\beta\) is the discount factor, \(\delta\) the depreciation rate,

\[
Y_t = \tilde{Z}_t K_t^\alpha N_t^{1-\alpha}
\]

is output, and \(Z_t\) is the productivity parameter. At steady-state, technology \(\tilde{Z}_t\) grows at rate \(g\). The stochastic fluctuations of technology around the growth path \(Z_t\) are assumed to follow an AR(1) in logs:

\[
z_t = \bar{z} + \psi z_{t-1} + \epsilon_t, \quad \epsilon_t \sim \text{i.i.d. } \mathcal{N}(0; \sigma^2_e),
\]

where \(z_t = \log Z_t\) and \(0 < \psi \leq 1\).

Preferences are assumed to be of the form

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Asset Market Facts</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Real Returns, US data, % per quarter, 1948Q1–96Q1</td>
</tr>
<tr>
<td>Asset</td>
<td>Mean</td>
</tr>
<tr>
<td>---------</td>
<td>------</td>
</tr>
<tr>
<td>3-month T-Bills</td>
<td>0.19</td>
</tr>
<tr>
<td>Long Gov’t Bonds</td>
<td>0.37</td>
</tr>
<tr>
<td>SP500</td>
<td>2.17</td>
</tr>
<tr>
<td>Long Gov’t Bond Premium</td>
<td>0.18</td>
</tr>
<tr>
<td>S&amp;P500 Premium</td>
<td>1.99</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>0.27</td>
</tr>
</tbody>
</table>

Notes: Returns are measured at quarterly frequency, units are per cent per quarter. The maturity of long bonds is 30 years. Real returns are deflated using the CPI. Risk premia are computed as the difference between the asset return and the 30-day T-bill rate. The Sharpe ratio is calculated as the mean of the S&P500 premium divided by its standard deviation.

Source: Ibbotson Associates.
\[
U(C_t, 1 - N_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma} + A \frac{(1 - N_t)^{1-\gamma} - 1}{1-\gamma} ,
\]
where \(A\) is a parameter.\(^2\)

Campbell first approximates all first-order conditions of consumers and firms and budget constraints to achieve a log-linear form. Using the method of undetermined coefficients, the endogenous variables (in logs) can be expressed as a linear function of the logs of the state variables. The solution for the model has the following form. The state variables are capital and technology. The detrended endogenous variables (in logs) are linear function of the state variables (in logs), e.g. for consumption:

\[
c_t = \eta_{ck} k_t + \eta_{cz} z_t,
\]

where \(\eta_{xy}\) denotes the elasticity of variable \(x\) with respect to variable \(y\). These elasticities are complicated functions of the deep parameters of the model. Similarly, next period’s capital can be written as

\[
k_{t+1} = \eta_{kk} k_t + \eta_{kz} z_t.
\]

All endogenous variables (e.g., investment and labour input) can be written in this form. It is useful to restate these equations using the lag-operator \(L\). Technology follows an AR(1) process

\[
z_t = \frac{1}{1 - \psi L} \epsilon_t
\]

while capital follows an AR(2) process

\[
k_{t+1} = \frac{\eta_{kk}}{(1 - \psi L)(1 - \eta_{kk} L)} \epsilon_t,
\]

consumption follows an ARMA(2,1) process

\[
c_t = \frac{\eta_{cz} + (\eta_{ck} \eta_{kk} - \eta_{ck} \eta_{kz}) L}{(1 - \psi L)(1 - \eta_{kk} L)} \epsilon_t.
\]

\(^2\) An alternative to the additive structure of consumption and leisure in (3), is the following multiplicative specification of the period utility function:

\[
U(C_t, (1 - N_t)) = \left[ \frac{C_t^{1-\nu} (1 - N_t)^{1-\nu} - 1}{1-\nu} \right]^{1-\gamma},
\]

where \(0 \leq \nu \leq 1\). Whereas the macroeconomic effects of the multiplicative and additive specifications have been studied for example in Campbell (1994), the implications for asset prices are less favourable. The reason is that the marginal rate of substitution contains not only consumption but also leisure:

\[
M_{t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{(1-\gamma)l-1} \left( \frac{1 - N_{t+1}}{1 - N_t} \right)^{(1-\gamma)(1-\nu)}.
\]

In the data, labour input and consumption are positively correlated. If \(\gamma > 1\), both exponents are negative. Therefore, a positive (conditional) correlation of \(C\) and \(1 - N\) lowers the conditional volatility of the marginal rate of substitution, which in turns lowers asset premia. So the multiplicative preferences have even more counterfactual asset pricing implications than the additive preferences considered in (3).

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Extensions to the basic models are easily incorporated in this log-linear framework. Campbell (1994) considers nonseparable preferences in labour and leisure as well as shocks to tax rates; Lettau and Uhlig (1999) allow for habit formation; and Campbell and Ludvigson (1999) study household production. Other possible extensions include adjustment costs in investment, which we will consider later, and home production; see Benhabib et al. (1991).

3. Asset Prices

An often ignored aspect of RBC models concerns their implications for asset prices. Usually only real variables are used to evaluate the performance of these models. In this Section we explore the asset pricing implications of RBC models in detail. The goal is to derive explicit solutions of important financial variables, such as expected risk premia for equity and long-term bonds, as well as the stochastic process of the risk-free short-term interest rate. The equations are written in terms of the deep parameters of the RBC model, e.g., as risk aversion and the persistence of the technology shock, and the elasticities of the endogenous variables from the log-linear solution of the RBC model.

The only asset in positive net supply is capital. Firms rent capital and hire labour and solve a one-period optimisation problem that yields the marginal products as factor prices. We will denote the return on capital as short-term equity. Its return is simply the marginal product of capital net of depreciation. Bonds and long-term equity do not exist explicitly in this economy. But Lucas (1978) has shown that shadow prices of any security that is in zero net supply can be computed easily using the intertemporal first-order condition of households. We use this approach to price the following three securities, in addition to short-term equity: (i) a short-term (riskfree) real bond, (ii) a long-term real bond and (iii) long-term equity. We will define these assets in detail below; first we show how the expected return of an arbitrary asset can be computed.

Let \( R_{t,t+1} \) be the gross return on an asset held from period \( t \) to \( t + 1 \). If the price and the cash flow of the asset in period \( t \) are denoted by \( P_t \) and \( F_t \), then

\[
R_{t,t+1} = \frac{P_{t+1} + F_{t+1}}{P_t}.
\]

Following Lucas (1978), asset prices can be calculated from the first-order condition

\[
E_t \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t,t+1} \right] = 1.
\]

Only consumption appears in the pricing equation since preferences are assumed to be additively separable in consumption and labour input.

Let \( R^f \) be the return of the one-period real bond, that is, the risk-free rate. The riskless interest rate is just the inverse of the expected marginal rate of substitution, or (since we assume CRRA preferences), expected consumption growth raised to the power of relative risk aversion:

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\[ R_{t,t+1}^f = \left\{ \beta \mathbb{E}_t \left[ \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] \right\}^{-1}. \] (10)

Since technology shocks are assumed to be lognormal and the RBC model is approximately linear in logs, we can rewrite the risk-free rate in logs as

\[ r_{t,t+1}^f = \gamma \mathbb{E}_t \Delta c_{t+1} - \frac{1}{2} \sigma^2 \text{Var}_t \Delta c_{t+1} - \log \beta. \] (11)

The consumption CAPM model implies that the (log) expected excess return of an asset is determined by the covariance of the asset return with consumption:

\[ r_{t,t+1}^{rb} \equiv r_{t,t+1}^f - r_{t,t+1}^f = \text{Cov}_t(\gamma \Delta c_{t+1}, r_{t,t+1}), \] (12)

where \( r_{t,t+1}^f = \log \mathbb{E}_t R_{t,t+1} \) is the logarithm of the expected gross return.\(^4\) In the linearised RBC model unexpected consumption growth and asset returns are linear functions of stochastic shocks to technology. Risk premia can therefore be expressed in terms of the underlying technology shock as

\[ r_{t,t+1}^{rb} = \text{Cov}(\eta_{c z} c_{t+1}, \eta_{r z} r_{t+1}) = \gamma \eta_{c z} \eta_{r z} \sigma^2, \] (13)

where \( \eta_{c z} \) and \( \eta_{r z} \) are, respectively, the elasticities of consumption and the asset return with respect to the technology shock. In addition, risk premia depend on risk aversion and the variance of the technology shock. If the return of an asset reacts in the same direction as consumption to a technology shock, then its risk premium is positive because agents require a premium to hold assets that pay off well in times of high expected consumption growth. For most parameter sets, consumption increases after a technology shock (i.e. \( \eta_{c z} > 0 \)), but if agents are almost risk-neutral and shocks are persistent, agents will decrease consumption because the substitution effect dominates the income effect; see Campbell (1994, p.475). In these cases, assets with positive \( \eta_{r z} \) carry a negative risk premium.

The elasticities are, of course, functions of risk aversion itself. Changing risk aversion directly affects risk premia, but the effects on \( \eta_{c z} \) and \( \eta_{r z} \) will turn out to be important as well. In particular, increasing risk aversion does not unambiguously increase risk premia, since lower \( \eta_{c z} \) and \( \eta_{r z} \) can offset the direct effect of higher \( \gamma \).

The consumption elasticity \( \eta_{c z} \) is given by the solution of the RBC model in (4). Next we compute the elasticity of various assets with respect to the technology shock. The log-linear asset pricing framework of Campbell and Shiller (1988) is ideally suited since it complements the log-linear equation from the solution to the RBC model. Campbell and Shiller (1988) decompose unexpected asset returns in an approximate log-linear framework. In our notation, the decomposition is

\[ r_{t,t+1} - \mathbb{E}_t r_{t,t+1} \approx (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^{j-1} \Delta f_{t+j} - (E_{t+1} - E_t) \sum_{j=2}^{\infty} \rho^{j-1} \eta_{t+j-1,t+j}; \] (14)

\(^3\) If \( x \) is normal, then \( \mathbb{E} e^x = e^{\mathbb{E} x + \sigma^2/2} \).

\(^4\) This notation is somewhat nonstandard since \( r_{t,t+1}^f \equiv \log \mathbb{E}_t R_{t,t+1} \neq \mathbb{E}_t \log R_{t,t+1} \) due to Jensen’s inequality. By using this notation we avoid cluttering the equation with Jensen terms.

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where $\rho = 1/(1 + \frac{F}{P})$.\textsuperscript{5} Equation (14) is a log-linear approximation of the accounting identity (9). The approximation error is fairly small as long as the dividend-price ratio of the asset does not vary too much (Campbell and Kuen Koo, 1997) which will be the case for all assets considered in this paper. It is easy to show that, in this model, $\rho = \exp (g - r) \approx 0.99$. Unexpected returns can occur either through revisions in expectations in future dividend growth (with a positive sign) or via news about future expected returns (with a negative sign). For example, a real bond has no dividend uncertainty, so unexpected returns cannot occur without changes in future expected returns. Dividend news is important for equity, however. Equation (14) can be used to compute the elasticities of asset returns with respect to technology shocks. In particular, we consider a long-term real bond and equity of the representative firm. But first we solve for the stochastic process of the risk-free one-period interest rate. Using this process, we can express the future expected returns of equity and the long bond in terms of the parameters of the solution of the RBC model.

3.1. The Risk-free Rate

As mentioned above, bonds do not exist in this economy. But the Lucas asset pricing equation allows us to compute the shadow price and return of any asset that is in zero net supply. The risk-free interest rate is determined by expected consumption growth and risk aversion from (11):

$$ r^f_{t,t+1} = \gamma E_t \Delta c_{t+1}, $$

where we ignored the constant terms involving the discount factor and the variance of consumption growth. Recall that log-consumption follows an ARMA(2,1) process, as shown in (8). Given this process for consumption, we can compute the expected consumption growth and express the risk-free rate in terms of exogenous parameters and elasticities of the RBC solution:

$$ r^f_{t,t+1} = \frac{\gamma}{1 - \psi L} \left[ -(1 - \psi) \eta c_x + \frac{1 - L}{1 - \eta_{kk} L} \eta c_k \eta k_x \right] \epsilon_t. $$

The risk-free rate follows an ARMA(2,1) process with the same AR roots but different MA roots than consumption.

A technology shock affects the risk-free rate through two channels. The term involving $\eta c_x$ measures the direct effect of the shock on consumption growth, holding capital constant. In the period of the shock consumption goes up by $\eta c_x$.\textsuperscript{6} Unless the shock is permanent, consumption is expected to revert to its long-term mean (holding capital constant), hence the direct effect is of opposite sign to $\eta c_x$. But the technology shock also increases the capital stock next period, which in turn causes consumption to grow. Therefore, the indirect effect through the capital stock causes the risk-free rate to increase. Depending on the model parameters, either effect can dominate; it is clear, however, that the persistence

\textsuperscript{5} Jermann’s (1998) ‘multiple strip’ asset valuation is closely related to this decomposition.

\textsuperscript{6} Recall that $\eta c_x$ can be negative for certain parameter values.
parameter $\psi$ plays a crucial role. Of course, the $\eta$s are functions of the persistence parameter $\psi$ so that it is difficult to do comparative statics. However, additional insights can be gained by looking at two extreme cases: completely transitory ($\psi = 0$) and permanent ($\psi = 1$) shocks.

First, consider the case of permanent shocks to technology. After simplifying (16), it is easy to check that an AR root cancels with the MA root. Hence the process for the risk-free rate becomes an AR(1) process:

$$r_{t+1}^f = \gamma \left( -\eta_{cz} + \frac{1 - L}{1 - \eta_{kk} L} \eta_{cz} \right) \epsilon_t. \quad (17)$$

The initial reaction of the risk-free rate after a positive technology shock is unambiguously positive, since $\eta_{cz}$ and $\eta_{kk}$ are positive. After the initial jump upward the risk-free rate will decrease back to its steady-state level at rate $\eta_{kk}$.

As a second special case, consider a completely transitory shock. Here, we obtain

$$r_{t+1}^f = 1 + \theta L (1 - \eta_{zz} L) \omega_t = 1 + \theta L \left( \frac{1 - \psi L}{1 - \eta_{kk} L} \right) \omega_t$$

$$= \frac{1}{\psi - \eta_{kk}} \sum_{s=0}^{\infty} \left[ \psi^{s+1} - \eta_{kk}^{s+1} + \Theta (\psi - \eta_{kk}^{s+1}) \right] \omega_{t-s}. \quad (20)$$

The MA representation is also useful for computing the unconditional volatility of the risk-free rate:

$$\sigma(r_{t+1}^f) = \gamma \left[ \frac{(\psi - 1) \eta_{cz} + \eta_{dk} \eta_{ck}}{\psi - \eta_{kk}} \right] \left[ \frac{(\psi + \Theta)^2}{1 - \psi^2} - \frac{2(\psi + \Theta)(\eta_{kk} + \Theta)}{1 - \psi \eta_{kk}} + \frac{(\eta_{kk} + \Theta)^2}{1 - \eta_{kk}^2} \right]^{1/2} \sigma_\epsilon. \quad (22)$$

The volatility of the risk-free rate in post-war data is fairly low, with a standard deviation of 0.86%. We will use (22) to compute the variability of the risk-free rate in the RBC model below.
3.2. Long-term Real Bond

A long-term real bond has a certain dividend (normalised to unity) from time \( t + 1 \) to infinity. Hence, according to (14), the only source for unexpected returns is a change in expectations in future expected returns. Moreover, since the model is homoscedastic and preferences are time-separable, it follows that risk premia are constant. Only news of the expected future risk-free rate can lead to an unexpected return of a real bond. Using the MA(\( \infty \)) representation (20) of the risk-free rate, the unexpected return of the long bond can be written as a function of the technology shock and the elasticities of the solution to the RBC model. Plugging (20) into (14) yields

\[
\frac{r_{t+1}^{lb} - E_t r_{t+1}^{lb}}{C_0} = -\frac{\rho \Theta_{\psi t+1}}{\psi - \eta_{kk}} \sum_{s=0}^{\infty} \rho^s \left[ \psi^{s+1} - \eta_{kk}^{s+1} + \Theta (\psi^s - \eta_{kk}^s) \right] \tag{23}
\]

\[
= -\frac{\rho (1 + \Theta \rho)}{(1 - \rho \psi)(1 - \rho \eta_{kk})} \Theta_{t+1}. \tag{24}
\]

After simplifying, we obtain for the elasticity of the long bond return with respect to the technology shock

\[
\eta_{rc}^{lb} = \frac{\gamma \rho}{1 - \rho \psi} \left[ (1 - \psi) \eta_{cz} - \frac{1 - \rho}{1 - \rho \eta_{kk}} \eta_{ck} \eta_{kz} \right]. \tag{25}
\]

The reaction of the long bond return to a technology shock depends again on a direct effect via consumption and an indirect effect via capital accumulation. Unless shocks are permanent the direct effect has the same sign as \( \eta_{cz} \) because consumption is expected to revert to its mean. If \( \eta_{cz} > 0 \), expected future interest rates are lower than their steady-state level, yielding a capital gain for the long bond. If consumption decreases after a positive shock (i.e., if \( \eta_{cz} < 0 \)) the effect has a negative sign. On the other hand, an increasing capital stock increases future consumption, which in turn increases expected interest rates and causes a capital loss for the long bond. This channel is described in (25) by the term involving \( \eta_{ck} \eta_{kz} \) which has a negative sign. The net effect is ambiguous and depends on the parameters of the model. As we will see later, the negative effect from capital accumulation can dominate, causing a negative long bond premium for reasonable parameters. The multiplier involving \( \gamma, \psi, \) and \( \rho \) discounts the long-run effects appropriately.

The risk premium for the long bond follows directly from the CCAPM (13):

\[
r_{t+1}^{lb} = \gamma \eta_{cz} \eta_{rc}^{lb} \sigma_e^2. \tag{26}
\]

Since the \( \eta \)'s are functions of exogenous parameters, such as \( \psi \) and \( \gamma \), a comparative statics analysis is difficult. But (25) is useful for developing an intuition of how the long bond premia are determined in the RBC model. First, consider changing the persistence of the technology shock \( \psi \). As Campbell (1994) points out, \( \eta_{kk} \) and \( \eta_{ck} \) do not depend on \( \psi \). Moreover, \( \eta_{kz} \) does not change much for different values of \( \psi \) as long as risk aversion is not extremely low. Thus, the effect through capital accumulation is not affected much by \( \psi \).
The direct effect through consumption diminishes as the technology shocks become more persistent, since $\eta_{cz}$ is bounded above by unity. This can be seen most clearly by considering the two extreme cases of completely transitory and permanent shocks. For permanent shocks, the direct effect through $\eta_{cz}$ vanishes because (holding capital constant), consumption is permanently at a new level and interest rates are not expected to change. Hence $\eta_{lb}^{rb}$ is negative, since the term describing the effect on capital accumulation is negative. In this case the long bond serves as a hedge against good shocks and therefore carries a negative risk premium (as long as $\eta_{cz}$ is positive). Simplifying (25) for the case of $\psi = 1$ confirms this intuition:

$$\eta_{lb}^{rb} = -\gamma \rho \eta_{cz} \frac{\eta_{ck} \eta_{kz}}{1 - \rho \eta_{kk}}.$$  (27)

A positive technology shock of size 1 in period $t$ leads to higher consumption of $\eta_{cz}$ in period $t + 1$. The factor $-\gamma \rho / (1 - \rho \eta_{kk})$ is the present value of the long-run effect that higher consumption will have on the future expected interest rate.\footnote{This intuition helps to understand Jermann’s (1998) finding that risk premia are often negative if technology shocks are persistent.}

Consider next the case of transitory shocks. The elasticity of the long bond return (25) simplifies to

$$\eta_{lb}^{rb} = \gamma \rho \left( \eta_{cz} - \frac{1 - \rho}{1 - \rho \eta_{kk}} \eta_{ck} \eta_{kz} \right)$$  (28)

$$\approx \gamma \rho \eta_{cz}.$$  (29)

where we used the approximation $\eta_{kz} \approx 0$. The unexpected return of the long bond is unambiguously positive after a positive technology shock. Interest rates are below their steady-state value for only one period since consumption quickly returns to its steady-state level. The temporarily lower expected interest rate causes a small but positive unexpected return for the long bond.

The effect of risk aversion on the long bond premium is harder to analyse because all the elasticities depend on $\gamma$. It turns out, however, that the direct linear effect of $\gamma$ on $\eta_{lb}^{rb}$ is usually stronger than the indirect effects via the elasticities. The long bond premium increases with $\gamma$ if the term in brackets in (25) is positive. If it is negative (e.g., when shocks are permanent) then increasing risk aversion actually causes the long bond premium to be more negative.

3.3. Equity

Since firms solve only one-period optimisation problems, long-term equity does not exist in this economy. Nevertheless, we can compute the shadow price of an asset that has the characteristics of long-term equity assuming that the asset is in zero net supply. We have to specify a sequence of cash flows. To proxy for equity cash flows, we assume that long-term equity pays the marginal product of capital in every period as cash flow:
\[ F_{eq}^t = (1 - \alpha) \frac{Y_t}{K_t}; \] 

(30)

in logs we may write (ignoring constants, as usual)

\[ f_{eq}^t = (y_t - k_t). \] 

(31)

Hence, log-cash flows are proportional to the difference of log-output and log-capital. Consider a share of an infinitely-lived firm. Future cash flows are uncertain owing to changes in output and capital that are driven by technology shocks. According to the Campbell-Shiller equation (14), the unexpected return of equity has two components: news to expected dividend growth and news to expected returns. Since risk premia are constant, the news to expected returns for equity are the same as the news to expected returns for the long bond. Hence equity is a long bond plus the claim to cash flows. In order to compute the equity premium, we have to compute the news to expected dividend growth. First, rewrite the log-cash flow in terms of the state variables:

\[ f_t = \eta_{fk} k_t + \eta_{fz} z_t, \] 

(32)

where \( \eta_{fk} = (\eta_{yk} - 1) \) and \( \eta_{fz} = \eta_{yz}. \) Cash flows can be written as an ARMA(2,1) process with the following parameters:

\[ f_t = \frac{\eta_{fz} + \left( \eta_{fk} \eta_{kk} - \eta_{fk} \eta_{kk} \right) L}{(1 - \psi L)(1 - \eta_{kk} L)} \epsilon_{t+1}. \] 

(33)

Using this ARMA representation, the news to expected discounted dividend growth in (14) can be computed. A similar calculation as for the news to expected returns yields\(^8\)

\[ (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^{j-1} \Delta f_{t+j} = \eta_{rz}^f \epsilon_t, \] 

(34)

where

\[ \eta_{rz}^f = \frac{1 - \rho}{1 - \rho \psi} \left( \eta_{fk} + \frac{\rho}{1 - \rho \eta_{kk}} \eta_{fk} \eta_{kk} \right). \] 

(35)

This equation is similar to the elasticity of the long bond (25). As in the case for news to expected returns, the technology shock affects news to dividend growth through a direct channel and an indirect channel via capital accumulation. The direct effect of a shock is always positive, since \( \eta_{fz} \) is positive as firms are more productive after a positive shock. The second channel through capital accumulation is always negative since \( \eta_{fk} = \eta_{yk} - 1 < 0. \)

---

\(^8\) This derivation ignores depreciation of capital. To be precise, one must use a slightly different definition of returns: \( R_{t+1} = [F_{t+1} + (1 - \delta)P_{t+1}] / P_t \). This yields a slightly different linearisation constant, \( \rho' = 1/[1 + (F/P)/(1 - \delta)] \) instead of \( \rho \) in (14). However, the numerical difference between \( \rho \) and \( \rho' \) is tiny for the depreciation rate used here (0.9901 compared to 0.9903). Hence we proceed using \( \rho \) in the following equation for equity.

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The elasticity of output with respect to capital is less than unity in RBC models, so cash flows decrease if capital increases. The net effect is ambiguous. Another way to see that the discounted news to dividend growth can fall after a shock is to note that log-cash flows are proportional to the difference between log-output and log-capital. Output increases substantially after a positive shock, while capital accumulation is slow. Hence, during the first few periods, cash flows are above steady-state level but then decrease as output reverts to its steady-state level. The net effect on discounted news of dividend growth can be positive or negative depending on the deep parameters of the model.

Finally, the risk premium for equity is given by

$$r_{i,t+1}^{eq} = \gamma \eta_{rz} (\eta_{rz}^f + \eta_{rz}^b) \sigma^2 \varepsilon.$$ \hspace{1cm} (36)

This implies that the wedge between the risk premium for equity and the long real bond is determined by $\eta_{rz}$. In the data, the equity premium is much larger than the long bond premium implying a large and positive $\eta_{rz}$. As we will see, the RBC model is not able to generate reasonable values for $\eta_{rz}$ so that the equity premium is close to the bond premium. For many sensible parameter values, $\eta_{rz}$ is even negative, so that the equity premium is smaller than the long bond premium. The analytical solutions presented here also suggest that the RBC model cannot generate high equity premia when technology shocks are permanent, since $\eta_{rz}$ is strongly negative.

To summarise, in this Section we have computed (1) analytical expressions for the stochastic process of the risk-free interest rate, (2) the risk premia for a real bond with infinite maturity and (3) claims to equity of a firm with infinite lifetime in a generic RBC model. In the next section, we compute these variables for a variety of deep parameters of the RBC model. In addition to the risk premia, we also report values of the Sharpe-ratio in the models. Following Hansen and Jagannathan (1991), Lettau and Uhlig (1998) provide an analytic expression for the Sharpe-ratio in models with log-normal distributions. As Hansen and Jagannathan show, the Sharpe-ratio provides a general measure for the risk-return trade-off in a model. In particular, it does not depend on any assumptions on dividend streams of asset. In this model, the Sharpe-ratio is determined as follows:

$$SR = \gamma |\eta_{rz}| \sigma \varepsilon.$$ \hspace{1cm} (37)

In exchange economies, (Lucas, 1978), increasing risk aversion unambiguously increases the Sharpe-ratio because agents consume their endowment. However, in economies with a storage technology, agents can save part of their endowment. In particular, $\eta_{rz}$ depends on the preferences of the agents. Increasing risk aversion can decrease $\eta_{rz}$ so that the net effect of higher risk aversion on the Sharpe-ratio becomes ambiguous.

3.4. Term Structure

The log-linear structure of the models enables the use of existing techniques to solve for a closed-form solution of the term structure. With respect to term
structure models, the RBC model yields a homoscedastic single-factor model studied e.g. in Campbell (1986) and Backus and Zin (1994). Here we follow the textbook treatment in Campbell et al. (1997, ch. 11). For simplicity, we assume that technology follows a random walk ($\psi = 1$). In this case, the first difference in log consumption follows an ARMA(1,1) process (8); expected consumption growth is an AR(1) (17) and therefore fits in the class of models covered in Campbell et al. (1997, pp. 429). They solve for the price of any $n$-period discount bond and $n$-period-ahead forward rates. The two most important parameters are the persistence of expected consumption growth (here $\eta_{kk}$) and the covariance of innovations in consumption and revisions in expected future consumption growth. In the RBC model, this covariance is equal to the elasticity of consumption with respect to the technology shock, $\eta_{cz}$. For the derivation of the term structure, we refer to Campbell et al. (1997).

4. Calibration

We follow Campbell (1994) in calibrating the model to quarterly data. The growth rate of technology $g$ is set to 0.5% and the steady-state level of the return to capital $r$ (net of depreciation) is fixed at 1.5%. Given a value for $\gamma$, the condition for a balanced growth path pins down the discount factor $\beta$. We choose $\sigma$, so that $z\gamma$ has a standard deviation of 0.7%; this follows Prescott (1986) and much of the literature thereafter. The remaining parameters are chosen as follows: $\delta = 0.025$ and $\alpha = 0.667$.

Campbell (1994) computes the various elasticities $\eta$ for a variety of parameter values of the model. For reasonable parameter values, i.e a fairly persistent technology shock $\psi = 0.95$ and log utility, a typical pattern for the $\eta$s is as follows: $\eta_{kk} = 0.95$, $\eta_{kc} = 0.05$, $\eta_{ch} = 0.55$, $\eta_{cz} = 0.10$. See Campbell (1994) for a detailed interpretation in terms of the deep parameters of the model.

5. Results with Fixed Labour Supply

In this Section we use the analytical solutions in Section 3 to compute the risk premia for a variety of parameters of the RBC model when labour supply is fixed. A more realistic version of the model with variable labour supply will be considered in the next Section. Agents derive utility only from consumption, and the period utility function is assumed to be of CRRA form:

$$U(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}. \tag{38}$$

Labour is normalised to unity, $N_t = 1$. We will choose different values for the parameters of risk aversion and the persistence of technology shock below. Table 2 presents the results for this case. We vary the persistence of the technology shock from completely transitory ($\psi = 0$) to intermediate values ($\psi = 0.5, 0.95$) to permanent ($\psi = 1$). Risk aversion is set to 0.01, 1, 10 and 100 corresponding to an elasticity of intertemporal substitution of 100, 1, 0.1, and

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Consider first a benchmark case with risk aversion of unity and fairly persistent technology shock \( w = 0.95 \). The Sharpe-ratio is 0.0026, which is too small (by a factor of 100) compared to postwar data. This suggests immediately that risk premia of financial assets will generally be small in the model since the marginal rate of substitution is not volatile enough. In other words, there is no hope that the model can lie inside the Hansen and Jagannathan (1991) volatility bounds. Of course this is expected because a model with production and storage technologies will not perform better than an exchange economy. However, it is still interesting to compare risk premia of bonds and equity. First, the risk premium on short-term equity has the same sign as \( g_{cz} > 0 \). This is intuitive, since the return of short-term equity depend only on the marginal product of capital in the next period. Unless consumption moves in the opposite direction as the MPK, the risk premium on this short-claim will be positive. Furthermore, the risk premium of the long bond is slightly higher than that of equity (0.00024\% compared to 0.00023\%). Both are much too small, as already indicated by the low Sharpe-ratio. Long bonds are riskier than stocks in this version of the RBC model!

First, consider the long bond. Equations (26) and (25) show that the risk premium of the long bond is driven by the response of the risk-free rate to a shock, which in turn is driven by the consumption response. In this benchmark case, since agents increase their consumption after a positive shock \( \eta_{cz} > 0 \) and \( \psi < 1 \), the direct effect of the technology shock on \( \eta_{cz}^b \) is positive. It dominates the second

---

**Table 2**

**Fixed Labour Supply**

US data: \( SR = 0.27 \), EqPrem = 1.99\%, Long Gov’t BoPrem = 0.15\%, \( \sigma(r^f) = 0.86\% \)

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**Notes:** This table shows the conditional asset premia and the conditional Sharpe ratio \( (SR) \). \( ST EqPrem \) is the short-term equity premium, \( LT EqPrem \) is the long-term equity premium and \( LT BoPrem \) is the long-term bond premium. Premia are calculated as excess returns over the riskless short bond. All numbers are in percent per quarter. An asterisk denotes a parameter set which implies that consumption is decreasing after a positive technology shock \( \eta_{cz} < 0 \).
indirect effect through capital accumulation, causing the long bond premium to be small but positive.

These channels can also be understood by studying the dynamics of consumption and the risk-free rate after a shock, see Figure 1. First consumption is increasing owing to the technology and the accumulation of capital; after about five years, consumption starts to revert to its steady-state level. Since the risk-free rate is determined by the expected growth in consumption, the risk-free rate is above its steady-state level as long as consumption is increasing. Once consumption falls, the risk-free rate is below steady-state. Higher interest rates in the future results in a capital loss of the long bond today, while lower rates cause a capital gain. Hence, depending on the relative sizes of the higher interest rates immediately after a shock and the lower rates when consumption is decreasing, the unexpected return of the long bond can be positive or negative after a positive technology shock. For the benchmark values of $\gamma = 1$ and $\psi = 0.95$, the positive effect slightly dominates, causing a small positive risk premium. However, when the persistence of the shock increases, the risk premium of the bond can become negative.

In contrast to real-world financial markets, long-term bonds carry a higher risk premium than equity in the RBC model with these parameter values. From (36) it follows that long bonds are riskier than stocks if $\eta^d_{\delta}$ is negative. Recall from (35) that $\eta^d_{\delta}$ is the sum of two terms: a positive term due to the direct effect of the

![Impulse responses to shock in technology](image)

**Fig. 1. Consumption and Risk-free Rate**

Figure shows impulse responses to a positive 1% shock in technology. Parameters used: $\gamma = 1$, $\psi = 0.95$. © Royal Economic Society 2003
technology shock and a negative effect due to capital accumulation. For the parameter values of the benchmark case, the negative effect slightly dominates the positive effect. The response of cash flows to a technology shock can also be understood from the impulse response function plotted in Figure 2. Output increases more than capital immediately after a positive technology shock. Since log-cash flows are proportional to the difference between log-output and log-capital, log-cash flows are above steady-state as long as log-output is higher than log-capital. For the benchmark case plotted here, the positive effect for cash flows lasts about five years. Afterwards cash flows are below steady-state but revert to their long-term means when the shock dies out. If the negative effect is large enough, the elasticity of cash flows news $\eta_fz$ is negative and equity is less risky than long bonds. In these cases, cash flows provide a hedge against positive technology shocks. The net effect is very small, however, since the two effects in (35) almost cancel each other out. Hence the difference of the equity premium and the long bond premium is small in absolute terms. As we will see, this is true for almost all parameter values. This result was not highlighted in Rouwenhorst (1995) or Jermann (1998), but these authors report many cases in which long bonds are riskier than equity.

Next, consider varying the persistence parameter while keeping risk aversion at unity. The Sharpe-ratio is increasing in $\psi$ because the consumption reaction to shocks $\eta_{cz}$ is increasing. This translates directly into a higher Sharpe-ratio

![Impulse responses to shock in technology](image_url)

**Fig. 2. Dividends**

Figure shows impulse responses to a positive 1% shock in technology. Parameters used:

$\gamma = 1, \psi = 0.95.$

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according to (37). However, the Sharpe-ratio is too small even for permanent shocks. Although the Sharpe-ratio increases with $\psi$, the risk premia for long-term assets are negative when shocks are permanent. As shown in (27), the elasticity of the long bond return is unambiguously negative in this case owing to higher interest rates in the future; equity has a slightly less negative risk premium since $n_{t/\tau}$ is small but positive. Hence, increasing persistence increases the Sharpe-ratio but reduces risk premia for long-term assets.

Now consider changing risk aversion. Higher risk aversion implies a lower elasticity of substitution and so consumers prefer a smoother consumption path. For given persistence $\phi$, increasing risk aversion also increases the Sharpe-ratio. For $\gamma = 100$ and permanent shocks, the Sharpe-ratio is about unity, four times the value from postwar data. However, long-term risk premia are negative, about $-2.47\%$ for the long bond and $-1.84\%$ for equity. The Sharpe-ratio is high, so that the models passes the Hansen-Jagannathan bounds test, yet risk premia are actually negative. The reason, of course is that increasing the persistence of technology shocks has a negative effect on risk premia. In the extreme case of permanent shocks, risk premia are unambiguously negative, as shown before; increasing risk aversion only strengthens this effect. Results for the intermediate case of $\gamma = 10$ are in between those for $\gamma = 1$ and $\gamma = 100$, confirming that the RBC model produces a very small Sharpe-ratio and minuscule (often negative) risk premia for reasonable levels of risk aversion.

The opposite extreme – relative risk aversion of almost zero – is also interesting. When shocks are somewhat persistent, agents decrease their consumption after a positive technology shock (indicated by an asterisk in Table 2). As Campbell (1994) explains, this is due to strong substitution effects that dominate income effects. A negative $n_{t/\tau}$ has strong implications for asset prices, since in this case assets that respond positively to a shock now carry a negative risk premium; see (13). These assets have a high payoff when consumption is low (namely, after positive technology shocks), so agents are willing to pay a negative premium in order to hold such assets. Both types of long-term assets react negatively to a technology shock and thus have a positive risk premium but the short-term equity premium is negative since its cash flow increases in response to a positive technology shock.

The last column in Table 2 shows the standard deviation of the risk-free rate. In postwar data, the risk-free rate was fairly stable, with a standard deviation of 0.86%. Any set of parameters of the RBC model implies an even lower volatility of the risk-free rate in the model. The model is not able to generate enough time-series volatility even for high risk aversion and persistence. The positive effect of risk aversion and persistence are offset by decreasing capital accumulation as can be seen from (22).

Summarising the results of this section, the RBC model requires very high risk aversion and persistent technology shocks to achieve a sensible Sharpe-ratio. However, risk premia of long-term assets are decreasing in the persistence parameter and are negative for permanent shocks. In particular, the wedge between risk premia of long bond and equity is very small for any set of parameters. Hence the RBC model is inconsistent with the substantially higher return of equity over long bonds in the data. Moreover, the model’s risk-free rate is too stable over time.
6. Extensions

6.1. An RBC Model with Variable Labour Supply

Following Campbell (1994), we consider next a more realistic model in which agents derive utility from consumption of goods and leisure:

\[ U(G_t, 1 - N_t) = \log(G_t) + A \frac{(1 - N_t)^{1-\gamma_n} - 1}{1 - \gamma_n}. \] (39)

As noted by King et al. (1988), log-utility in consumption is needed to achieve a balanced growth path when utility is additively separable in leisure. Note that the model with variable labour input converges to the model with fixed labour input as \( \gamma_n \to \infty \). This extension of the model can be solved using the log-linear approximation technique, as in the fixed-labour case. Campbell (1994) describes the effects for the real variables in detail. Table 3 presents the results for financial assets. As in the case of fixed labour supply, risk premia and the Sharpe-ratio are much too small. For given persistence, the Sharpe-ratio is decreasing in \( \gamma_n \). The reason is that agents increase their labour supply after a positive shock to take advantage of higher wages and can afford to increase their consumption more than in the case with fixed labour supply. Higher \( \eta_{uz} \) implies a higher Sharpe-ratio. But, the effect is rather small. For \( \psi = 0.95 \), the Sharpe-ratio increases by about 50\% in a model with almost fixed labour (\( \gamma = 100 \)) compared to a model with (almost) linear disutility of labour (\( \gamma = 0.01 \)).

Risk premia are also increasing in absolute value when labour becomes more flexible. Note that risk premia for long-term assets become more negative in the case of permanent shocks. As for the Sharpe-ratio, the effects are not very large; the equity and long bond premia roughly triple for the case of linear disutility of labour. The orders of magnitude are not affected by \( \gamma_n \). The volatility is very similar to the model with fixed labour, so we do not report the numbers.

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</tr>
</tbody>
</table>

Note: See Table 2.

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6.2. Leverage

It is possible to introduce leverage in a similar fashion as in Campbell (1986) and Abel (1999). Let cash flows of a leveraged firm be given by

$$f_t^{eq} = \xi(y_t - k_t), \quad (40)$$

where $\xi$ is the leverage factor. The solution for the equity premium derived in the previous Section is still applicable, but the coefficients in (32) must be replaced by $\eta_{yk} = \xi(\eta_{yk} - 1)$ and $\eta_{yz} = \xi\eta_{yz}$. Although leverage does not affect real allocations (since the Modigliani-Miller theorem holds), it does affect risk premia of equity because it makes cash flows more volatile. Table 4 reports the effect of leverage on the equity premium in the RBC model. Risk aversion is set to 100 to illustrate the effect as clearly as possible. Leverage ratios are set to 3, reported in Benninga and Protopapadakis (1990), 10, and to an extreme case of 100.

Table 4 shows that leverage can affect the equity premium in different ways. For $\psi = 0.95$, the long bond premium is 0.107 percentage points higher than the equity premium for a firm without leverage. Introducing leverage decreases the equity premium further relative to the bond premium (which is, of course, not affected by leverage). This effect can also be seen by $\eta_{eq}^l$; when $\eta_{eq}^l$ is negative in the no-leverage case, increasing $\xi$ causes $\eta_{eq}^l$ to be even more negative since it is linear in $\xi$.

Leverage has just the opposite effect for $\psi = 1$. The equity premium for an all-equity firm is slightly higher than the long bond premium. Introducing leverage widens this gap. However, the effect is small unless leverage becomes extremely high. For a realistic value of around 3, the equity premium is 1.9 percentage points higher than the bond premium. In absolute terms, it is still negative. The leverage effect becomes more important as $\xi \to \infty$. In the limit, cash flows become infinitely volatile and the equity premium approaches infinity. Moreover, leverage is not able to drive a substantial wedge between equity and bond premia.

Table 4

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>$\xi$</th>
<th>EqPrem %</th>
<th>EqPrem–LT BoPrem %</th>
<th>$\eta_{eq}^l$</th>
</tr>
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<td>0.95</td>
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<td></td>
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<td>6.329</td>
<td>5.574</td>
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<tr>
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<td>100</td>
<td>60.818</td>
<td>63.295</td>
<td>55.741</td>
</tr>
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</table>

Note: See Table 2.
6.3. Adjustment Costs

In this Section we extend the model with fixed labour input to allow for adjustment costs of investment. The capital accumulation equation is

\[ K_{t+1} = (1 - \delta)K_t + K_t F\left(\frac{I_t}{K_t}\right), \]

where \( I_t = Y_t - C_t \) denotes investment and \( F \) is a concave installation function with \( F(g + \delta) = g + \delta \) and \( F'(g + \delta) = 1 \). \( F' \) can be interpreted as the reciprocal of Tobin’s \( q \). Given these properties of \( F \), the steady-state is the same as in the model without adjustment costs. Define \( \gamma_i = -(g + \delta)F''(g + \delta)/F'(g + \delta) \). Then \( 1/\gamma_i \) is the elasticity of the investment-capital ratio with respect to Tobin’s \( q \) at the steady-state. Note that \( \gamma_i = 0 \) reduces to the model without adjustment costs. It is straightforward to solve the model using the log-linear method of Campbell (1994). Table 5 presents the elasticities computed from the linearised model; Table 6 presents the asset pricing implications.

For given risk aversion, increasing \( \gamma_i \) implies more rigid investment and hence lower reaction of capital to a shock, so \( \eta_{kz} \) decreases. Since less is invested, consumption reacts more to shocks; i.e., \( \eta_{cz} \) increases in \( \gamma_i \). The elasticities of the long-term assets are also strongly affected by variations in \( \gamma_i \). Consider first \( \eta_{lb}^{rz} \), which is computed in (25). For fixed risk aversion, increasing \( \gamma_i \) increases \( \eta_{lb}^{rz} \) because the consumption elasticity \( \eta_{cz} \) is increasing and the capital accumulation channel \( \eta_{ck} \) is decreasing. The last column in Table 5 shows that the effect of \( \gamma_i \) on \( \eta_{lb}^{rz} \) is positive but fairly small. Equation (35) shows that the negative effect involving capital accumulation decreases since \( \eta_{kz} \) is lower with adjustment costs. Here \( \eta_{kz} \) equals \( z \), since labour input is assumed to be fixed.

It is instructive to consider the limiting case \( \gamma_i \to \infty \). In this case of extreme adjustment costs, the elasticity of capital with respect to the shock converges to

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( \gamma_i )</th>
<th>( \eta_{cz} )</th>
<th>( \eta_{ck} )</th>
<th>( \eta_{kz} )</th>
<th>( \eta_{kh} )</th>
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<th>( \eta_{lz}^{rz} )</th>
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<tr>
<td></td>
<td>( \infty )</td>
<td>0.89</td>
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<td>0.00</td>
<td>1.00</td>
<td>0.01</td>
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</tr>
<tr>
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<tr>
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<tr>
<td></td>
<td>( \infty )</td>
<td>0.89</td>
<td>0.11</td>
<td>0.00</td>
<td>1.00</td>
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<td>0.11</td>
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<td>1.00</td>
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<tr>
<td></td>
<td>( \infty )</td>
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<td>0.11</td>
<td>0.00</td>
<td>1.00</td>
<td>73.76</td>
<td>0.11</td>
</tr>
</tbody>
</table>

Notes: This table shows the elasticities from the log-linear solution of the RBC model. Persistence of the technology shock is set to 0.95.

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The setup of the model in this section draws on Baxter and Crucini (1993) and lecture notes by John Campbell.

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zero and $\eta_{\text{rc}}$ tends to 0.89. In other words, the economy behaves like an exchange economy. For $\psi = 0.95$ and $\rho = 0.99$ it follows from (25) that $\eta^{lb}_{\text{rc}} = 0.74\gamma$. Similarly, $\eta^{f}_{\text{rc}} = \alpha(1 - \rho)/(1 - \rho\psi) = 0.11$ This means that high adjustment costs in combination with high risk aversion can increase the elasticity of the long bond with respect to the shock, but the elasticity of cash flows is still small even for extremely rigid investments.

Table 6 reports the resulting risk premia. Since $\eta_{\text{rc}}$ increases with $\gamma_r$, the Sharpe-ratio increases when investment becomes more rigid, for a given level of risk aversion. However, the effect for moderate levels of risk aversion is rather small. For log-utility, the Sharpe-ratio increases from 0.0026 without adjustment cost to 0.01 with high adjustment costs. To reach the value (from postwar data) of 0.26, high risk aversion is required even in the presence of extremely rigid investment. Since $\eta^{lb}_{\text{rc}}$ is increasing strongly in $\gamma_r$, the risk premium of the long bond also increases. But the effect is still too small – even for extreme adjustment costs – unless risk aversion is high. For risk aversion of 100, the premia for bonds is very large for high $\gamma_r$. Since $\eta^{lb}_{\text{rc}}$ is small regardless of the adjustment costs, the wedge between the risk premia of long bonds and equity is very small for any value of $\gamma_r$. Hence, adjustment costs cannot solve this puzzle of the RBC model.

### 7. Conclusion

The goal of this paper was to gain more insight into the determination of asset prices in RBC models. We presented closed-form solutions for expected risk premia of long-term bonds and equity based on the approximate log-linear solution of the RBC model in Campbell (1994). The analytical expressions for the asset prices were written as functions of the deep parameters of the model and the elasticities of the endogenous real variables with respect to the state variables, which can be computed using Campbell’s method.

Technology shocks affect the returns of long bonds and equity through two channels: directly through the shock and indirectly via capital accumulation. In the case of long bonds, expected interest rates are determined by the reaction of
consumption to technology shocks. For most parameter values consumption increases in response to a positive technology shock. In these cases the direct effect of the technology shock on the return of the long bond is positive while the effect through capital accumulation is negative. The sign of the net effect is ambiguous, but for permanent shocks the negative effect always dominates. So in this case, the risk premium of a long bond is negative. In addition to future interest rates, equity returns are also affected by changes in expectations about future dividend growth. The response of dividend news to a technology can also be decomposed into a direct channel due to the shock and an indirect channel from capital accumulation. For cash flows, the first effect is always positive whereas the capital effect is negative. Depending on the parameter values of the RBC model, either effect can dominate. It is therefore possible that long bonds carry a higher risk premium than equity.

The risk-free short-term interest rate follows an ARMA(2,1) process with the same autoregressive roots as consumption. However, the moving-average roots differ. The reaction of the risk-free rate after a technology shock can be positive or negative, although for most parameter values it increases following a positive shock. We also solved for the unconditional volatility of the risk-free rate and computed the Sharpe-ratio as a general measure of the risk-return trade-off in the model.

Next, we used the analytical results to compute asset prices for a variety of values of risk aversion and persistence of the technology shock. The Sharpe-ratio increases in risk aversion and persistence. In order to obtain roughly the same Sharpe-ratio as in postwar data, shocks have to be permanent and risk aversion must be high. Risk premia are also generally very small. Unless shocks are permanent, the risk premium of long bonds exceeds the equity premium. For permanent shocks, both premia are negative unless risk aversion is very low.

We considered two extensions to the basic model. First, we allowed labour to be variable. For given risk aversion, risk premia are somewhat higher than in the fixed-labour model because agents choose to work more after a positive shock since wages are higher. This makes consumption more volatile, which increases risk premia; however, the effect is rather small. Next we considered leverage in the capital structure of firms. Leverage actually decreases the equity premium if dividend growth reacts negatively after a positive shock. In any case, leverage rates must be extremely high to have a major impact on the equity premium. As a third extension, we allowed for adjustment costs in investment. Rigid investment causes the economy to behave more like an exchange economy in which agents consume their endowment. For extremely costly investment, risk premia can increase substantially. However, risk aversion still must be high in order to generate premia of the order of magnitude seen in the data. The wedge between risk premia for long bonds and equity is very small, even for extreme adjustment costs.

This paper is further evidence that the profession is still a long way away from understanding the determination of asset prices and how they are related to macroeconomic fundamentals. For example, the recent run-up in stock prices is very hard to reconcile with the current generation of models. Advances in
technology, at least as modelled in RBC models, cannot account for the extremely high premia to equity observed in the last few years. Moreover, from a macroeconomic perspective, it is interesting to study the relationship of consumption and asset prices. Lettau and Ludvigson (2000) observe that the ratio of consumption to wealth depends on expectations of future returns to wealth. Incorporating such links present interesting challenges for future research.

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References


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