Survival with Ambiguity

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Abstract

We analyze a market populated by expected utility maximizers and smooth ambiguity-averse consumers. We study conditions under which ambiguity-averse consumers survive and affect prices in the limit. If ambiguity vanishes with time or if the economy exhibits no aggregate risk, ambiguity-averse consumers survive, but have no long-run impact on prices. In both scenarios, ambiguity-averse consumers are fully insured against ambiguity in equilibrium and, thus, behave as expected utility maximizers with correct beliefs. If ambiguity-averse consumers are not fully insured against ambiguity, their behavior mimics expected utility maximizers with wrong beliefs and a discount factor which might be higher or lower than their actual discount factor. We use this insight to analyze a Markov economy with large persistent ambiguity. Consumers with decreasing absolute ambiguity aversion whose prudence with respect to ambiguity exceeds twice their absolute ambiguity aversion a.s. survive in the presence of expected utility maximizers with correct beliefs. If the economy further exhibits aggregate risk, they drive the expected utility maximizers out of the market and determine prices in the limit. In contrast, consumers with increasing or constant absolute ambiguity aversion ambiguity aversion only survive in the absence of aggregate risk and have no impact on limit prices.

Keywords: ambiguity, ambiguity-aversion, survival.

JEL Codes: D50, D81.

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1 Introduction

The simplicity and tractability of the representative consumer model has established it as the work-horse of macroeconomics and finance. A growing body of literature, however, suggests that (under reasonable assumptions on the parameters) a representative expected utility maximizer cannot account for the historically observed prices and dividends in the financial markets. This, in turn, has created interest for models which consider a representative agent with alternative preference specifications and try to account for the observed price anomalies, see, for example, Backus, Routledge and Zin (2005). Among the models that have been proposed, theories of ambiguity aversion have established themselves as a viable alternative to expected utility maximization. They capture experimentally observed behavior and have been used to explain some of the empirical phenomena documented in financial markets, such as the home bias, as in Uppal and Wang (2003), the equity premium puzzle, as in Epstein and Schneider (2008), Collard et al. (2011), negative correlation between asset prices and returns, as in Ju and Miao (2012).

The limitation of this approach consists in the fact that it replaces one representative agent by another without seriously considering the impact of heterogeneity of preferences on prices and market allocations. The literature on market selection pioneered by Sandroni (2000) and Blume and Easley (2006) provides the tools to examine the long-term effects of heterogeneity in markets. In particular, it allows us to analyze the robustness of such explanations by examining whether agents with alternative preference specifications can survive in the presence of expected utility maximizers and exert persistent influence on market prices.

It has been established that in economies with bounded endowments, when markets are complete, Sandroni (2000), or when the allocation is Pareto optimal, Blume and Easley (2006), investors’ survival only depends on the accuracy of beliefs and the size of discount factors. In particular, among equally patient investors, only those with correct beliefs survive; among investors with correct beliefs, only the most patient survive. Risk attitudes, and more generally preferences, do not matter.

These results pose a serious caveat for any of the behavioral approaches which are alternative to expected utility maximization. In fact one can typically represent non-expected utility max-
imizing behavior as expected utility maximizing behavior for some wrong beliefs. Hence any deviation from expected utility maximization can only be seen as a short term phenomenon, with no long term effect on market prices. This would suggest that this type of preference heterogeneity can be disregarded provided one is only interested in long term outcomes. Indeed, so far existing studies of survival of non-expected utility maximizing agents in the presence of expected utility maximizers with correct beliefs (max-min expected utility in Condie (2008), variational preferences in Da Silva (Unpublished Results), loss-aversion in Easley and Yang (Unpublished Results); see Section 2 for a discussion of these results) have failed to identify a persistent impact of such agents on market outcomes.

In this paper we argue that even though ambiguity-averse investors behave as expected utility maximizers with wrong beliefs, they may nevertheless survive and affect market outcomes in the long run. Hence, differently from other alternatives to expected utility maximization and in contrast to the results obtained in Condie (2008), ambiguity aversion might represent an important source of heterogeneity in financial markets, which cannot be ignored even in the long run.

We address these issues by examining a market populated by expected utility maximizers and smooth ambiguity-averse investors, as in Klibanoff, Marinacci and Mukerji (2009), henceforth KMM (2009). We choose this model, because it allows us to separate the objective ambiguity present on the market, to which all investors are exposed, from the subjective attitude towards ambiguity. Furthermore, it also allows us to vary the degree of ambiguity aversion and relate it to the investor’s chances to survive. Finally, two recent papers, Collard et al. (2011) and Ju and Miao (2012), show that asset prices in an economy with a representative smooth ambiguity-averse agent replicate the historically observed patterns in asset returns.

We assume that the market exhibits two levels of uncertainty: the first is the uncertainty about the investors’ endowments, the second is the uncertainty about the probability distribution determining the evolution of endowments. We refer to the first type of uncertainty as risk and to the second type of uncertainty as ambiguity. Ambiguity is described by the set of probability distributions which can govern the endowment process and by a second-order probability distribution on this set of priors. The main difference between ambiguity and risk in our model consists in the fact that the realization of the risky state (realization of endowments) is interpersonally verifiable, while the realization of the ambiguous state (the distribution of endowments)
is not. Hence, asset payoffs and prices can only depend on the realization of the risky variables, but not on the realization of the ambiguous ones. Similarly, trades cannot be made contingent on the ambiguous states, i.e., on the distribution governing the endowment streams. We assume that the economy has a complete set of Arrow securities with payoffs contingent on the realization of the risky state. No assets with payoffs contingent on the realization of the ambiguous states are available.

In our model, both types of investors have the same information about the structure of uncertainty. Both ambiguity-averse investors and expected utility maximizers are averse towards risk. However, while ambiguity-averse investors prefer to reduce their exposure to ambiguity, expected utility maximizers are indifferent towards it. Hence, if both types of investors have identical discount factors and correct beliefs, differences in their ability to survive can only be attributed to the difference in their attitude towards ambiguity.

We restrict the set of economies to those, in which there is a unique objectively correct second-order probability distribution consistent with the observed process of state realizations. The main finding of our paper is that if the ambiguity that the economy faces is sufficiently large and persistent such that the consumer cannot fully insure himself against it, not even in the long run, survival is not independent of the degree of ambiguity aversion. This is true, even though all investors in the economy are assumed to have correct beliefs and identical discount factors. The intuition behind this result is as follows: a smooth ambiguity-averse investor with correct beliefs and a constant discount factor effectively behaves as an expected utility maximizer with incorrect beliefs and a time-dependent discount factor. We name the beliefs and the discount factor we would have to attribute to the ambiguity-averse investor in order to reconcile his behavior with expected utility maximization, his effective beliefs and his effective discount factor. The factors modifying the beliefs and the discount rate depend on the decision maker’s equilibrium consumption and on the function describing his attitude towards ambiguity. In particular, if the ambiguity-averse investor were completely insured against ambiguity, he would be indistinguishable from an expected utility maximizer with a constant discount factor and correct beliefs. However, we show that if the economy faces large and persistent ambiguity and aggregate risk, the ambiguity-averse investor will not be completely insured against ambiguity and, hence, his ambiguity aversion will influence both his effective beliefs and his effective discount factor. His effectively wrong beliefs always inhibit his chances of survival compared to
an expected utility maximizer with correct beliefs. However, changes in his effective discount factor can offset this effect.

We analyze three classes of functions representing the investor’s attitude towards ambiguity: functions with constant, decreasing and increasing absolute ambiguity aversion. For these three classes of functions, we compute the effective discount factor of the ambiguity-averse investor. The effective discount factor is equivalent to the actual discount factor for the class of functions exhibiting constant absolute ambiguity aversion. It is larger (smaller) than the actual discount factor for the class of functions with decreasing (increasing) absolute ambiguity aversion. We then use these results to derive implications for the survival of ambiguity-averse investors. Since the effective discount factor for ambiguity-averse investors with decreasing absolute ambiguity aversion is larger than their actual discount factor, it forces them to save more, and thus, enhances their chances of survival. We show that if these investors’ prudence with respect to ambiguity exceeds twice their absolute ambiguity aversion, this effect offsets the effect caused by wrong beliefs. Such investors a.s. survive. If furthermore, they are not completely insured against ambiguity in equilibrium, they drive expected utility maximizers with correct beliefs out of the market. On such paths, allocations and prices in the limit are identical to those in an economy with a representative ambiguity-averse agent. In particular, such paths will exhibit higher saving rates and distorted state prices compared to those predicted by using standard models of expected utility maximization.

For the case of constant and increasing absolute ambiguity aversion, the effective discount factor of the ambiguity-averse investors either remains unchanged or is less than their actual discount factor, while their effective beliefs differ from the truth. If they are not fully insured against ambiguity, such investors a.s. vanish from the market, even though their actual beliefs are correct and their discount factor is identical to the one of the expected utility maximizers.

The intuition behind these results is simple: in the smooth model of ambiguity, ambiguity aversion has an intertemporal effect, and may force the investor to save more relative to an ambiguity-neutral investor. Osaki and Schlesinger (Unpublished Results) refer to this effect as precautionary savings. In a two-period model with exogenous uncertainty, they show that when the investors’ prudence with respect to ambiguity exceeds their ambiguity aversion, or equivalently, when they exhibit decreasing absolute ambiguity aversion, precautionary savings are positive. In our model, the same condition corresponds to ambiguity-averse investors being
effectively more patient than their expected utility counterparts. If furthermore their prudence exceeds twice their absolute ambiguity aversion, the resulting increase in precautionary savings offsets the effect of wrong beliefs preventing their level of consumption from converging to 0. In contrast, increasing absolute ambiguity aversion (which is equivalent to prudence being lower than absolute ambiguity aversion) makes ambiguity-averse investors effectively less patient and results in negative precautionary savings, eventually driving them out of the market.

It is important to note that the dependence of survival on ambiguity aversion arises only for cases in which: (i) the economy faces aggregate risk, and (ii) the ambiguity is large and persistent (in a sense to be made clear in Sections 3 and 5 respectively). To indicate the importance of aggregate risk, we analyze the case in which the total endowment of the economy is certain. We show that in this scenario, all investors will be fully insured against risk, and thus, also against ambiguity. It follows that ambiguity-averse investors with correct beliefs will survive, but their ambiguity aversion will not matter for prices and allocations\(^3\). To highlight the effect of persistent ambiguity, we study the case in which investors can learn the probability distribution of asset payoffs as time evolves by observing the state realizations. In this case, only beliefs and discount factors determine survival. In particular, ambiguity-averse investors with correct beliefs and discount factors equal to those of the expected utility maximizers survive. However, since ambiguity vanishes in the limit, ambiguity aversion has no long run impact in this scenario. Finally, to show that large ambiguity is necessary for our results, we construct an example of an economy with persistent ambiguity and aggregate risk, in which ambiguity-averse agents are fully insured against ambiguity and thus, survive, but market outcomes are indistinguishable from those generated by a representative expected utility maximizer.

In economies in which expected utility maximizers with correct beliefs are present, survival of ambiguity-averse investors with effectively wrong beliefs is possible only if these investors are effectively more patient than expected utility maximizers. This condition is no longer necessary, when all investors in the economy have identical wrong beliefs and the economy exhibits large persistent ambiguity with aggregate risk. In an example, we show that in such an economy, the effective beliefs of ambiguity-averse expected utility maximizers can be closer to the truth than the beliefs of expected utility maximizers. Ambiguity-averse investors a.s. survive and ensure that market beliefs are closer to the truth than in a market populated only by expected utility maximizers.

\(^3\) Similar results are derived in Condie (2008) and Da Silva (Unpublished Results).
maximizers.

The remainder of the paper is organized as follows: the next section provides a short overview of the related literature. Section 3 presents the model of a market with expected utility maximizers and smooth ambiguity-averse consumers. Section 4 defines and shows the existence of an interior equilibrium for such an economy. Section 5 analyzes the question of survival with ambiguity aversion with correct beliefs and states our main results. Section 6 presents an example of an economy, in which all investors have wrong beliefs. Section 7 concludes. All proofs are collected in the Appendix.

2 Related Literature

The paper which is most closely related to our work is Condie (2008), which analyzes the issue of survival of max-min expected utility maximizers as axiomatized by Gilboa and Schmeidler (1989). Condie shows that when there is an expected utility maximizer with correct beliefs and when the true probability distribution is in the interior of the max-min consumer’s set of priors, this consumer vanishes, unless he is completely insured. The intuition behind this result is simple: at any period, a max-min consumer can be represented as an expected utility maximizer by choosing beliefs in such a way that they support the equilibrium consumption stream at the equilibrium prices. These effective beliefs will correspond to the truth only if the max-min consumer is completely insured, but will be wrong otherwise. Hence, max-min expected utility maximizers can survive only in economies in which there is no persistent aggregate risk. The case of no aggregate risk, however, is special in that ambiguity-averse investors are fully insured and, thus, have no effect on prices. Hence, even when ambiguity-averse investors survive, market outcomes look as if all investors were expected utility maximizers. Using the max-min expected utility to determine whether ambiguity-averse investors can survive is problematic for several reasons: first, the max-min expected utility does not allow for a distinction between ambiguity and ambiguity attitude. Hence, it is not clear whether the fact that max-min investors vanish should be attributed to their ambiguity aversion, or to information asymmetries: while max-min investors face uncertainty about the actual distribution of returns, expected utility maximizers "know" the correct distribution. Second, even if one were to attribute the effect to ambiguity aversion, the max-min expected utility only allows for a very extreme form of ambiguity aversion: the decision maker always chooses the worst probability distribution to evaluate a given
act. Finally, the max-min expected utility is a limit case of the smooth ambiguity model with constant absolute ambiguity aversion. This raises the question of whether the degree of ambiguity aversion and the way ambiguity aversion changes with wealth can influence survival. We extend the result of Condie (2008) by considering a more general class of ambiguity-averse consumers, clearly differentiating between objective ambiguity and subjective ambiguity attitude and highlighting the role of differences in ambiguity aversion for survival.

Da Silva (Unpublished Results) looks at survival with variational preferences as in Maccheroni et al. (2006a, b) and shows that agents with such preferences survive whenever their effective beliefs converge to the truth. As in Condie (2008), instances of survival of ambiguity-averse agents are without any impact on market allocations and prices.

Rigotti, Shannon and Strzalecki (2008) show how effective beliefs can be derived for all known models of ambiguity aversion. While we use a similar technique to analyze the conditions for the survival of a smooth ambiguity-averse consumer, in our infinite-horizon model there are two effects at work: ambiguity aversion causes the consumer to behave as an expected utility maximizer with incorrect beliefs, but it may also force him to save more thereby increasing his effective discount factor. The latter effect can compensate for the former and result in survival. More generally, our paper contributes to the literature on survival in financial markets by re-examining the question of whether correct beliefs are the only determinant of survival. As it is well-known from the work of Sandroni (2000) and Blume and Easley (2006), in economies with bounded endowments and complete markets populated by expected utility maximizers, market participants with identical discount factors survive if and only if they have correct beliefs. Rader (1981), Kogan et al. (2006, 2011) and Yan (2008) prove that the consumer’s risk attitude influences his ability to survive, together with his discount factor and beliefs, for the case of a growing economy with unbounded aggregate endowments. In particular, Rader (1981) and Yan (2008) use homothetic preferences and show that, ceteris paribus, lower degrees of risk aversion are beneficial for survival. This effect, however disappears for bounded economies.

Kogan et al. (2011) study an economy with unbounded endowment and identical risk preferences and show that agents with incorrect beliefs can survive for specific choices of the utility function for risk. These results are however quite sensitive to the choice of the utility function for risk: in the commonly used HARA-class, the only utility functions, for which this result obtains are those of the CARA type.
In this paper, we retain the bounded economy assumption common to the earlier literature on market selection and, similarly to Rader (1981) and Yan (2008) demonstrate that preferences and specifically, ambiguity aversion, can matter for survival. The results we obtain are, however, quite different: while ceteris paribus, lower degrees of risk aversion are beneficial for survival, this is not true in general for lower degrees of ambiguity aversion. What matters is the way in which absolute ambiguity aversion changes with wealth and the resulting impact on precautionary savings.

Our framework deviates from both the bounded and unbounded economy market selection literature cited above in two respects: first, markets are incomplete in that they do not allow for bets on ambiguous events; second, decision makers’ preferences deviate from expected utility maximization and in particular are not time-separable. The market incompleteness prevents ambiguity-averse agents from insuring completely against ambiguity. The time-inseparability of preferences leads to the difference between the actual and the effective discount factor used by ambiguity-averse agents.

We consider two special cases in which market incompleteness and time-inseparability do not matter: the case of vanishing ambiguity, in which betting on infinite endowment streams coincides with betting on the ambiguous states of the economy; and the case of no aggregate risk, in which insuring everyone against risk in equilibrium automatically guarantees that all agents are also completely insured against ambiguity. In these two cases, the only relevant characteristic for survival are the consumer’s beliefs. Ambiguity-averse investors behave exactly as expected utility maximizers.

In general, however, market incompleteness and time-inseparability will have an effect on the equilibrium allocations. When the ambiguity-averse consumers are not able to fully insure themselves against ambiguity, survival is dependent on the ambiguity attitude. In this sense, our paper is related to the research on survival in incomplete financial markets. Coury and Sciubba (2012) show that it is always possible to construct an equilibrium in which an agent with incorrect beliefs survives. Beker and Chattopadhyay (2010) demonstrate that the dynamics of an economy with incomplete markets is highly non-trivial: in some cases an agent with correct beliefs can vanish, in others, the economy might exhibit cycles in which the consumption of each of the agents approaches 0 infinitely often. While these papers look at rather general forms of incompleteness, in our paper the incompleteness arises from the presence of ambiguity and
is, therefore, quite specific, in that it only matters for the ambiguity-averse consumers. If all consumers were expected utility maximizers, equilibrium allocations in our setting would be Pareto-optimal and the survival results for complete markets would go through.

There are few works in the literature, which consider preference specifications other than expected utility maximization in the context of survival. Borovicka (Unpublished Results) examines survival in the context of Epstein and Zin (1989) preferences. He also shows that time-nonseparability has an effect on survival as compared to the case of time-separable preferences. However, his results are derived for the case of identical preferences and heterogeneous beliefs and show that investors with wrong beliefs can survive even when agents with correct beliefs are present in the market. In this sense, they are more similar to the approach of Kogan et al. (2011) than to our paper. The question of whether heterogeneity of preferences matters is not discussed in his work. Furthermore, he considers an economy with unbounded endowment, whereas our results are for the case of a bounded economy.

Easley and Yang (Unpublished Results) analyze the survival of loss-averse decision makers and show that these agents disappear in the presence of investors with Epstein-Zin preferences, who do not exhibit loss-aversion. Hence, even though loss-aversion has been shown to generate some of the asset pricing phenomena observed in the data, it cannot account for them in the long-run.

3 The Model

3.1 Modelling the Uncertainty

Let \( \mathbb{N} = \{0; 1; 2; \ldots \} \) denote the set of time periods. Uncertainty is modelled through a sequence of random variables \( \{S_t\}_{t \in \mathbb{N}} \) each of which takes value from a finite set \( S_t \). We set \( S_0 = \{s_0\} \), i.e., no information is revealed in period 0. Denote by \( s_t \in S_t \) the realization of random variable \( S_t \). Denote by \( \Sigma = \prod_{t \in \mathbb{N}} S_t \) the set of all possible observation paths, with representative element \( \sigma = (s_0; s_1; s_2; \ldots s_t; \ldots) \). Finally denote by \( \Sigma_t = \prod_{\tau=0}^{t} S_t \) the collection of all finite paths of length \( t \), with representative element \( \sigma_t = (s_0; s_1; s_2; \ldots s_t) \). Each finite observation path \( \sigma_t \) identifies a decision/observation node and the set of all possible observation paths \( \Sigma \) can also be seen as the

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4 Mathematically, Epstein-Zin preferences represent a special case of the KMM’s (2009) recursive model of smooth ambiguity aversion. We thank Viktor Tsyrennikov for pointing this out.
set of all nodes.

We can represent the information revelation process in this economy through a sequence of finite partitions of the state space $\Sigma$. In particular, define the cylinder with base on $\sigma_t \in \Sigma$, $t \in \mathbb{N}$ as $Z(\sigma_t) = \{ \sigma \in \Sigma | \sigma = (\sigma_t) \}$. Let $\mathbb{F}_t = \{ Z(\sigma_t) : \sigma_t \in \Sigma \}$. Be a partition of the set $\Sigma$. Clearly, $\mathbb{F} = (\mathbb{F}_0 \ldots \mathbb{F}_t \ldots)$ denotes a sequence of finite partitions of $\Sigma$ such that $\mathbb{F}_0 = \Sigma$ and $\mathbb{F}_t$ is finer than $\mathbb{F}_{t-1}$. We assume that all agents have identical information and that the information revelation process is represented by the sequence $\mathbb{F}$.

Let $\mathcal{F}_t$ be the $\sigma$-algebra generated by partition $\mathbb{F}_t$. $\mathcal{F}_0$ is the trivial $\sigma$-algebra. Let $\mathcal{F}$ be the $\sigma$-algebra generated by $\cup_{t \in \mathbb{N}} \mathcal{F}_t$. It can be shown that $\{ \mathcal{F}_t \}_{t \in \mathbb{N}}$ is a filtration.

We define on $(\Sigma; \mathcal{F})$ a family of probability distributions $\{ \pi^n \}_{n=1}^N$ and throughout we assume $\pi^n(Z(\sigma_t)) > 0$ for all $n$ and all $\sigma_t$. Intuitively, $\{ \pi^n \}_{n=1}^N$ denotes the set of models which could describe the evolution of the state process in the economy. In what follows, for brevity, we abuse notation slightly by denoting $\pi^n(Z(\sigma_t)) = \pi^n(\sigma_t) = \pi^n(\sigma_t; s_0; s_1; s_2 \ldots s_t)$. The one-step-ahead probability distribution $\pi^n(s_{t+1} | \sigma_t)$ at node $\sigma_t$ is determined by:

$$
\pi^n(s_{t+1} | \sigma_t) = \pi^n(\sigma_t; s_0; s_1; s_{t+1} | s_0 \ldots s_t) = \frac{\pi^n(s_0 \ldots s_t; s_{t+1})}{\pi^n(s_0 \ldots s_t)} \quad \text{for any } s_{t+1} \in S_{t+1}
$$

In words, $\pi^n(s_{t+1} | \sigma_t)$ is the probability under distribution $\pi^n$ that the next observation will be $s_{t+1}$ given that we have reached node $\sigma_t$.

Agents in the economy are uncertain about the realization of $\pi^n$. Hence, there are two sources of uncertainty: uncertainty about the realization of the state of the world $s_t$, captured by the probability distributions $\pi^n$, and uncertainty about the actual probability measure which governs the realization of the state of the world. We will refer to the first source of uncertainty as risk, while the term ambiguity is used with regard to the second. The important distinction is that whereas $s_t$ (and thus $\sigma_t$) are publicly observable, the realizations of $\pi$ are not observable.

To describe ambiguity formally, we define an extended state space $\Theta = \{ s_0 \} \times \{ \{ \pi^1 \ldots \pi^N \} \times S_t \}_{t \in \mathbb{N} \setminus \{0\}}$ with a representative path $\theta = (s_0; (\pi_1; s_1); (\pi_2; s_2) \ldots (\pi_t; s_t) \ldots)$. The interpretation is that at time $t$, $\pi_t$ is the unobservable realization of an element of $(\pi^n)$ over the observable states $s_t$. The truncated paths $\theta_t$ and the corresponding sets $\Theta_t$ are defined by analogy to $\sigma_t$ and $\Sigma_t$. Since only $\sigma_t$ and not $\theta_t$ is observable at each time $t$, we define the projection of $\theta_t$ on $\Sigma_t$ by $\sigma_t(\theta_t)$. Also by analogy, we define the $\sigma$-algebra $\hat{\mathcal{F}}$ generated by the natural filtration on $\Theta$. Denote by $\mu$ the "true" probability distribution on $(\Theta; \hat{\mathcal{F}})$ with $\mu(\theta_t) = \mu(s_0; (\pi_1; s_1); (\pi_2; s_2) \ldots (\pi_t; s_t))$. 

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Conditional probabilities are defined in the usual way. A natural restriction imposed on $\mu$ is that
\[ \mu(s \mid \theta_{t-1}; \pi^n) = \frac{\mu(\theta_{t-1}; (\pi^n; s))}{\sum_{s' \in S_t} \mu(\theta_{t-1}; (\pi^n; s'))} = \pi^n(s \mid \sigma_{t-1}(\theta_{t-1})), \tag{2} \]
i.e., the one-step-ahead probability distribution over observable states in period $t$ is indeed given by the realization of $\pi$ in $t$. The probability of an observable path $\sigma_t$ according to $\mu$ is $\mu(\sigma_t) =: \sum_{\{\theta_t|\sigma_t(\theta_t) = \sigma_t\}} \mu(\theta_t)$.

Finally, given any $\pi^n \in \{\pi^n\}_{n=1}^N$ and any $\sigma_t \in \Sigma_t$, the probability assigned by the Bayesian posterior of $\mu$ conditional on $\sigma_t$ to $\pi^n$ is denoted by $\mu_{\sigma_t}(\pi^n)$ and is given by:
\[ \mu_{\sigma_t}(\pi^n) =: \frac{\sum_{s_{t+1} \in S_{t+1}} \sum_{\{\theta_t|\sigma_t(\theta_t) = \sigma_t\}} \mu(\theta_t; (\pi^n; s_{t+1}))}{\sum_{\{\theta_t|\sigma_t(\theta_t) = \sigma_t\}} \mu(\theta_t)}. \tag{3} \]
The formula is a direct application of the Bayes rule: the numerator is the probability of the joint occurrence of $\pi^n$ at $t + 1$ and $\sigma_t$ (these are all paths $\theta_t$, compatible with the observed state $\sigma_t$, on which $\pi^n$ is realized together with any possible observation of $s_{t+1}$); in the denominator is the probability of all paths $\theta_t$, compatible with the observed state $\sigma_t$.

For the analysis of survival in Section 5, we will assume that the set of states of the world is identical across all periods, i.e., $S_t = S$ for all $t \in \mathbb{N} \setminus \{0\}$. Two benchmark cases will be of particular interest: vanishing ambiguity and persistent ambiguity.

**Definition 3.1** In an economy with vanishing ambiguity, in period 0, a probability distribution $\pi^n$ is drawn according to a distribution $\mu = (\mu_1 \ldots \mu_N) \gg 0$ and the observable path $\sigma$ is determined according to $\pi^n$. Furthermore, the set of observable paths $\Sigma$ can be partitioned into subsets $\{\Sigma^1 \ldots \Sigma^N\}$ such that $\pi^n(\Sigma^n) = 1$.

An economy with vanishing ambiguity is thus defined by the fact that: (i) the true model is determined in period 0; (ii) observing $\sigma \in \Sigma^n$ a.s. allows the investor to identify the correct model as being $\pi^n$. In this case, it is possible to learn the true probability distribution $\pi^n$ by observing the state of the world $s_t$ in each period and using Bayesian updating on the prior $\mu$. The posteriors satisfy $\mu_{\sigma_t}(\pi^n) = \frac{\pi^n(\sigma_t)\mu_{\sigma_t}}{\sum_{\sigma_t'=1}^{\pi^n(\sigma_t')\mu_{\sigma_t'}}}$ for $n \in \{1 \ldots N\}$ and $\lim_{t \to \infty} \mu_{\sigma_t}(\pi^n) = 1$ a.s. on the set of paths $\Sigma^n$, i.e., whenever $\pi^n$ is the realization of the initial draw.

Now consider a situation, in which the uncertainty about $\pi^n$ does not vanish along a given path $\sigma$. We will call this class of economies, economies with persistent ambiguity:

**Definition 3.2** In an economy with persistent ambiguity, there is a $\tilde{\delta} > 0$ such that for all $\theta_t \in \Theta$, $\sigma_t \in \Sigma$ and all $n \in \{1 \ldots N\}$, $\mu_{\theta_t}(\pi^n) = \mu_{\sigma_t(\theta_t)}(\pi^n) > \tilde{\delta}$.

Since the conditional probabilities of all $\pi^n$ remain bounded away from 0, in such an economy
it is impossible to learn the correct model, even upon observing the entire observable path $\sigma$.

An important class of economies with persistent ambiguity are those economies in which both $\mu$ and $(\pi^n)_{n=1}^N$ are Markov processes$^5$.

**Definition 3.3** In an economy with Markov persistent ambiguity, $\mu$ is a Markov process with

$$
\mu_{\theta_t}(\pi^n) = \mu_{\sigma_t(\theta_t)}(\pi^n) =: \pi_s(\pi^n) > 0
$$

for all $n \in \{1...N\}$, all $t \in \mathbb{N}$, all $\theta_t \in \Theta_t$ and all $\sigma_t \in \Sigma_t$. Also, for all $n \in \{1...N\}$, $\pi^n$ is a Markov process with $\pi^n(s | \sigma_t) := \pi^n(s | s_t) > 0$.

Note that in such an economy, $\tilde{\delta}$ can be defined as $\tilde{\delta} := \min_{s \in S} \mu_s(\pi^n) > 0$.

### 3.2 Preferences and Beliefs

There is a single good and $I$ infinitely lived consumers, each with consumption set $\mathbb{R}_+$. A consumption plan $c : \Sigma \to \prod_{t \in \mathbb{N}} \mathbb{R}_+$ is a sequence of $\mathbb{R}_+$-valued functions $\{c(\sigma_t)\}_{t \in \mathbb{N}}$ in which each $c(\sigma_t)$ is $\mathcal{F}_t$-measurable. Each consumer is endowed with a particular consumption plan, called $i$'s endowment stream and denoted $e^i$. $e$ stands for the total endowment of the economy.

Let $\mu^i$ be a probability distribution on $\left( \Theta; \bar{\mathcal{F}} \right)$ representing consumer $i$'s prior. Given any $\pi^n \in \{\pi^n\}_{n=1}^N$ and any $\sigma_t \in \Sigma_t$, agent $i$’s posterior probability of $\pi^n$ is denoted by $\mu^i_{\pi^n}(\pi^n)$

$$
\mu^i_{\sigma_t}(\pi^n) = \mu^i(\pi^n | \sigma_t) = \frac{\sum_{s_{t+1} \in S_{t+1}} \sum_{\theta_t | \sigma_t(\theta_t) = \sigma_t} \mu^i(\theta_t; (\pi^n; s_{t+1}))}{\sum_{\theta_t | \sigma_t(\theta_t) = \sigma_t} \mu^i(\theta_t)}.
$$

Let $\succeq_i$ denote agent $i$’s preference ordering over consumption plans. Preferences $\succeq_i$ are represented by the following recursive functional:

$$
V^i_{\sigma_t}(c^i) = u_i(c^i(\sigma_t)) + \beta_i \phi_i^{-1} \left( \sum_{n=1}^N \phi_i \left( \sum_{s_{t+1} \in S_{t+1}} V^i_{(\sigma_t; s_{t+1})}(c^i) \pi^n(s_{t+1} | \sigma_t) \right) \mu^i_s(\pi^n) \right).
$$

This representation of preferences was suggested by KMM (2009). Here $\beta_i \in (0; 1)$ is agent $i$'s intertemporal discount factor; $u_i : \mathbb{R}_+ \to \mathbb{R}$ and $\phi_i : \mathbb{R} \to \mathbb{R}$ are continuous and strictly increasing functions. The interpretation of $V^i$ is as follows: at time $t$, on path $\sigma$, consumer $i$ receives an instantaneous utility from consumption $u_i(c^i(\sigma_t))$. From the next period on, he expects a state-contingent consumption stream which, depending on the state realization in period $t+1$, $s_{t+1}$, will generate a discounted utility equal to $V^i_{(\sigma_t; s_{t+1})}(c^i)$. The consumer faces two types of uncertainty: first, he does not know which state will occur in period $t+1$, second

---

$^5$ This type of processes is used in Miao and Ju (2012) to replicate asset returns in a model with smooth ambiguity aversion.
he is uncertain on which probability distribution determines the realization of the state at \( t+1 \).

The first type of uncertainty — risk — is captured by taking the expectation of the discounted payoffs with respect to a probability measure \( \pi^n(s_{t+1} \mid \sigma_t) \). The second type of uncertainty — ambiguity — is captured by a probability distribution over \( \pi^n, \mu_{\sigma_t}^{i}(\pi^n) \) and a concave function \( \phi_i \). While the distribution \( \mu_{\sigma_t}^{i} \) captures the perceived ambiguity, \( \phi_i \) expresses consumer \( i \)'s attitude towards this ambiguity. Finally, applying the inverse of \( \phi_i \) to the expression in square brackets and multiplying by \( \beta_i \) corresponds to finding the certainty equivalent of the expected future consumption stream in terms of present utility. Note that when \( \phi_i \) is a linear function (e.g., the identity), the representation above reduces to intertemporal expected utility maximization.

Our choice of the preference representation is motivated by the following considerations: first, differently from most other forms of representation of ambiguity-averse preferences, the KMM (2009) smooth model of ambiguity allows for a clear separation between ambiguity and ambiguity attitude. In particular, the function \( \phi \) controls the degree of ambiguity aversion and allows us to compare decision makers which differ according to this characteristic. Second, the smooth model of ambiguity allows for a recursive formulation. This means that the beliefs of the decision maker are updated according to the Bayesian rule and the modelled behavior is dynamically consistent\(^6\).

We impose the following assumptions on the primitives of the model:

**Assumption 1** The functions \( u_i : \mathbb{R}_+ \to \mathbb{R} \) are twice continuously differentiable, strictly concave, \( u_i(0) = 0, \lim_{c \to 0} u_i'(c) = \infty \) and \( \lim_{c \to \infty} u_i'(c) = 0 \).

**Assumption 2** Each of the functions \( \phi_i : \mathbb{R} \to \mathbb{R} \) is either linear or strictly concave, twice continuously differentiable and \( \lim_{y \to 0} \phi_i'(y) > 0 \).

**Assumption 3** Endowments are uniformly bounded away from zero and aggregate endowments are uniformly bounded. Formally, there is an \( m > 0 \) such that \( e_i(\sigma_t) > m \) for all \( i, \sigma_t \); moreover, there is an \( m' > m > 0 \) such that \( \sum_{i=1}^{I} e^i(\sigma_t) < m' \) for all \( \sigma_t \).

**Assumption 4** There is a \( \delta > 0 \) such that for all paths \( \sigma \), dates \( t \) and states of the world \( s_{t+1} \in S_{t+1}, \pi^n(s_{t+1} \mid \sigma_t) > 0 \) for some \( n \in \{1...N\} \) implies \( \pi^n(s_{t+1} \mid \sigma_t) \geq \delta \) for all \( n \in \{1...N\} \).

**Assumption 5** For every \( \sigma_t \), the set of one-step-ahead probability distributions \( \Pi(\sigma_t) = \)

---

\(^6\) In general, in models of ambiguity aversion, there is an intrinsic tension between dynamic consistency and standard generalizations of the Bayesian updating rule. The recursive smooth model of ambiguity formulated by KMM (2009) is one of the few existing in the literature, which simultaneously satisfy dynamic consistency and allow for Bayesian updating of beliefs. KMM (2009) provide a second formulation of the representation, which is time-separable, but violates dynamic consistency. The analysis of survival for such preferences (and more generally, for ambiguity-averse preferences which violate dynamic consistency) is a question of independent interest.

14
\[\{\pi^n(s \mid \sigma_t)\}_{n=1}^N\] consists of \(N\) linearly independent vectors.

Assumptions 1 and 3 appear in Blume and Easley (2006). Assumption 2 is necessary, since we extend their model to the case of ambiguity aversion. Assumption 1 implies that all consumers are strictly risk-averse\(^7\). Assumption 2 allows for both ambiguity aversion and ambiguity neutrality, hence the case of expected utility maximization is covered by our model. Taken together, Assumptions 1 and 2 exclude the case in which a consumer chooses 0 consumption in an (observable) state of the world in which the consumer has a positive endowment and which has a positive probability according to this consumer’s beliefs. Assumption 3 requires that each consumer’s endowment in all states of the world is uniformly bounded above and uniformly bounded away from 0. Assumption 4 states that for every \(\sigma_t\), the one-step-ahead probability distributions \(\pi^n(\cdot \mid \sigma_t)\) are mutually absolutely continuous and that the minimal positive probability they assign to a given state in the next period conditional on the history \(\sigma_t\) is uniformly bounded away from 0.

Taken together, Assumptions 1—4 guarantee that the solution to the consumer’s maximization problem will be interior. Hence, they preclude the possibility that a consumer would vanish in finite time.

The role of Assumption 5 is to ensure that at each \(\sigma_t\), we can uniquely identify "correct" beliefs for our economy. If this condition is satisfied, the compound one-step-ahead state probabilities of consumer \(i\), \(\sum_{n=1}^N \mu^i_{\sigma_t}(\pi^n)\pi^n(s \mid \sigma_t)\), coincide with the compound one-step-ahead probabilities under the true probability distribution \(\sum_{n=1}^N \mu_{\sigma_t}(\pi^n)\pi^n(s \mid \sigma_t)\) if and only if \(\mu^i_{\sigma_t} = \mu_{\sigma_t}\).

In particular, if all consumers are expected utility maximizers with identical discount factors, those whose priors \(\mu^i\) are not absolutely continuous with respect to \(\mu\) will vanish almost surely as long as an investor with a prior \(\mu\) is present.

This might not hold in absence of Assumption 5. Consider an economy with Markov persistent ambiguity, \(S_t = S = \{s_1; s_2\}\), \(N = 3\) and such that \(\mu_{\sigma_t}(\pi^n) = \mu^n\) and \(\pi^n(s \mid \sigma_t) = \pi^n(s)\) for all \(\sigma_t\), \(n \in \{1; 2; 3\}\) and \(s \in S\). \(\pi^n\) are given by: \((\pi^n)_{n=1}^3 = (\frac{1}{2}; \frac{1}{2}), (\pi^2(s_1); \pi^2(s_2)) = (\frac{1}{3}; \frac{2}{3}), (\pi^3(s_1); \pi^3(s_2)) = (\frac{2}{3}; \frac{1}{3})\). Suppose the truth is that the two states are equally likely: \(\mu_1 = 1\). Any belief \(\mu^i\) with \(\mu^i_2 = \mu^i_3\) is going to generate the same probabilities of reaching states \(s_1\) and \(s_2\). Hence, any expected utility maximizer with this type

\(^7\) The restriction \(u(0) = 0\) is necessary only for those functions \(\phi\), which have a non-negative domain (e.g., \(\phi = \ln\)).
of beliefs will survive, even though his beliefs may be wrong in that they are not absolutely
continuous with respect to the truth (for example, \( \mu_1' = 0, \mu_2' = \mu_3' = \frac{1}{2} \)). This is not the type of
wrong beliefs we are interested in. Hence, we eliminate such redundancy by Assumption 5.

4 The Equilibrium of the Economy

We assume that markets are complete with respect to the observable states of the world, i.e.
there is a complete system of Arrow securities contingent on the realization of \( \sigma_t \) for all \( \sigma_t \in \Sigma \).
However, agents are not able to trade on the realization of the probability distribution \( \pi^n \), i.e. the
probability distribution over states is non-contractible. Since both endowments and consumption
streams are assumed to be \( \mathcal{F}_t \)-measurable, at any time \( t \), the only information available
about \( \pi^n \) is the realization of \( \sigma_t \). Hence, the restriction that trades can only be conditioned on
\( \sigma_t \) appears fairly natural.

Definition 4.1 An equilibrium of the economy is an integrable price system \( (p_t(\sigma_t))_{\sigma_t \in \Sigma} \) and
a consumption stream \( c^i \) for every consumer \( i \) such that for all \( t \), at all nodes \( \sigma_t \in \Sigma_t \), all consumers \( i \in \{1...I\} \) are maximizing their utility given the price system and markets clear:

\[
c^i = \arg \max_{c^i} V_{\sigma_t}^i (c^i) = \arg \max_{c^i} \{ u_i (c^i (\sigma_t)) + + \beta_t \phi_t^{-1} \sum_{n=1}^{N} \phi_n \left( \sum_{n+1 \in S_{n+1}} V_{(\sigma_{t+n+1})}^i (c^i) \pi^n (s_{t+n+1} | \sigma_t) \right) \mu_{\sigma_t}^i (\pi^n) \}
\]

s.t. \( \sum_{\sigma_t \in \Sigma_t} p(\sigma_t) c^i (\sigma_t) \leq \sum_{\sigma_t \in \Sigma_t} p(\sigma_t) e^i (\sigma_t), \forall t \}

\[
\sum_{i=1}^{I} c^i (\sigma_t) = \sum_{i=1}^{I} e^i (\sigma_t).
\]

Since markets in our economy are incomplete, we cannot directly use the Pareto-optimality
conditions as in Blume and Easley (2006). Instead, we first show that an equilibrium of the
economy exists and then use the properties of this equilibrium to discuss survival.

Proposition 4.1 Under Assumptions 1–4, an equilibrium of the economy exists.

Our next Proposition ensures that the equilibrium can be described by a system of first-order
conditions. The result of this Proposition is a direct consequence of the Inada conditions im-
posed on the function \( u \), the concavity of \( \phi \) and the mutual absolute continuity of the one-step-
ahead distributions \( \Pi (\sigma_t) \).

To state the Proposition, we introduce the following notation: for a given consumption stream
\( c^i, E^\pi_{n}(V^i_{\sigma_{t+1}}(c^i)) \) denotes the expectation of the value function \( V^i_{(\sigma_t;S_{t+1})} \) conditional on \( \sigma_t \) with respect to the probability distribution \( \pi^n \),

\[
E^\pi_{n}(V^i_{\sigma_{t+1}}(c^i)) =: \sum_{s_{t+1} \in S_{t+1}} V^i_{(\sigma_t;S_{t+1})}(c^i)(s_{t+1} | \sigma_t). 
\] (9)

**Proposition 4.2** Under Assumptions 1–4, the equilibrium of the economy satisfies for all \( i \in \{1,...I\} \), all \( t \in \mathbb{N} \), and all \( \sigma_t \in \Sigma_t \) and \( s_{t+1} \in S_{t+1} \) such that \( \mu(\sigma_t) > 0 \) and such that \( \sum_{n=1}^{N} \mu^i_n(\pi^n)(s_{t+1} | \sigma_t) > 0 \):

\[
\frac{u'_i(c^i(\sigma_t))}{\beta_i u'_i(c^i(\sigma_t); s_{t+1})} \frac{\sum_{n=1}^{N} \phi_i^n(E_{n}(V^i_{\sigma_{t+1}}(c^i))\mu^i_n(\pi^n)|s_{t+1} | \sigma_t)}{\phi_i^n(\sum_{n=1}^{N} \phi_i^n(E_{n}(V^i_{\sigma_{t+1}}(c^i))\mu^i_n(\pi^n)))} = \frac{p(\sigma_t)}{p(\sigma_t; s_{t+1})} \] (10)

This result allows us to use techniques similar to Blume and Easley (2006) to analyze the conditions under which ambiguity-averse consumers can survive.

## 5 Survival with Ambiguity Aversion and Correct Beliefs

**Definition 5.1** Consumer \( i \) vanishes on a path \( \sigma \) if \( \lim_{t \to \infty} c^i(\sigma_t) = 0 \). Consumer \( i \) survives on \( \sigma \) if \( \lim_{t \to \infty} \sup c^i(\sigma_t) > 0 \).

The survival of a consumer can in general depend on his preferences, on his discount factor and on his beliefs. In this paper, we concentrate on the impact of ambiguity aversion on survival, while keeping the discount factors and the beliefs of the decision makers identical for most of the discussion. For a given function \( \phi \), the coefficient of absolute ambiguity aversion is given by: \(-\frac{\phi''}{\phi'}\). We distinguish between constant, decreasing and increasing absolute ambiguity aversion, depending on the monotonicity properties of \(-\frac{\phi''}{\phi'}\). We will use the abbreviations CAAA, DAAA and IAAA, respectively, to denote these cases. We note that functions which exhibit DAAA, also exhibit prudence with respect to ambiguity, i.e., \( \phi''' > 0 \). In analogy to absolute prudence with respect to risk, see Kimball (1990), the absolute prudence with respect to ambiguity is given by \(-\frac{\phi'''}{\phi''}\). Note that DAAA (IAAA) obtains if and only if prudence with respect to ambiguity exceeds (is lower than) absolute ambiguity aversion, \(-\frac{\phi'''}{\phi''} > -\frac{\phi''}{\phi'}\). Just as in the case of risk aversion, we can define the class of hyperbolic absolute ambiguity aversion (HAAA) preferences as

\[
\phi(y) = \frac{\alpha y}{1-\gamma} + d \] ,

where \( \alpha \) and \( d \) are chosen in such a way that \( \frac{\alpha y}{1-\gamma} + d > 0 \).

8 Here, \( \gamma = 1 \) corresponds to the ambiguity-neutral case; \( \gamma = 0 \) to \( \phi(y) = \ln y \) and \( \gamma = \pm \infty \), to the CAAA class with a coefficient of absolute ambiguity-aversion given by \( \frac{\phi''}{\phi'} \). For finite \( \gamma \), the condition \( \gamma < 1 \) characterizes those HAAA functions which exhibit DAAA, whereas \( \gamma > 1 \) defines the class of those HAAA functions with IAAA.
5.1 Survival in Economies with Vanishing Ambiguity

We start by analyzing whether ambiguity aversion has an impact on survival for the case of vanishing ambiguity described in Definition 3.1. Recall that in such an economy consumer $i$'s beliefs can be described by a prior $\mu^i = (\mu^i_1 \ldots \mu^i_N)$ over $(\pi^N)_{n=1}^N$.

**Proposition 5.1** Suppose that Assumptions 1–4 hold. Consider an economy with vanishing ambiguity, and suppose that all consumers have identical discount factors, $\beta_i = \beta$ for all $i \in \{1 \ldots I\}$. Suppose that for a given consumer $i$, $\mu^i$ is absolutely continuous with respect to the truth $\mu$, i.e., $\mu^i_n > 0$ implies $\mu^i_n > 0$;

1. the function defined by

$$G^i(\tilde{\mu}^i_1(\pi^1) \ldots \tilde{\mu}^i_N(\pi^N)) = \frac{\sum_{n=1}^N \phi^i_n (y_n) \tilde{\mu}^i_\sigma_i (\pi^n) \pi^n (s_{t+1} | \sigma_t)}{\phi^i_1 (\phi^{-1}_i \left( \sum_{n=1}^N \phi^i_1 (y_n) \tilde{\mu}^i_\sigma_i (\pi^n) \right))}, \quad (11)$$

where $y_n$ are parameters bounded between $[0; \frac{1}{1-\beta} u(m')]$, is continuously differentiable and its total derivative is uniformly bounded for all values of the parameters.

Then $i$ survives almost surely. In particular, a consumer $i$ whose prior is absolutely continuous w.r.t. the truth survives a.s. whenever $\phi_i$ exhibits HAAA.

Proposition 5.1 is in line with the main result in Blume and Easley (2006). With identical discount factors only beliefs matter for survival, while preferences are immaterial. In particular, the absolute degree of ambiguity aversion plays no role in determining which of the consumers will survive, as long as the priors are absolutely continuous with respect to the truth. The additional condition (ii) we have to impose simply requires that a slight change in the posteriors, $\mu^i_\sigma_i (\pi^n)$ leads to a uniformly bounded change in the factor

$$\frac{\sum_{n=1}^N \phi^i_1 (E_{\pi^n} (V^{i}_{\sigma_{t+1}} (c^i))) \mu^i_\sigma_i (\pi^n) \pi^n (s_{t+1} | \sigma_t)}{\phi^i_1 (\phi^{-1}_i \left( \sum_{n=1}^N \phi^i_1 (E_{\pi^n} (V^{i}_{\sigma_{t+1}} (c^i))) \mu^i_\sigma_i (\pi^n) \right))} \quad (12)$$

which takes the place of beliefs in the first-order condition of the smooth ambiguity-averse consumers. This implies that when $\mu_\sigma_i$ is close to the truth, i.e., assigns a probability close to 1 to the "true" model on this path $\pi^{n*} \in \{\pi^1 \ldots \pi^N\}$ such that $\lim_{t \to \infty} \mu_\sigma_i (\pi^{n*}) = 1$, (12) is close to the Dirac measure $\delta_{\pi^{n*}}$ assigning a probability of 1 to $\pi^{n*}$. It guarantees that (12) converges to $\delta_{\pi^{n*}}$ at the same rate as the beliefs of an expected utility maximizer updated according to the Bayesian rule. The latter implies survival, as shown in Blume and Easley (2006). As our result demonstrates, all most commonly used functional forms satisfy this condition.
5.2 Survival in Economies with No Aggregate Risk

Our next result establishes the survival of ambiguity-averse consumers for the case of persistent ambiguity and no aggregate risk. In this case, the ambiguity-averse consumer is completely insured against ambiguity and behaves as an expected utility maximizer with correct beliefs.

Proposition 5.2 Suppose that Assumptions 1–4 hold. Suppose that all consumers \( i \in \{1...I\} \) have identical discount factors \( \beta_i = \beta \) and correct beliefs, \( \mu^i \sigma_t (\pi^n) = \mu \sigma_t (\pi^n) \) for all \( \sigma_t \) and all \( n \in \{1...N\} \). In an economy with persistent ambiguity and no aggregate risk, i.e. \( e(\sigma_t; s_{t+1}) = e(\sigma_t; s'_{t+1}) \) for all \( t \in \mathbb{N} \), all \( \sigma_t \in \Sigma_t \) and all \( s_{t+1} \) and \( s'_{t+1} \in \Sigma_{t+1} \), all consumers survive a.s..

In the two cases discussed in Propositions 5.1 and 5.2, ambiguity-averse consumers effectively mimic expected utility maximizers, either because ambiguity vanishes with time, or because full insurance against ambiguity coincides with full insurance against risk which is available to everyone in the economy.

5.3 Economies with Persistent Ambiguity and Aggregate Risk

We now turn to the case of persistent ambiguity with aggregate risk. We will assume that all states in \( S \) have strictly positive one-step-ahead probabilities, i.e., \( \pi^n(s | \sigma_t) \geq \delta \) for all \( s \in S, \sigma_t \in \Sigma \) and \( n \in \{1...N\} \). Since, by assumption, for all \( \theta_t \in \Theta \) and all \( n \in \{1...N\} \), \( \mu_{\sigma_t(\theta_t)} (\pi^n) = \mu_{\theta_t} (\pi^n) \), by Assumption 5, consumer \( i \) has correct beliefs iff \( \mu^i \sigma_t (\pi^n) = \mu \sigma_t (\pi^n) \) for all \( \sigma_t \in \Sigma \) and all \( n \in \{1...N\} \). For some of our results, we will consider the following special class of economies referred to as Markov economies:

Definition 5.2 A Markov economy is an economy with Markov persistent ambiguity, in which the total initial endowment depends only on the current state realization, i.e., \( e(\sigma_t; s) = e(\sigma'_t; s) := e_s \) for all \( t, t' \in \mathbb{N}, \sigma, \sigma' \in \Sigma \) and all \( s \in S \).

To understand the conditions which determine whether ambiguity-averse consumers can survive in an economy in which ambiguity matters in a non-trivial way, we state:

Lemma 5.3 Suppose that Assumptions 1–4 hold. Consider an economy with persistent ambiguity. If \( i \) is a smooth ambiguity-averse consumer, while \( j \) is an expected utility maximizer with \( \phi_j(y) = y \) and correct beliefs, then on any given path \( \sigma \in \Sigma \), the equilibrium consumption streams of \( i \) and \( j \) satisfy:

\[
\lim_{T \to \infty} \frac{1}{T+1} \ln \left( \frac{u^i_j (c^j (\sigma_T; s_{T+1}))}{u^i_j (c^i (\sigma_T; s_{T+1}))} \right) = \left[ \ln \beta_j - \ln \beta_i \right] - \\
- \lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} \ln \phi_i \left( \frac{\sum_{n=1}^{N} \phi_i^i [E_{\pi^n} (V_{\sigma_{t+1}^i} (c^i))] \mu^i \sigma_t (\pi^n)}{\phi_i^i \left( \phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i \left[ E_{\pi^n} (V_{\sigma_{t+1}^i} (c^i)) \right] \mu^i \sigma_t (\pi^n) \right) \right)} \right) \tag{13}
\]
The remaining factor is given by

\[ \lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} \left[ \ln \frac{1}{\sum_{n=1}^{N} \mu_{\sigma_{t}} (\pi^{n}) \pi^{n} (s_{t+1} \mid \sigma_{t})} - \frac{\sum_{n=1}^{N} \phi_{n} [E_{\pi^{n}} (V_{\sigma_{t+1}} (c^{i}))] \mu_{\sigma_{t}} (\pi^{n}) \pi^{n} (s_{t+1} \mid \sigma_{t})}{\sum_{n=1}^{N} \phi_{n} [E_{\pi^{n}} (V_{\sigma_{t+1}} (c^{i}))] \mu_{\sigma_{t}} (\pi^{n})} \right] \]

This Lemma is key to our following results. The sign of the l.h.s of (13) identifies the cases in which an ambiguity-averse investor survives or vanishes in the presence of an expected utility maximizer with correct beliefs. Since consumption is bounded above, \( u_{j}' (c^{i} (\sigma_{T}; s_{T+1})) \neq 0 \). It follows that \( \lim_{T \to \infty} \frac{1}{T+1} \ln \frac{u_{j}' (c^{i} (\sigma_{T}; s_{T+1}))}{u_{j}' (c^{i} (\sigma_{T}; s_{T+1}))} \) will be positive on a given path if and only if \( u_{j}' (c^{i} (\sigma_{T}; s_{T+1})) \to \infty \), i.e. if the consumption of \( i \) on this path converges to 0 and \( i \) vanishes. If \( \lim_{T \to \infty} \frac{1}{T+1} \ln \frac{u_{j}' (c^{i} (\sigma_{T}; s_{T+1}))}{u_{j}' (c^{i} (\sigma_{T}; s_{T+1}))} \) is negative or zero, consumer \( i \) will not vanish relative to \( j \).

The r.h.s. of (13) highlights the factors which determine whether \( i \) survives. As in Blume and Easley (2006), the first factor is the difference in the discount factors of \( j \) and \( i \) — the higher \( i \)'s discount factor \( \beta_{i} \), the more \( i \) is going to save, hence, the more wealth \( i \) will accumulate relative to \( j \) and the higher \( i \)'s chances for survival.

To understand the second and the third term on the r.h.s. of (13), it is useful to look at the MRS of an expected utility maximizer and a smooth ambiguity-averse decision maker. To do so, fix a \( \sigma_{t} \) and an \( s_{t+1} \). In equilibrium,

\[ \beta_{j} u_{j}' (c^{i} (\sigma_{t})) \frac{\sum_{n=1}^{N} \phi_{n} [E_{\pi^{n}} (V_{\sigma_{t+1}} (c^{i}))] \mu_{\sigma_{t}} (\pi^{n}) (s_{t+1} \mid \sigma_{t})}{\phi_{i}^{-1} \left( \frac{\sum_{n=1}^{N} \phi_{n} [E_{\pi^{n}} (V_{\sigma_{t+1}} (c^{i}))] \mu_{\sigma_{t}} (\pi^{n})}{\sum_{n=1}^{N} \phi_{n} [E_{\pi^{n}} (V_{\sigma_{t+1}} (c^{i}))] \mu_{\sigma_{t}} (\pi^{n})} \right)} \]

Note that the factor

\[ \sum_{n=1}^{N} \phi_{n} [E_{\pi^{n}} (V_{\sigma_{t+1}} (c^{i}))] \mu_{\sigma_{t}} (\pi^{n}) (s_{t+1} \mid \sigma_{t}) \]

\[ \phi_{i}^{-1} \left( \frac{\sum_{n=1}^{N} \phi_{n} [E_{\pi^{n}} (V_{\sigma_{t+1}} (c^{i}))] \mu_{\sigma_{t}} (\pi^{n})}{\sum_{n=1}^{N} \phi_{n} [E_{\pi^{n}} (V_{\sigma_{t+1}} (c^{i}))] \mu_{\sigma_{t}} (\pi^{n})} \right) \]

in the MRS of an ambiguity-averse decision maker takes the place of beliefs

\[ \sum_{n=1}^{N} \mu_{\sigma_{t}} (\pi^{n}) (s_{t+1} \mid \sigma_{t}) \]

for an expected utility maximizer. While the expression in (15) is not necessarily a probability distribution, we can normalize it to obtain the effective beliefs of the ambiguity-averse agent:

\[ \frac{\sum_{n=1}^{N} \phi_{n} [E_{\pi^{n}} (V_{\sigma_{t+1}} (c^{i}))] \mu_{\sigma_{t}} (\pi^{n}) (s_{t+1} \mid \sigma_{t})}{\sum_{n=1}^{N} \phi_{n} [E_{\pi^{n}} (V_{\sigma_{t+1}} (c^{i}))] \mu_{\sigma_{t}} (\pi^{n})} \]

The remaining factor is given by

\[ \frac{\sum_{n=1}^{N} \phi_{n} [E_{\pi^{n}} (V_{\sigma_{t+1}} (c^{i}))] \mu_{\sigma_{t}} (\pi^{n})}{\phi_{i}^{-1} \left( \frac{\sum_{n=1}^{N} \phi_{n} [E_{\pi^{n}} (V_{\sigma_{t+1}} (c^{i}))] \mu_{\sigma_{t}} (\pi^{n})}{\sum_{n=1}^{N} \phi_{n} [E_{\pi^{n}} (V_{\sigma_{t+1}} (c^{i}))] \mu_{\sigma_{t}} (\pi^{n})} \right)} \]
and does not depend on the next-period-state, \( s_{t+1} \). It can be interpreted as an additional discount factor, which multiplies the actual discount factor of \( i, \beta_i \). We will refer to the expression

\[
\beta_i \frac{\sum_{n=1}^{N} \phi_i \left[ E_{\pi^n} \left( V_{\sigma_{t+1}} (c^i) \right) \right] \mu_{\sigma_i} (\pi^n)}{\phi_i \left( \sum_{n=1}^{N} \phi_i \left[ E_{\pi^n} \left( V_{\sigma_{t+1}} (c^i) \right) \right] \mu_{\sigma_i} (\pi^n) \right)}
\]

as the effective discount factor of the ambiguity-averse decision maker \( i \).

In what follows, we will discuss the additional discount factor and the effective beliefs separately. Here, we merely state the following useful lemma:

**Lemma 5.4** If Assumptions 1–4 hold, then, in an economy with persistent ambiguity, each of the factors (15), (17) and (18) is uniformly bounded away from 0 and uniformly bounded above.

Lemma 5.4 demonstrates that just as, by Assumption 4, the one-step-ahead beliefs of an expected utility maximizer are uniformly bounded away from 0, so are the one-step-ahead effective beliefs of the ambiguity-averse agent. Similarly, his additional discount factor is also uniformly bounded away from 0 and uniformly bounded above.

### 5.3.1 The Effective Beliefs of an Ambiguity-Averse Consumer

As in Blume and Easley (2006), the beliefs of the ambiguity-averse agent play a crucial role for his survival relative to an expected utility maximizer. This is reflected in the last term on the r.h.s. of (13), which contains the log of the difference of the effective beliefs of \( i \) and the (correct) beliefs of \( j \). In expectations, this term will equal the relative entropy of \( i \)'s beliefs with respect to the true probability distribution. In particular, if both consumers have correct actual beliefs, but the effective beliefs (17) of the ambiguity-averse consumer differ from the truth, this will naturally inhibit his chances for survival. In this subsection we examine whether the effective beliefs of the ambiguity-averse consumer will differ from the truth in the limit.

Consider expression (17) which describes the effective beliefs of an ambiguity-averse consumer \( i \). Note that \( i \)'s effective beliefs will in general differ from \( i \)'s actual beliefs. As the following lemma shows, the two will coincide if and only if \( i \) is fully insured against ambiguity.

**Lemma 5.5** Suppose that Assumptions 1–5 hold. In an economy with persistent ambiguity, the effective beliefs of a smooth ambiguity-averse consumer \( i \) coincide with his actual beliefs if and only if \( i \) is fully insured against ambiguity. Formally, for a given \( \sigma_t \),

\[
\sum_{n=1}^{N} \phi_i \left[ E_{\pi^n} \left( V_{\sigma_{t+1}} (c^i) \right) \right] \mu_{\sigma_i} (\pi^n) = \sum_{n=1}^{N} \mu_{\sigma_i} (\pi^n) \pi^n (s | \sigma_t)
\]

holds for all \( s \in S \) if and only if for all \( n, n' \in \{1...N\}, E_{\pi^n} \left( V_{\sigma_{t+1}} (c^i) \right) = E_{\pi^{n'}} \left( V_{\sigma_{t+1}} (c^i) \right) \).
Hence, even if i's actual beliefs are correct, i.e., $\mu_{i_{\sigma_i}}(\pi^n) = \mu_{\sigma_i}^{\pi} (\pi^n)$ for all $n$, his effective beliefs will differ from the truth, unless he is fully insured against ambiguity.

In fact, whenever ambiguity-averse decision makers are fully insured against ambiguity, they will not perceive the consumption stream to be ambiguous. They will thus behave as expected utility maximizers and will have no impact on prices. If the sets of probability distributions $\Pi(\sigma_t)$ are small relative to the set of states of the world $S$, it is easy to generate economies with persistent ambiguity and aggregate risk, in which full insurance against ambiguity obtains in equilibrium.

**Example 5.1** Consider a Markov economy such that for each $\pi^n$, the variables $S_t$ are i.i.d. Let $\pi_t$ also be i.i.d. with $\mu_{\sigma_i}(\pi^n) =: \mu_n$ for all $\sigma_t \in \Sigma$. Suppose that all consumers $i \in \{1...I\}$ have identical discount factors $\beta_i = \beta$, correct beliefs and identical utility functions $u_t = u$. Suppose also that $\Pi(\sigma_t) \equiv \Pi$ contains less than $|S| - 1$ linearly independent probability distributions. By the Rouché-Capelli Theorem, the set of solutions of

$$\left( \sum_{s \in S} u\left(e^i(s)\right) \pi^n(s) = \sum_{s \in S} u\left(e^i(s)\right) \pi^{n+1}(s) \right)^N$$

is an affine space of $\mathbb{R}^n$ of dimension at least 2. In particular, there exists an initial endowment process for $i$ such that $e^i(\sigma; s) = e^i(s)$ for all $\sigma_t$, (21) holds and $e^i(s) \neq e^i(s')$ for some states $s$ and $s'$. Set $e^j(s) = e^i(s)$ for all $s \in S \cup \{s_0\}$ and all $j \in \{1...I\}$ to obtain an economy with persistent ambiguity and aggregate risk. Note that

$$u\left(e^i(s_0)\right) + \sum_{t=1}^{\infty} \beta^t \sum_{s \in S} u\left(e^i(s)\right) \pi^n(s) = u\left(e^i(s_0)\right) + \sum_{t=1}^{\infty} \beta^t \sum_{s \in S} u\left(e^i(s)\right) \pi^{n'}(s)$$

for all $n, n' \in \{1...N\}$ and all $i \in \{1...I\}$. Hence, at the initial endowment $e$, all agents are fully insured against ambiguity, and, since the agents are symmetric, the initial endowment coincides with the equilibrium allocation. Furthermore, by Lemma 5.5, consumers’ effective beliefs coincide with the truth and hence, all agents a.s. survive, regardless of their attitude towards ambiguity. Therefore, ambiguity attitude has no impact on prices and allocations.

Example 5.1 requires that the sets of one-step-ahead probability distributions $\Pi(\sigma_t)$ are "small" relative to the set of contractible states $S$. In this case, full insurance against ambiguity can be obtained for all agents by trading on $S$ alone, even though the economy exhibits aggregate risk. In general, however, since the probability distributions over states $\pi^n$ are non-contractible, an ambiguity-averse agent might have limited possibilities to insure himself against ambiguity.

Specifically, in economies, which exhibit "large" Markov persistent ambiguity, i.e., in which the sets $\Pi(\sigma_t) \equiv \Pi(s_t)$ are sufficiently rich, an agent will be fully insured against ambiguity only if he is fully insured against risk.

**Definition 5.3** An economy exhibits *large* Markov persistent ambiguity if all sets of one-step-
ahead probability distributions \( \Pi(s_t) \) contain at least \( |S| - 1 \) linearly independent vectors.

**Lemma 5.6** Let Assumptions 1–4 hold. In an economy which exhibits large Markov persistent ambiguity, an agent is fully insured against ambiguity at a node \( \sigma_t \) if and only if he is fully insured against risk at \( \sigma_t \).

Combined with Lemma 5.5, Lemma 5.6 suggests that in economies with large Markov persistent ambiguity which also exhibit aggregate risk, the effective beliefs of ambiguity-averse consumers will differ from the truth. We would like to show that in this type of economies, effective beliefs of ambiguity-averse consumers can remain distant from the truth even in the limit. We first provide a formal definition of convergence of effective beliefs towards the actual beliefs of a given investor. For a given consumption stream \( c^i \), write \( b^i (s | \sigma_t) \) for the effective beliefs (17) of consumer \( i \) on node \( \sigma_t \).

**Definition 5.4** In equilibrium, the effective one-step-ahead beliefs of agent \( i \) converge towards his actual beliefs on a path \( \sigma \) if

\[
\lim_{T \to \infty} \frac{1}{T + 1} \sum_{t=0}^{T} \left[ \sum_{s \in S} \sum_{n=1}^{N} \mu_{\sigma_t} (\pi^n) \pi^n (s | \sigma_t) \ln \frac{\sum_{n=1}^{N} \mu_{\sigma_t} (\pi^n) \pi^n (s | \sigma_t)}{b^i (s | \sigma_t)} \right] = 0. \tag{23}
\]

Definition 5.4 says that \( i \)'s effective one-step-ahead beliefs on \( \sigma \) converge to his actual one-step-ahead beliefs \( \sum_{n=1}^{N} \mu_{\sigma_t} (\pi^n) \pi^n (s | \sigma_t) \) if, as \( t \) goes to \( \infty \), the average relative entropy (with respect to the truth) of \( b^i \) converges towards the relative entropy of \( i \)'s actual beliefs. In Lemma A.1 in the Appendix we show that this definition is equivalent to requiring that the effective beliefs of \( i \) converge towards his actual beliefs along \( \sigma \) on all but a sparse set of time periods. It turns out that this is the definition of belief convergence which is relevant for survival.

We next show that in the class of Markov economies with large persistent ambiguity and aggregate risk the set of paths on which an ambiguity-averse agent with correct beliefs does not vanish and his effective one-step-ahead beliefs converge towards the truth has measure 0.

**Proposition 5.7** Suppose that Assumptions 1–5 hold. Consider a Markov economy with large persistent ambiguity and aggregate risk. Let \( I = \{i; j\} \), where \( j \) is an expected utility maximizer, with \( \phi_j (y) = y \) and \( i \) is a smooth ambiguity-averse consumer. Suppose that both consumers have identical discount factors \( \beta_i = \beta_j = \beta \) and identical correct beliefs. Then the set of paths, on which \( i \) does not vanish and \( i \)'s one-step-ahead effective beliefs converge towards his actual beliefs (i.e., the truth) has measure 0.

Proposition 5.7 states that in Markov economies with large persistent ambiguity and aggregate risk, ambiguity-averse consumers can survive only if their beliefs do not converge to the truth.
Hence, if such a consumer survives, he will have a non-trivial impact on prices even in the limit. However, the fact that his beliefs are distinct from the truth inhibits his chances to survive. To understand whether such a consumer can indeed survive in the presence of an expected utility maximizer with correct beliefs, in the next subsection, we will examine the second factor which influences survival, the effective discount factor of the ambiguity-averse consumer.

5.3.2 The Effective Discount Factor of an Ambiguity-Averse Consumer

The decomposition (13) implies that the value of the effective discount factor of an ambiguity-averse consumer has an impact on his chances for survival. In particular, if (18) exceeds, is equal to or is lower than 1, the effective discount factor of the ambiguity-averse decision maker will be higher, equal or lower than his actual discount factor. If further, $\beta_i = \beta_j$ and if (18) exceeds (is lower than) one, the additional discount factor will enhance (inhibit) the ambiguity-averse consumer’s ability to survive.

Our next results show that the additional discount factor (18) exactly equals 1 for consumers with constant absolute ambiguity aversion; it is less than one in the case of increasing absolute ambiguity aversion; it is greater than one for the case of decreasing absolute ambiguity aversion.

Lemma 5.8  Under Assumptions 1–4, the additional discount factor in (18) of a CAAA-consumer satisfies

$$\frac{\sum_{n=1}^{N} \phi_i^t \left[ E_{\pi_n} \left( V_{\sigma_{t+1}}^i \left( c^i \right) \right) \right] \mu_{\sigma_i}^t \left( \pi^n \right)}{\phi_i^t \left( \phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i \left[ E_{\pi_n} \left( V_{\sigma_{t+1}}^i \left( c^i \right) \right) \right] \mu_{\sigma_i} \left( \pi^n \right) \right) \right)} = 1.$$  

(24)

Hence, the effective discount factor for such a consumer is $\beta_i$.

Lemma 5.9  Under Assumptions 1–4, the additional discount factor in (18) of an IAAA-consumer satisfies

$$\frac{\sum_{n=1}^{N} \phi_i^t \left[ E_{\pi_n} \left( V_{\sigma_{t+1}}^i \left( c^i \right) \right) \right] \mu_{\sigma_i}^t \left( \pi^n \right)}{\phi_i^t \left( \phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i \left[ E_{\pi_n} \left( V_{\sigma_{t+1}}^i \left( c^i \right) \right) \right] \mu_{\sigma_i} \left( \pi^n \right) \right) \right)} \leq 1$$  

(25)

and the equality is obtained only if $E_{\pi_n} \left( V_{\sigma_{t+1}}^i \left( c^i \right) \right) = E_{\pi_{n'}} \left( V_{\sigma_{t+1}}^i \left( c^i \right) \right)$ for all $n, n' \in \{1...N\}$. Hence, the effective discount factor for such a consumer is smaller than $\beta_i$.

Lemma 5.10  Under Assumptions 1–4, the additional discount factor in (18) of a DAAA-consumer satisfies

$$\frac{\sum_{n=1}^{N} \phi_i^t \left[ E_{\pi_n} \left( V_{\sigma_{t+1}}^i \left( c^i \right) \right) \right] \mu_{\sigma_i}^t \left( \pi^n \right)}{\phi_i^t \left( \phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i \left[ E_{\pi_n} \left( V_{\sigma_{t+1}}^i \left( c^i \right) \right) \right] \mu_{\sigma_i} \left( \pi^n \right) \right) \right)} \geq 1$$  

(26)

and the equality is obtained only if $E_{\pi_n} \left( V_{\sigma_{t+1}}^i \left( c^i \right) \right) = E_{\pi_{n'}} \left( V_{\sigma_{t+1}}^i \left( c^i \right) \right)$ for all $n, n' \in \{1...N\}$. Hence, the effective discount factor for such a consumer is greater than $\beta_i$. 

24
The intuition behind these results is simple: in the smooth model of ambiguity, ambiguity aver-
sion has an intertemporal effect, and may force the investor to save more or less relative to an
(otherwise identical) ambiguity-neutral investor. Osaki and Schlesinger (Unpublished Results)
refer to this effect as precautionary savings. In a two-period model, in which the decision maker
can only make a decision about savings and satisfies $u'' = 0$, they show that precautionary sav-
ings are positive or negative depending on whether the decision maker exhibits decreasing or
increasing absolute ambiguity aversion. Lemmas 5.8, 5.9 and 5.10 state the analogous result
for the effective discount factor, but pertain to an infinite horizon problem, in which the con-
sumer can allocate consumption arbitrary across states. Notably, they do not depend on the
properties of the utility function for risk. They imply that consumers with decreasing (increas-
ing) absolute ambiguity aversion, and thus a level of prudence with respect to ambiguity higher
(lower) than their absolute ambiguity aversion, will be effectively more (less) patient than an
otherwise identical ambiguity-neutral consumer.

We next discuss the implication of these results for the survival of ambiguity-averse consumers.

5.3.3 Survival in Economies with Large Persistent Ambiguity and Aggregate Risk

The decomposition in (13) allows us to identify two effects which will influence the chances
of survival for an ambiguity-averse consumer: his effective beliefs, which, in economies where
agents perceive large persistent ambiguity and aggregate risk, differ from the truth and have a
negative impact on survival, and his additional discount factor, which has a positive or negative
impact on survival, depending on whether it is larger or smaller than 1. The trade-off between
these two effects will determine whether ambiguity-averse agents will survive and have impact
on prices and allocations.

By Lemmas 5.8 and 5.9, when agents are either CAAA or IAAA, the effective discount factor
of the ambiguity-averse agent is either equal (in the CAAA case) or smaller (in the IAAA case)
than the discount factor of the expected utility maximizer. We thus obtain:

Proposition 5.11  Let Assumptions 1–4 hold. Consider an economy with persistent ambiguity.
Let $I = \{i; j\}$, where $j$ is an expected utility maximizer and $i$ is a smooth ambiguity-averse
consumer $i$, whose $\phi_i(y)$ is either CAAA or IAAA. Suppose that both $i$ and $j$ have correct
beliefs and identical discount factors. Then $j$ survives a.s. Furthermore, $i$ vanishes a.s. on the
set of paths, on which his effective one-step ahead beliefs do not converge to the truth as in (23).

Proposition 5.11 demonstrates that, even when controlling for discount factors and correct be-
liiefs, ambiguity aversion matters for survival, when ambiguity is large and persistent and the
decision makers cannot be fully insured against it. In particular, a CAAA / IAAA investor can survive in the presence of an expected utility maximizer with correct beliefs only on those paths, on which his effective beliefs converge towards the truth in the sense of (23). While large deviations of their effective beliefs from the truth might not disappear on such paths, they become more and more rare with time.

If Assumption 5 holds, we can combine this result with Proposition 5.7. We thus obtain that in Markov economies with large and persistent ambiguity and aggregate risk, the effectively incorrect beliefs of the ambiguity-averse consumers, combined with their effectively lower discount factor a.s. drive them out of the market.

We next turn to the issue of survival of DAAA consumers. By Lemma 5.10, the effective discount factor of such consumers exceeds their actual discount factor and may compensate for their effectively wrong beliefs. Our next proposition shows that this is indeed true:

**Proposition 5.12** Let Assumptions 1–4 hold. Consider an economy with persistent ambiguity. Let $I = \{i; j\}$, where $j$ is an expected utility maximizer and $i$ is a smooth ambiguity-averse consumer, whose $\phi_i(y)$ exhibits DAAA such that $-\frac{\phi''_i(y)}{\phi'_i(y)} \geq -2 \frac{\phi''_i(y)}{\phi'_i(y)}$. Suppose that both $i$ and $j$ have correct beliefs and identical discount factors. Then $i$ survives a.s. Furthermore, $j$ vanishes a.s. on the set of paths, on which $i$’s effective one-step-ahead beliefs do not converge to the truth as in (23).

Proposition 5.12 establishes that DAAA consumers survive in the presence of expected utility maximizers with correct beliefs, as long as their absolute prudence with respect to ambiguity $\left(-\frac{\phi''_i(y)}{\phi'_i(y)}\right)$ exceeds twice their absolute ambiguity aversion $\left(-\frac{\phi''_i(y)}{\phi'_i(y)}\right)$. This is a strengthening of the DAAA property which requires the ambiguity-averse consumer to be sufficiently prudent (relative to his ambiguity aversion) so that the resulting increase in precautionary savings compensates for his wrong effective beliefs. For instance, all HAAA functions with $\gamma \in [0; 1)$ satisfy this condition. Under Assumption 5, Proposition 5.7 implies that in Markov economies with large persistent ambiguity and aggregate risk, such ambiguity-averse consumers drive expected utility maximizers out of the market, despite having effectively wrong beliefs in the limit.

In the limit, such an economy looks as an economy with a representative DAAA consumer. We will thus observe deviations of state prices of consumption from those which would obtain in

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9 The analogous property, but with respect to risk is extensively discussed in Gollier (2001, p. 210, pp.285-289) and used, e.g., to characterize utility functions, for which introducing the possibility to invest in risky assets leads to an increase in savings.
an economy with a representative expected utility maximizer. As Lemma 5.10 demonstrates, DAAA consumers are effectively more patient than their expected utility counterparts, even though they have identical discount factors. This implies that such economies will exhibit levels of savings which would appear to be excessive and hard to reconcile, given the actual discount factor, with expected utility maximization, but which can be attributed to ambiguity aversion. The Markov economies considered by Ju and Miao (2012) exhibit persistent ambiguity, but with unbounded endowments. As their paper shows, ambiguity aversion may explain a range of pricing phenomena which are deemed ‘anomalies’ under expected utility maximization, such as excessive equity premium and negative correlation between asset prices and returns. Our analysis suggests that the same phenomena might obtain and be persistent in an economy with both expected utility maximizers with correct beliefs and investors exhibiting decreasing absolute ambiguity aversion, provided that ambiguity is large and the economy exhibits aggregate risk. Note that the results of Lemma 5.5 and of Proposition 5.7 depend critically on Assumption 5, i.e., on the existence of a unique objectively correct probability distribution $\mu$ consistent with the stochastic process in the economy. This assumption, together with the assumptions of large ambiguity and aggregate risk identifies a class of economies, in which ambiguity-averse consumers exhibit beliefs and behavior distinct from those of the expected utility maximizers and thus either drive the latter out of the market or vanish depending on their absolute ambiguity aversion. If there are only two states of the world, ($|S| = 2$), Assumption 5 is not necessary for the result in Proposition 5.7. However, if $|S| \geq 3$ and if not all of the vectors $(\pi^n(s))_{s \in S}$ are linearly independent, in violation of Assumption 5, it is possible to generate examples of economies, in which ambiguity-averse consumers would have effectively correct beliefs, even though they are not fully insured against ambiguity. Their survival would then depend only on their effective discount factor, as described in Lemmas 5.8, 5.9 and 5.10. The analysis of such economies would however add nothing substantially new to the results obtained so far.

6 Survival with Wrong Beliefs: An Example

So far, we have considered economies in which all agents have correct beliefs. We showed that in certain cases, ambiguity-averse investors might have an advantage over expected utility maximizers. This advantage arises, because the effective discount factor of investors with DAAA is higher than their actual discount factor. It is obvious that in the case in which expected util-
ity maximizers with correct beliefs are present, this is the only way in which ambiguity-averse investors can survive, while at the same time having wrong effective beliefs.

The question then naturally arises: can ambiguity aversion be superior to expected utility maximization in situations, in which no investor has correct beliefs? Just as above, as long as ambiguity-averse investors are not fully insured against ambiguity, their effective beliefs will differ from their actual (wrong) beliefs. Hence, if we can show that ambiguity aversion "corrects" for the initially wrong beliefs, survival of ambiguity-averse investors can occur even when the offsetting effect of larger effective discount factor is absent. In such economies, ambiguity-averse investors will not only survive, but also help move the initially wrong aggregate beliefs closer to the truth.

To demonstrate that this can indeed occur, consider the following example.

**Example 6.1** Consider a Markov economy, in which \( N = 2 \), \( |S| = 2 \), \( \mu_{\pi_i}(\pi^n) =: \mu_n > 0 \) and \( \pi^n(s \mid s_t) =: \pi^n(s) > 0 \) for \( n \in \{1; 2\} \) and all \( s_t \in S \). Let \( e_{s1} < e_{s2} \). Suppose that \( \pi^1(s^1) = \pi^2(s^2) =: \pi > \frac{1}{2} \). Intuitively, \( \pi^1 \) denotes the "bad" unobservable state, in which the state \( s^1 \) with the lower initial endowment is more likely, whereas \( \pi^2 \) is the "good" unobservable state.

There are two agents in the economy, \( i \) is an ambiguity-averse agent with \( ^{10} \phi_i(y) = -e^{-\alpha y} \) and \( u_i(c) = \ln c \), while \( j \) is an expected utility maximizer with \( u_j(c) = \ln c \). The endowments of the two agents are equal, i.e., \( e_i = e_j = \frac{n}{2} \) for \( s \in \{s^1; s^2\} \).

Assume that the probability of the "bad" unobservable state \( \mu_1 \) satisfies \( \mu_1 \in \left( \frac{1}{2}; 1 \right) \) and the agents have identical wrong beliefs given by \( \mu^1_i = \mu^1_j = \frac{1}{2} \). Hence, both agents underestimate the probability of the "bad" unobservable state \( \pi^1 \).

It is easy to show that in equilibrium, \( \frac{e^i(\sigma_i; s^1)}{e^i(\sigma_i; s^2)} \in \left( \frac{e_{s1}}{e_{s2}} ; 1 \right) \) holds for all \( \sigma_i \) and, hence, \( i \)'s one-step-ahead effective beliefs satisfy:

\[
\begin{align*}
& b_i(s^1 \mid \sigma_i) = \sum_{k=1}^{2} \mu^i_k \cdot \left( e^i(\sigma_i; s^1)^{-\pi^h(s^1)} \cdot (e^i(\sigma_i; s^2)^{-\pi^h(s^2)} \cdot \pi^k(s^1) - \pi^h(s^1)) \cdot \pi^k(s^1) \right) > \sum_{k=1}^{2} \mu^j_k \pi^k(s^1) = 1 \quad \frac{1}{2}.
\end{align*}
\]

(27)

Intuitively, \( i \)'s ambiguity aversion forces him to overestimate the probability of the "bad" unobservable state relative to his actual beliefs, as long as he is not fully insured against it. In particular, at the initial endowment, \( i \)'s effective beliefs are such that he prefers to buy some consumption in state \( s^1 \) in exchange for giving up consumption in state \( s^2 \). Hence, the effective one-step-ahead beliefs of consumer \( i \) satisfy:

\[
\begin{align*}
& b_i(s^1 \mid \sigma_i) \in \left( \frac{1}{2}; \frac{e_{s1}}{e_{s2}} \right)^{\pi} \left( 1 - \pi \right) + \left( \frac{e_{s1}}{e_{s2}} \right)^{1-\pi} \left( \pi \right).
\end{align*}
\]

(28)

\[^{10} \text{In this example, for the sake of the simplicity of the resulting representation, we deviate from the assumption that } u(0) = 0. \text{ An economy with such preferences is analyzed in Collard et al. (2011).} \]
In the Appendix, we show that i’s effective beliefs are uniformly bounded away from his actual beliefs on all equilibrium paths. I.e., there is a $\xi > 0$ such that $b_i(s^1 | \sigma_t) > \frac{1}{2} + \xi$ for all $\sigma_t$. If, furthermore, the true probability of $\pi^1$ satisfies:

$$
\mu_1 > \frac{\left( \frac{e_i^1}{e_j^1} \right)^{1-\pi} (2\pi - 1)}{\left( \frac{e_i^1}{e_j^2} \right)^\pi + \left( \frac{e_i^1}{e_j^2} \right)^{1-\pi}},
$$

we have that $b^j(s^1 | \sigma_t) \in \left( \frac{1}{2} + \xi; \mu_1 \right)$, and, thus, closer to the truth than $\frac{1}{2}$, the one-step-ahead probability assigned to $s^1$ by $j$. Using the fact that i’s beliefs are uniformly bounded away from j’s beliefs in combination with Theorem 6 of Blume and Easley (2006), we conclude that i will almost surely survive in this economy, while j will almost surely vanish.

The example presented above suggests that ambiguity aversion per se might be a valuable trait in financial markets, especially when the market underestimates the probability of (unobservable) bad states. Since ambiguity-averse investors are sensitive to variation of utility not only across observable, but also across unobservable states, they are willing to pay a higher insurance premium in order to increase their consumption in the "bad" state than expected utility maximizers. If the probability of such a bad event is being underestimated by the market, ambiguity-averse investors’ effective beliefs are closer to the truth than the average market belief. In the limit, expected utility maximizers vanish a.s., while the ambiguity-averse investors survive and bring aggregate market beliefs closer to the truth.

7 Conclusion

In this paper, we analyzed the question of whether smooth ambiguity-averse consumers can survive in the presence of expected utility maximizers with correct beliefs. We showed that the answer to this question will depend both on the nature and persistence of ambiguity and risk in the economy and on the degree and type of ambiguity aversion. We identified situations, in which ambiguity-averse consumers can survive by completely insuring against ambiguity and mimicking the behavior of expected utility maximizers with correct beliefs. However, in this case, ambiguity aversion will have no impact on prices. When there is aggregate risk and ambiguity is large and persistent, ambiguity-averse consumers cannot be completely insured against it. Their effective beliefs are thus wrong even in the limit and their survival depends on the form of the function characterizing their ambiguity aversion and on the interplay between their ambiguity aversion and prudence with respect to ambiguity. In particular, consumers with decreasing absolute ambiguity aversion whose prudence with respect to ambiguity exceeds twice
their absolute ambiguity aversion will a.s. survive, regardless of whether they are completely insured against ambiguity. Hence, prices in a market in which ambiguity-averse investors are present can deviate from those in a market populated by expected utility maximizers with correct beliefs.

We also provided an example, which illustrates that when all agents have wrong beliefs, which underestimate the "bad" state in the economy, ambiguity-averse investors’ beliefs will be closer to the truth, which helps them survive. Note that differently from markets with correct beliefs, this result does not rely on the increased propensity to save on the side of ambiguity-averse investors.

In this paper, we assumed that even though some features of the economy (the realizations of $\pi$) are unobservable, the structure of the uncertainty is such that all investors in the market can have correct beliefs, i.e., $\mu_{\theta_1}(\pi^n) = \mu_{\sigma_1(\theta_1)}(\pi^n)$. While this is true for a large class of stochastic processes, it is easy to generate examples, in which the unobservability of $\theta$ would preclude any agent in the economy from having correct beliefs. Each agent will then form his best forecast conditional on the observable variables in the economy. It would then be interesting to examine whether the effective beliefs of ambiguity-averse investors could be closer to the truth than those of the expected utility maximizers. Our last example suggests that it should be possible to construct economies in which this is the case. Combined with their increased propensity to save, this could give further advantage to DAAA investors in terms of survival.

**Appendix A. Proofs**

**Proof of Proposition 4.1:**

An equilibrium of the economy exists under the following conditions, see Bewley (1972):

1. the consumption sets are convex, Mackey closed and contained in the set of essentially bounded measurable functions;
2. the preferences of the consumers are complete and transitive;
3. the better sets are convex and Mackey closed;
4. the worse sets are closed in the norm topology;
5. there exists a set of paths with strictly positive measure such that the preferences of all consumers satisfy strict monotonicity on this set, i.e. adding a constant to the payoff in each state and each period makes the consumer strictly better off;
6. for all consumers, the initial endowments are in the interior of the consumptions sets.
W.l.o.g., we can assume that the consumption set of a consumer $i \in \{1...T\}$ is given by the sets of all essentially bounded measurable functions and, hence, satisfies condition 1. Assumption 2 is trivially satisfied, since consumers’ preferences are represented by the utility function $V^i$. In particular, KMM (2009) show that $V^i$ exists and is unique for every consumption stream $c'$. To prove convexity, as required by Assumption 3, first compare two streams of consumption $c$ and $c'$ such that $c(\sigma_t) = c'(\sigma_t)$ for all $\sigma_t \neq \sigma_0$, $c(\sigma_0) \neq c'(\sigma_0)$. Consider the stream $\alpha c + (1 - \alpha) c'$ for some $\alpha \in (0;1)$. Since

$$V^i_{\sigma_0}(\alpha c + (1 - \alpha) c') = u_i(\alpha c(\sigma_0) + (1 - \alpha) c'(\sigma_0)) +$$

$$+ \beta_i \phi^{-1}_i \left[ \sum_{n=1}^{N} \phi_i \left( \sum_{a_t \in S_t} V^i_{(\sigma_t;\sigma_0)}(c) \pi^n(s_1 \mid \sigma_0) \right) \mu^i_{\sigma_0}(\pi^n) \right]$$

$$\geq \alpha u_i(c(\sigma_0)) + (1 - \alpha) u_i(c'(\sigma_0))$$

$$+ \beta_i \phi^{-1}_i \left[ \sum_{n=1}^{N} \phi_i \left( \sum_{a_t \in S_t} V^i_{(\sigma_t;\sigma_0)}(c) \pi^n(s_1 \mid \sigma_0) \right) \mu^i_{\sigma_0}(\pi^n) \right]$$

$$= \alpha V^i_{\sigma_0}(c) + (1 - \alpha) V^i_{\sigma_0}(c')$$

it follows that the convexity of the better sets for such mixtures is implied by the strict concavity of $u_i$. Consider an arbitrary $T$ and take two consumption streams $c$ and $c'$ such that $c(\sigma_t) = c'(\sigma_t)$ for all $\sigma_t \notin \cup_{t=0}^{T} \Sigma_t$ and such that $c(\sigma_t) \neq c'(\sigma_t)$ for some $\sigma_t$ such that $t \leq T$. Just as above, the concavity of $u_i$ implies that for every $\sigma_T \in \Sigma_T$,

$$V^i_{\sigma_T}(\alpha c + (1 - \alpha) c') \geq \alpha V^i_{\sigma_T}(c) + (1 - \alpha) V^i_{\sigma_T}(c')$$

(A-2)

By the strict monotonicity of $\phi_i$ (and thus, of $\phi^{-1}_i$), (and using, once again the concavity of $u_i$), we thus conclude that

$$V^i_{\sigma_{T-1}}(\alpha c + (1 - \alpha) c') \geq \alpha V^i_{\sigma_{T-1}}(c) + (1 - \alpha) V^i_{\sigma_{T-1}}(c').$$

(A-3)

Proceeding by induction, we obtain that for all $t \in \{1...T\}$,

$$V^i_{\sigma_t}(\alpha c + (1 - \alpha) c') \geq \alpha V^i_{\sigma_t}(c) + (1 - \alpha) V^i_{\sigma_t}(c')$$

(A-4)

Since $T$ was chosen arbitrary, but finite, have shown that convexity holds w.r.t. any two consumption streams which are constant after some finite time period $T$. Furthermore, since we assumed that $c \neq c'$, at least one of the inequalities is strict, and, hence,

$$V^i_{\sigma_0}(\alpha c + (1 - \alpha) c') > \alpha V^i_{\sigma_0}(c) + (1 - \alpha) V^i_{\sigma_0}(c')$$

Now note that each pair of consumption streams $c$ and $c'$ can be represented as a limit of two sequences of consumption streams $(c^T)_{T \in \mathbb{N}}$ and $(c'^T)_{T \in \mathbb{N}}$ such that for each $T \in \mathbb{N}$, $c^T$ coincides with $c$ on all paths of length $T$ and is constant for all possible continuations and similarly for $c'^T$: $c^T = (c(\sigma_t))_{t \leq T; k...k...}$ and $c'^T = (c'(\sigma_t))_{t \leq T; k...k...}$. We then have that the pointwise limits of the sequences satisfy: $\lim_{T \to \infty} c^T = c$, $\lim_{T \to \infty} c'^T = c'$ and $\lim_{T \to \infty} [\alpha c^T + (1 - \alpha) c'^T] = \ldots$
\[ \alpha c + (1 - \alpha) c'. \]

If \( c \neq c' \), then there is a \( \tilde{T} \) such that for all \( T \geq \tilde{T} \), we have:

\[ V_{\sigma_0}^i (\alpha c^T + (1 - \alpha) c'^T) > \alpha V_{\sigma_0}^i (c^T) + (1 - \alpha) V_{\sigma_0}^i (c'^T). \]  

(A-5)

The function \( V^i \) is a contraction, see Marinacci and Montrucchio (2007, pp. 7-9), and hence, continuous, implying that

\[ V^i \cdot \alpha c + (1 - \alpha) c' \leq \alpha V^i (c) + (1 - \alpha) V^i (c'). \]  

We also have that \( V^i \) is uniformly continuous, hence, \( V^i \) is continuous w.r.t. the Mackey topology. This means that both the better and the worse sets are closed with respect to the Mackey topology, and, hence, also in the norm topology and assumptions 3. and 4. are satisfied.

For condition 5, take the set of paths to be \( \Sigma \). Note that \( V^i \) is monotonic, see KMM (2009). Take any consumption stream \( c \). Clearly, adding a positive amount to \( c (\sigma_0) \), strictly improves the act. But, similarly, adding a positive number to each of the \( c (\sigma_1) \) for \( \sigma_1 \in \Sigma_1 \) leads to a strict increase in \( V^i_{\sigma_1} \), and by the monotonicity of \( \phi^i \), to a strict increase in the evaluation of the act, etc. Hence, the preferences of all consumers are strictly monotonic on \( \Sigma \).

Finally, Assumption 3 ensures that the endowment stream of each consumer is uniformly bounded away from 0 and from infinity, and is, therefore, in the interior of this consumer’s consumption set, thus implying condition 6. We conclude that an equilibrium of the economy exists.

Proof of Proposition 4.2:

If \( p (\cdot) \) is an equilibrium price system, then condition (10) is the first-order condition of consumer \( i \)’s maximization problem at state \( \sigma_t \). Hence, it will be satisfied in any equilibrium, in which consumer \( i \) chooses an interior allocation on all finite paths with positive probabilities. We now show that Assumptions 1–4 imply that the optimal consumption streams of all consumers will be strictly positive on all finite paths which have positive probability. To show this, we demonstrate that the MRS between consumption at \( \sigma_t \) and at \( (\sigma_t; s_{t+1}) \) will always be strictly positive and finite, as long as the true probability of \( \sigma_t \) and the conditional probability of \( s_{t+1} \) given \( \sigma_t \) are both positive.

First note that since the initial endowment is uniformly bounded above, then so is any of the consumption streams in equilibrium and, hence, by Assumption 1, \( u' (0) \) is always strictly positive. Furthermore, setting \( c (\sigma_0) = 0 \) is not optimal, since endowment is uniformly bounded away from 0 and \( u' (0) = \infty \).

Let \( \mu (\sigma_t) > 0 \) and let \( u (c (\sigma_t)) > 0 \). By the argument above, at least one such \( \sigma_t \) exists. KMM (2009) demonstrate that if the consumption stream is bounded, so is \( V^i (c) \), hence, \( E_{\pi^n}^i (V_{\sigma_{t+1}}^i (c')) \) are bounded as well. It follows, by Assumption 2, that \( \phi^i_1 \left[ E_{\pi^n}^i (V_{\sigma_{t+1}}^i (c')) \right] \) and

\[ \phi^i_1 \left( \phi^i -1 \left( \sum_{n=1}^{N} \phi^i_1 \left[ E_{\pi^n}^i (V_{\sigma_{t+1}}^i (c')) \right] \mu_{\sigma_1}^i (\pi^n) \right) \right) \]  

(A-7)

are also strictly positive. We first show that it is not optimal to choose a consumption path on which
Indeed, assume that in the optimum, $E_{\pi^n}(V^i_{\sigma_{t+1}}(c^i)) = 0$ for all $n \in \{1...N\}$. It follows that the continuation of the consumption stream $c$ entails $c(\sigma_t; s) = 0$ for all $s \in S_{t+1}$ and $c(\sigma_t; s; s_{t+2}...s_{t+k}) = 0$ for any continuation of the path $(\sigma_t; s)$. Hence, at node $\sigma_t$, consumer $i$ envisions a constant consumption of $0$ at all following nodes. Consider a deviation at node $\sigma_t$ and at all nodes $(\sigma_t; s)$ with $s \in S_t$ such that consumption at $\sigma_t$ is given by $c^i(\sigma_t) - \epsilon$ and instead, $c^i(\sigma_t; s) = \epsilon > 0$, with $\epsilon p(\sigma_t) = \epsilon \sum_{s \in S_{t+1}} p(\sigma_t; s)$. Assume that consumption from $s_{t+2}$ on remains at $0$ for all continuation paths. Hence, consumer $i$ trades some of his (positive) consumption for all one-step-ahead states of the world. The utility of such a consumption stream at node $\sigma_t$ is given by:

$$
\frac{u_i(c^i(\sigma_t)) + \beta_i \phi^{-1}_i \left( \sum_{n=1}^{N} \phi_i \left( \sum_{s \in S_{t+1}} V^i_{(\sigma_t; s)}(c^i(s | \sigma_t)) \right) \right)}{1 - \phi_i} = u_i(c^i(\sigma_t)) + \beta_i \phi^{-1}_i \left( \sum_{n=1}^{N} \phi_i \left( \sum_{s \in S_{t+1}} u_i(\epsilon) \pi^n(s | \sigma_t) \right) \right) = u_i(c^i(\sigma_t) - \epsilon) + \beta_i u_i(\epsilon).
$$

(A-8)

It is obvious that the derivative w.r.t. $\epsilon$ at $\epsilon = 0$ is $\infty$, hence any small $\epsilon$ represents an improvement over the original plan, in contradiction to the assumption made above.

It follows that in the optimum, $E_{\pi^n}(V^i_{\sigma_{t+1}}(c^i)) > 0$ for all $n \in \{1...N\}$, and, hence, by Assumption 2, we can exclude the case in which

$$
\sum_{n=1}^{N} \phi_i \left[ E_{\pi^n}(V^i_{\sigma_{t+1}}(c^i)) \right] \mu^i_{\sigma_t}(\pi^n(s_{t+1} | s_t)) \quad (A-9)
$$

or

$$
\phi_i \left( \phi^{-1}_i \left( \sum_{n=1}^{N} \phi_i \left[ E_{\pi^n}(V^i_{\sigma_{t+1}}(c^i)) \right] \right) \right) \mu^i_{\sigma_t}(\pi^n) \quad (A-10)
$$

equals $\infty$. Furthermore, both expressions (A-9) and (A-10) are strictly positive for all $s_{t+1}$ such that $\pi^n(s_{t+1} | \sigma_t) > 0$, which, by Assumption 4 is true, whenever $\sum_{n=1}^{N} \mu^i_{\sigma_t}(\pi^n(s_{t+1} | \sigma_t)) > 0$. Since $u^i(0) = \infty$, this implies that $c^i(\sigma_t; s_{t+1}) \neq 0$, whenever $\sum_{n=1}^{N} \mu^i_{\sigma_t}(\pi^n(s_{t+1} | \sigma_t)) > 0$. According to assumption 4, however, this is implied by $\sum_{n=1}^{N} \mu^i_{\sigma_t}(\pi^n(s_{t+1} | \sigma_t)) > 0$. Hence, $c^i(\sigma_t; s_{t+1}) = 0$ can only occur if $\sum_{n=1}^{N} \mu^i_{\sigma_t}(\pi^n(s_{t+1} | \sigma_t)) = 0$, i.e., if state $s_{t+1}$ indeed has a probability of 0 conditional on $\sigma_t$. Therefore, conditional on being in a node $\sigma_t$ to which $i$ assigns positive consumption, consumer $i$ assigns positive consumption to all nodes $(\sigma_t; s_{t+1})$ which have positive one-step-ahead conditional probabilities given $\sigma_t$. Since $i$ will enjoy positive consumption in period 0, forward induction implies that $i$ will have strictly positive consumption on all finite paths which have positive probability with respect to the truth. This, in turn implies that the first order condition will hold.
on all such paths.

**Proof of Proposition 5.1:**

Consider a path $\sigma$ such that $\lim_{t \to \infty} \mu_{\sigma_t} (\pi_n) = 0$ for some $\pi_n \in \{\pi^1, \ldots, \pi^N\}$. Let $\hat{\mu}$ denote $i$’s prior (which by assumption is a.c. w.r.t. the true prior $\mu$) and denote by $\epsilon_t$ the sequence describing the rate of convergence of Bayesian updating on $\hat{\mu}$: $\epsilon_t (\sigma) = \left| \hat{\mu}_{\sigma_t} - \pi_n \right|$. Hence, if we can show that

$$G^i (\hat{\mu}_{\sigma_t}) = \frac{\phi_i' \left( \phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i (y_n) \hat{\mu}_{\sigma_t} (\pi_n) \right) \right)}{\phi_i' (y_n)}$$

converges uniformly to $\pi_n$ at a rate of at most $\epsilon_t (\sigma)$ on the set of all consumption streams, we would have shown that an ambiguity-averse investor learns the truth at least as fast as a Bayesian expected utility maximizer and, hence, according to Theorem 4 in Blume and Easley (2006) survives almost surely. From now on, (whenever $i, \sigma_t$ and $c^i$ are clear from the context), we will substitute $y_n = E_{\pi_{n+1}} (V_{\sigma_{t+1}} (c^i))$.

Suppose first that for every $n \in \{1, \ldots, N\}$, the total derivative of $G^i$ with respect to $\hat{\mu}_{\sigma_t}$ is continuous and uniformly bounded on the set of all possible values of $(y_n)_{n=1}^{N}$. Then, $G^i$ is Lipschitz, see Lee (2003, p. 595), and there exists a constant $\kappa$ such that for any $\mu'$, $|G^i (\hat{\mu}_{\sigma_t}) - G^i (\mu'_{\sigma_t})| \leq \kappa |\mu'_{\sigma_t} - \hat{\mu}_{\sigma_t}|$. In particular, setting $\mu' = \delta_{\pi_n}$ implies that for each path $\sigma$ and for all $t$,

$$|G^i (\hat{\mu}_{\sigma_t}) - G^i (\pi_n)| = \frac{\phi_i' (y_n, \hat{\mu}_{\sigma_t} (\pi_n), (s_{t+1} | \sigma_t))}{\phi_i' (y_n)} - \frac{\phi_i' (y_n, \pi_n, (s_{t+1} | \sigma_t))}{\phi_i' (y_n)} \leq \kappa \left| \hat{\mu}_{\sigma_t} - \delta_{\pi_n} \right| = \kappa \epsilon_t (\sigma)$$

Since $\kappa \epsilon_t (\sigma)$ has a rate of convergence of $\epsilon_t (\sigma)$, it follows that the beliefs of the smooth ambiguity-averse agents converge to the truth at the same rate as those of the Bayesian expected utility maximizers.

We will now show that the class of HAAA $\phi$-functions satisfy the condition of the proposition. Indeed, let $\phi (y) = \frac{1-\gamma}{\gamma} \left( \frac{\alpha y}{1-\gamma} + d \right)^{\gamma-1}$ and $\phi^{-1} (y) = \frac{1}{\gamma} \frac{1}{\gamma} \frac{\alpha y}{1-\gamma} - d$. Then,

$$G^i (\hat{\mu}_{\sigma_t}) = \frac{\phi_i' \left( \phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i (y_n) \hat{\mu}_{\sigma_t} (\pi_n) \right) \right)}{\phi_i' (y_n)}$$

Taking into account that $\mu_{\sigma_t} (\pi_N) = 1 - \sum_{n \neq N} \mu_{\sigma_t} (\pi_n)$, for $n \in \{1, \ldots, N - 1\},$

$$\left( G^i \right)' (\hat{\mu}_{\sigma_t}) = \frac{\phi_i' \left( \phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i (y_n) \hat{\mu}_{\sigma_t} (\pi_n) \right) \right)}{\phi_i' (y_n)} \left( \frac{\alpha y_n}{1-\gamma} + d \right)^{\gamma-1} \hat{\mu}_{\sigma_t} (\pi_n) \pi_n (s_{t+1} | \sigma_t) \pi_n (s_{t+1} | \sigma_t)$$

(A-13)
Now note that for any continuous, finite and uniformly bounded on the set of all possible values of sums of uniformly bounded terms, they are also uniformly bounded. Hence, the denominator is bounded between $\frac{\alpha}{1-\gamma} + d$ and $\frac{\alpha M}{1-\gamma} + d$. By the boundaries on $\sum_{n=1}^{N} \frac{\alpha y}{1-\gamma} + d$, we obtain

$$
\frac{\gamma-1}{\gamma} \left( \left( \frac{\alpha y}{1-\gamma} + d \right)^{\gamma} - \left( \frac{\alpha y}{1-\gamma} + d \right)^{\gamma-1} \right) \sum_{n=1}^{N} \left( \frac{\alpha y}{1-\gamma} + d \right)^{\gamma} \bar{\mu}_{\sigma_t} \pi^n (s_{t+1} | \sigma_t)
$$

Consider first the case of a finite $\gamma$ such that $\gamma \notin \{0; 1; \pm \infty\}$. According to the specification of HAAA, $d > 0$. Note that $y_n$ is bounded between 0 and an upper bound, $M$, given by the discounted value of the consumption stream assigning the maximal total endowment of the economy, $m^i$ to consumer $i$. Hence, the denominator is bounded between $(d)^{2(\gamma-1)}$ and $\left( \frac{\alpha M}{1-\gamma} + d \right)^{2(\gamma-1)}$. Since both numerators are finite sums of uniformly bounded terms, they are also uniformly bounded. Hence, $\left( G^{i} \right)^{\prime} \bar{\mu}_{\sigma_t} (\pi^n)$ is indeed continuous, finite and uniformly bounded on the set of all possible values of $(y_1 \ldots y_n)$, and, hence, on every path $\sigma$. It follows that the total derivative of $G^{i}$ is continuous and uniformly bounded, whenever $\gamma \notin \{0; 1; \pm \infty\}$. For the case of $\gamma \in \{+\infty; -\infty\}$, $\theta$ is in the class of CAAA, $\theta = -e^{-\frac{1}{2}}$ and we obtain

$$
\left( G^{i} \right)^{\prime} \bar{\mu}_{\sigma_t} (\pi^n) = \frac{\alpha e^{-\frac{1}{2}y_n} \sum_{n=1}^{N} \alpha e^{-\frac{1}{2}y_n} \bar{\mu}_{\sigma_t} (\pi^n) [\pi^n (s_{t+1} | \sigma_t) - \pi^n (s_{t+1} | \sigma_t)]}{\left( \sum_{n=1}^{N} \alpha e^{-\frac{1}{2}y_n} \bar{\mu}_{\sigma_t} (\pi^n) \right)^2}
$$

By the boundaries on $y_n$, established above, $\alpha e^{-\alpha y_n}$ is bounded between $\alpha e^{-\alpha M}$ and $\alpha$, and so the expression in the denominator is bounded between $\left[ (e^{-\alpha M})^2 ; (\alpha)^2 \right]$. Since both numerators are finite sums of uniformly bounded terms, they are also uniformly bounded. Hence, $\left( G^{i} \right)^{\prime} \bar{\mu}_{\sigma_t} (\pi^n)$ is indeed continuous, finite and uniformly bounded on the set of all possible values of $\theta^{i} (y_n)$, and, hence, on every path $\sigma$. It follows that the total derivative of $G^{i}$ is continuous and uniformly bounded.

Finally, consider the case $\gamma = 0$, or $\phi (y) = \ln y$. For this case,

$$
G^{i} \left( \bar{\mu}_{\sigma_t} \right) = \sum_{n=1}^{N} \frac{\bar{\mu}_{\sigma_t} (\pi^n) \pi^n (s_{t+1} | \sigma_t)}{y_{n}} \prod_{n=1}^{N} \frac{\bar{\mu}_{\sigma_t} (\pi^n)}{y_{n}}
$$

For $\bar{n} \in \{1 \ldots N-1\}$, $\left( G^{i} \right)^{\prime} \bar{\mu}_{\sigma_t} (\pi^n)$ is given by:

$$
\prod_{n=1}^{N} \frac{y_{n}}{y_{\bar{n}}} \bar{\mu}_{\sigma_t} (\pi^n) \left[ \frac{y_{\bar{n}} \pi^n (s_{t+1} | \sigma_t) - \pi^n (s_{t+1} | \sigma_t)}{y_{\bar{n}}} + \ln \frac{y_{\bar{n}}}{y_{n}} \sum_{n=1}^{N} \frac{y_{\bar{n}}}{y_{n}} \bar{\mu}_{\sigma_t} (\pi^n) \pi^n (s_{t+1} | \sigma_t) \right]
$$

Now note that for any $n \in \{1 \ldots N-1\}$,

$$
\frac{y_{n}}{y_{N}} = \frac{E_{\pi^n} (V^i_{\sigma_{t+1}} (c^i))}{E_{\pi^n} (V^i_{\sigma_{t+1}} (c^i))} = \sum_{s \in S} \pi^n (s | \sigma_t) V^i_{\sigma_t(s)} (c^i) = \sum_{s \in S} \pi^N (s | \sigma_t) V^i_{\sigma_t(s)} (c^i).
$$
Observe that by Assumption 4,
\[ \delta \leq \frac{\sum_{s \in S} \pi^n(s \mid \sigma_t) V_{i(\sigma; s)}^i(c')}{\sum_{s \in S} \pi^n(s \mid \sigma_t) V_{i(\sigma; s)}^i(c')} \leq \frac{1}{\delta} \] (A-19)
holds for all possible values of \( V_{i(\sigma; s)}^i(c') \) \( s \in S \). To see this, note that on all nodes \( \sigma_t, V_{i(\sigma; s)}^i(c') > 0 \) (see the proof of Proposition 4.2), and let \( V_{i(\sigma; s')}^i(c') \) =: \( \max_{s \in S} V_{i(\sigma; s)}^i(c') \) and \( V_{i(\sigma; s)}^i(c') =: \min_{s \in S} V_{i(\sigma; s)}^i(c') \) hence,
\[ \frac{\sum_{s \in S} \pi^n(s \mid \sigma_t) V_{i(\sigma; s)}^i(c')}{\sum_{s \in S} \pi^n(s \mid \sigma_t) V_{i(\sigma; s)}^i(c')} \leq \frac{[1 - (N - 1)\delta] V_{i(\sigma; s')}^i(c') + \delta \sum_{s \in S \setminus s'} V_{i(\sigma; s)}^i(c')}{\delta V_{i(\sigma; s')}^i(c')} = \frac{1}{\delta} \] (A-20)
\[ \frac{\sum_{s \in S} \pi^n(s \mid \sigma_t) V_{i(\sigma; s)}^i(c')}{\sum_{s \in S} \pi^n(s \mid \sigma_t) V_{i(\sigma; s)}^i(c')} \geq \frac{[1 - (N - 1)\delta] V_{i(\sigma; s')}^i(c') + \delta \sum_{s \in S \setminus s'} V_{i(\sigma; s)}^i(c')}{[1 - (N - 1)\delta] V_{i(\sigma; s')}^i(c') + \delta \sum_{s \in S \setminus s'} V_{i(\sigma; s)}^i(c')} (A-21) \]
\[ \frac{\sum_{s \in S} \pi^n(s \mid \sigma_t) V_{i(\sigma; s)}^i(c')}{\sum_{s \in S} \pi^n(s \mid \sigma_t) V_{i(\sigma; s)}^i(c')} \geq \frac{\delta V_{i(\sigma; s')}^i(c')}{V_{i(\sigma; s')}^i(c')} = \delta \]
Note that the inequalities remain valid, even as \( \lim_{t \to \infty} V_{i(\sigma; s')}^i(c') = 0 \). Thus, examining (A-17), we find that it is continuous and uniformly bounded on all paths \( \sigma \). It follows that the total derivative of \( G^i \) is continuous and uniformly bounded, as asserted.\)

**Proof of Proposition 5.2:**

We showed in the proof of Proposition 4.1 that the preferences of all consumers are strictly convex. Hence, in the absence of aggregate risk, all consumers will be fully insured against risk, i.e., \( c^i(\sigma_t; s_{t+1}) = c^i(\sigma_t; s'_{t+1}) \) for all \( i \in \{1...I\}, \) all \( t \in \mathbb{N}, \) all \( \sigma_t \in \Sigma_t \) and all \( s_{t+1}, s'_{t+1} \in S \). Hence, \( V_{i(\sigma; s_{t+1})}^i(c') = V_{i(\sigma; s_{t+1})}^i(c') \) for all \( s_{t+1}, s'_{t+1} \in S_{t+1}, \) and, therefore,
\[ \phi_t^{-1} \left[ \sum_{n=1}^{N} \phi_t \left( \sum_{n=1}^{N} V_{i(\sigma; s_{t+1})}^i(c') \pi^n(s_{t+1} \mid \sigma_t) \right) \mu_{\sigma_t}(\pi^n) \right] = \sum_{n=1}^{N} \sum_{s_{t+1} \in S_{t+1}} V_{i(\sigma; s_{t+1})}^i(c') \pi^n(s_{t+1} \mid \sigma_t) \mu_{\sigma_t}(\pi^n) (A-22) \]
for all \( t \in \mathbb{N}, \) all \( \sigma_t \in \Sigma \) and all \( s_{t+1}, s'_{t+1} \in S \). It follows that all consumers in the economy effectively behave as expected utility maximizers with correct beliefs and, therefore, a.s. survive.\]

**Proof of Lemma 5.3:**

By Proposition 4.2, we have
\[ u'_1(c'(\sigma_t)) \beta_i u'_i(c'(\sigma_t; s_{t+1})) \sum_{n=1}^{N} \phi_t^{-1} \phi_t \left[ \sum_{n=1}^{N} \phi_t \left( E_{\sigma_t} \left( V_{i(\sigma; s_{t+1})}^i(c') \right) \mu_{\sigma_t}(\pi^n) | \pi^n(s_{t+1} \mid \sigma_t) \right) \right] = \beta_j u'_j(c'(\sigma_t; s_{t+1})) \sum_{n=1}^{N} \mu_{\sigma_t}(\pi^n) \pi^n(s_{t+1} \mid \sigma_t) (A-23) \]
Hence,

\[ \prod_{t=0}^{T} \frac{u'_i(c^t(\sigma_t))}{u'_i(c^t(\sigma_t; s_{t+1}))} = \prod_{t=0}^{T} \frac{\sum_{n=1}^{N} \phi'_i \left[ E_{\pi^n}(V_{\sigma_{t+1}}^i(c^t)) \right] \mu_{\sigma_t}(\pi^n)(s_{t+1} | \sigma_t)}{\phi'_i \left( \sum_{n=1}^{N} \phi_i \left[ E_{\pi^n}(V_{\sigma_{t+1}}^i(c^t)) \right] \mu_{\sigma_t}(\pi^n) \right)} \]  

(A-24)

\[ \frac{u'_j(c^t(\sigma_T; s_{T+1}))}{u'_i(c^t(\sigma_T; s_{T+1}))} = \frac{u'_j(c^t(\sigma_0))}{u'_i(c^t(\sigma_0))} \prod_{t=0}^{T} \frac{\beta_j}{\beta_i} \frac{\sum_{n=1}^{N} \phi'_i \left[ E_{\pi^n}(V_{\sigma_{t+1}}^j(c^t)) \right] \mu_{\sigma_t}(\pi^n)(s_{t+1} | \sigma_t)}{\phi'_i \left( \sum_{n=1}^{N} \phi_i \left[ E_{\pi^n}(V_{\sigma_{t+1}}^j(c^t)) \right] \mu_{\sigma_t}(\pi^n) \right)} \]  

(A-25)

We are interested in the term \( \lim_{T \to \infty} \frac{1}{T+1} \ln \frac{u'_j(c^t(\sigma_T; s_{T+1}))}{u'_i(c^t(\sigma_T; s_{T+1}))} \). Since by Proposition 4.2 both \( u'_j(c^t(\sigma_0)) \) and \( u'_i(c^t(\sigma_0)) \) are strictly positive and finite, it follows that \( \lim_{T \to \infty} \frac{1}{T+1} \ln \frac{u'_j(c^t(\sigma_0))}{u'_i(c^t(\sigma_0))} = 0 \). Note that

\[ \sum_{n=1}^{N} \phi'_i \left[ E_{\pi^n}(V_{\sigma_{t+1}}^j(c^t)) \right] \mu_{\sigma_t}(\pi^n)(s_{t+1} | \sigma_t) \phi'_i \left( \sum_{n=1}^{N} \phi_i \left[ E_{\pi^n}(V_{\sigma_{t+1}}^j(c^t)) \right] \mu_{\sigma_t}(\pi^n) \right) \]

(A-26)

and hence,

\[ \lim_{T \to \infty} \frac{1}{T+1} \ln \frac{u'_j(c^t(\sigma_T; s_{T+1}))}{u'_i(c^t(\sigma_T; s_{T+1}))} = \left[ \ln \beta_j - \ln \beta_i \right] - \]

(A-27)

\[ \text{Proof of Lemma 5.4:} \]

For a given \( \sigma_t \) and \( s_{t+1} \), we will write \( y_n := E_{\pi^n}(V_{\sigma_{t+1}}^i(c^t)) \). For (17), we then have:

\[ \sum_{n=1}^{N} \phi'_i(y_n) \mu_{\sigma_t}(\pi^n)(s_{t+1} | \sigma_t) \leq \phi'_i \left( \min_{n \in \{1, \ldots, N\}} y_n \right) \left( 1 - \delta \right) = \frac{1 - \delta}{\delta} \]  

(A-28)

\[ \phi'_i \left( \max_{n \in \{1, \ldots, N\}} y_n \right) \delta \leq \phi'_i \left( \min_{n \in \{1, \ldots, N\}} y_n \right) \left( 1 - \delta \right) = \frac{\delta}{1 - \delta} \]

(A-29)

For (18),

\[ \frac{\sum_{n=1}^{N} \phi'_i(y_n) \mu_{\sigma_t}(\pi^n)}{\phi'_i \left( \sum_{n=1}^{N} \phi_i(y_n) \mu_{\sigma_t}(\pi^n) \right)} \leq \frac{\phi'_i \left( \min_{n \in \{1, \ldots, N\}} y_n \right) \left( 1 - \delta \right)}{\phi'_i \left( \min_{n \in \{1, \ldots, N\}} y_n \right)} \leq \frac{1 - \delta}{1 - \delta} = 1 - \delta. \]  

(A-30)
where the second inequality follows from the fact that $\phi^{-1}$ is monotone, and, therefore, quasi-concave.

$$
\frac{\sum_{n=1}^{N} \phi_i'(y_n) \mu_{\sigma_i}^i(\pi^n)}{\phi_i'(\phi^{-1}_i\left(\sum_{n=1}^{N} \phi_i(y_n) \mu_{\sigma_i}^i(\pi^n)\right))} \geq \frac{\phi_i'(\max_{n\in\{1...N\}} y_n) \delta}{\phi_i'(\phi^{-1}_i(\phi(\max_{n\in\{1...N\}} y_n)))} = \tilde{\delta},
$$

where the second inequality follows from the fact that $\phi^{-1}$ is monotone, and thus, quasi-convex. Hence, both (17) and (18) are uniformly bounded away from 0 and from above and so is therefore, their product (15).

**Proof of Lemma 5.5:**

As before, we write $y_n =: E_{\pi^n}(V_{\sigma_{s+1}}^i(c^i))$. Suppose that $y_n = y_{n'}$ holds for all $n, n' \in \{1...N\}$, then for every $s \in S$,

$$
\sum_{n=1}^{N} \phi_i'(y_n) \mu_{\sigma_i}^i(\pi^n)(s | \sigma_t) \geq \frac{\phi_i'(y_n) \sum_{n=1}^{N} \mu_{\sigma_i}^i(\pi^n)(s | \sigma_t)}{\phi_i'(\sum_{n=1}^{N} \mu_{\sigma_i}^i(\pi^n))} = \sum_{n=1}^{N} \mu_{\sigma_i}^i(\pi^n)(s | \sigma_t).
$$

Hence, the agent’s effective beliefs coincide with his actual beliefs. Assume now that the agent has effective beliefs, which coincide with his actual beliefs:

$$
\left(\sum_{n=1}^{N} \phi_i'(y_n) \mu_{\sigma_i}^i(\pi^n)(s | \sigma_t)\right) = \sum_{n=1}^{N} \mu_{\sigma_i}^i(\pi^n)(s | \sigma_t) |
$$

or

$$
\left(\sum_{n=1}^{N} \phi_i'(y_n) \mu_{\sigma_i}^i(\pi^n) - 1\right) \mu_{\sigma_i}^i(\pi^n)(s | \sigma_t) = 0 |
$$

Since by Assumption 5 all of the vectors $\pi^n$ are linearly independent, this system of equations has a unique solution given by $\phi_i'(y_n) = \sum_{n=1}^{N} \phi_i'(y_n) \mu_{\sigma_i}^i(\pi^n)$ for every $n \in \{1...N\}$, which by the strict monotonicity of $\phi'$ implies $y_n = y_{n'}$ for all $n, n' \in \{1...N\}$. We thus obtain full insurance against ambiguity, whenever actual and effective beliefs coincide.

**Proof of Lemma 5.6:**

Full insurance against risk at a given node $\sigma_t$ is equivalent to $V^i_{(\sigma_t; s)}(c^i) = V^i_{(\sigma_t; s')} (c^i)$ for all $s, s' \in S$.

Full insurance against ambiguity at $\sigma_t$ is equivalent to

$$
\sum_{s \in S} V^i_{(\sigma_t; s)}(c^i) \pi^n(s | \sigma_t) = \sum_{s \in S} V^i_{(\sigma_t; s)}(c^i) \pi^{n+1}(s | \sigma_t)
$$

for all $n \in \{1...N\}$. For a given $s \in S$, normalize, w.l.o.g., $V^i_{(\sigma_t; s)}(c^i) =: 0$ and note that the condition of full insurance against ambiguity is equivalent to the following system of equations:

$$
\left(\sum_{s \in S} V^i_{(\sigma_t; s)}(c^i) \pi^n(s | \sigma_t) - \pi^{n+1}(s | \sigma_t)\right) = 0, \quad \text{for all } n = 1, \ldots, N.
$$

38
Since at least \(|S| - 1\) of the vectors \(\pi^n (s | \sigma_t)\) are linearly independent, so are at least \(|S| - 1\) of the vectors \((\pi^n (s | \sigma_t) - \pi^{n+1} (s | \sigma_t))\) \(s \in S \setminus \tilde{s}\). Hence, this system of equations has a unique solution, \(V_{(\sigma_i; s)} (c') = V_{(\sigma_i; \tilde{s})} (c')\) for all \(s \in S \setminus \tilde{s}\), which is exactly the condition for full insurance against risk.

**Proof of Proposition 5.7:**

We start with the following Lemma:

**Lemma A.1** In equilibrium, the effective one-step-ahead beliefs of agent \(i\) converge towards his actual beliefs a.s. on a set of paths \(\Sigma\), iff there exists a.s. a sparse set of periods \(N (\sigma) \subset \mathbb{N}\) such that for any two states \(s\) and \(s' \in S\) and any \(\epsilon > 0\), there is a \(T_e (\sigma)\) such that for any \(t \geq T_e, t \in \mathbb{N} \setminus N (\sigma)\),

\[
\left| \frac{u_i' (c^i (\sigma; s))}{u_i' (c^i (\sigma; s'))} \cdot \frac{\sum_{n=1}^{N} \mu_{s_n} (\pi^n (s | \sigma_t)) \cdot p (\sigma_t; s')}{\sum_{n=1}^{N} \mu_{s_n} (\pi^n (s' | \sigma_t))} \cdot \frac{p (\sigma_t; s) \cdot p (\sigma_t; s')}{p (\sigma_t; s)} - 1 \right| < \epsilon. \tag{A-37}
\]

**Proof of Lemma A.1:**

First note that the expectation in (23) becomes 0 if and only if

\[
b_i (s | \sigma_t) = \left[ \sum_{n=1}^{N} \mu_{s_n} (\pi^n (s | \sigma_t)) \right]. \tag{A-38}
\]

By Lemma 5.4, \(b_i (s | \sigma_t)\) is uniformly bounded away from 0. Hence, all of the terms in the summation are finite. It follows that convergence as in (23) obtains if and only if there is a sparse set of periods \(N (\sigma) \subset \mathbb{N}\) such that for each \(s, s', t\), and each \(\epsilon\), there is a \(T_e (\sigma)\) such that \(t > T_e, t \notin N (\sigma)\),

\[
\left| \frac{\sum_{n=1}^{N} \mu_{s_n} (\pi^n (s | \sigma_t)) \cdot b_i (s | \sigma_t)}{\sum_{n=1}^{N} \mu_{s_n} (\pi^n (s' | \sigma_t)) \cdot b_i (s' | \sigma_t)} \cdot \frac{p (\sigma_t; s) \cdot p (\sigma_t; s')}{p (\sigma_t; s)} - 1 \right| < \epsilon. \tag{A-39}
\]

By the f.o.c. (10), which must hold in equilibrium, this is equivalent to (A-37).

Let \(\tilde{\Sigma}\) denote the set of paths, on which \(i\)'s effective one-step-ahead beliefs converge towards \(i\)'s actual beliefs and \(i\) survives. Assume, by a manner of contradiction that \(\mu (\tilde{\Sigma}) > 0\). The non-atomicity of \(\mu\) implies that there is an event \(B \subset \tilde{\Sigma}\) such that \(\mu (B) > 0\) and, hence, such that \(\mu (\tilde{\Sigma} | B) = 1\). Furthermore, there is a \(\tilde{\sigma}_T \in B\) such that \(\mu (Z (\tilde{\sigma}_T)) > 0\) and \(\mu (\tilde{\Sigma} | Z (\tilde{\sigma}_T)) = 1\).

In Claims 1 and 2 below, we show that for any path \(\sigma \in Z (\tilde{\sigma}_T),\)

(i) \(\lim_{t \to -\infty, t \notin N (\sigma)} c^i (\sigma_t; s)\) exists and depends only on \(s\), but not on \(\sigma\), i.e., \(\lim_{t \to -\infty, t \notin N (\sigma)} c^i (\sigma_t; s) = c^i (s)\);

(ii) \(\exists \xi > 0\) such that whenever \(s \neq s'\), either \(|c^i (s) - c^i (s')| \geq \xi\) or \(c^i (s) = c^i (s') = 0\).

Condition (A-37) together with Lemma 5.6 implies that for all \(s, s' \in S\), on \(Z (\tilde{\sigma}_T)\), \(\lim_{t \to -\infty, t \notin N (\sigma)} V_{(\sigma_i; s')}^i = \lim_{t \to -\infty, t \notin N (\sigma)} V_{(\sigma_i; s')}^i\), and, since the economy is Markov,

\[
\lim_{t \to -\infty, t \notin N (\sigma)} \left[ u_i (c^i (\sigma_t; s)) + \beta \phi^{-1} \left( \sum_{n=1}^{N} \phi (\sigma) \sum_{s'' \in S} V_{(\sigma; s'')^n} (c^i (\sigma_t; s'' | s)) \mu_{s''} (\pi^n) \right) \right] = \left( A-40 \right)
\]

\[
\lim_{t \to -\infty, t \notin N (\sigma)} \left[ u_i (c^i (\sigma_t; s')) + \beta \phi^{-1} \left( \sum_{n=1}^{N} \phi (\sigma) \sum_{s'' \in S} V_{(\sigma_i; s'')^n} (c^i (\sigma_t; s'' | s')) \mu_{s''} (\pi^n) \right) \right]
\]

39
The definition of a sparse set implies that for every path $\sigma \in Z(\bar{\sigma}_T)$ and every $\epsilon > 0$, and every $\epsilon > 0$ there is a $T(\epsilon;\sigma)$ such that for any $T \geq T(\epsilon;\sigma)$

$$\# \left\{ t \leq T \middle| \frac{u_i(c^i(\sigma;\bar{\sigma})))}{\mu(\sigma;\bar{\sigma})} - \frac{p(\sigma)}{p(\bar{\sigma})} > \epsilon \right\} < \epsilon, \quad (A-41)$$

i.e., the frequency of those periods, on which effective beliefs differ from actual beliefs by more than $\epsilon$ converges to 0 along every path in $Z(\bar{\sigma}_T)$. Since by (i), $\lim_{t \to \infty} c^i(\sigma_t; s) = c^i(s)$ everywhere on $Z(\bar{\sigma}_T)$, Egoroff’s (1911) Theorem states that for any small $\delta > 0$, there exists a set $B_\delta \subseteq Z(\bar{\sigma}_T)$ s.t. $\mu(Z(\bar{\sigma}_T) \setminus B_\delta) \leq \delta$, $\lim_{t \to \infty} c^i(\sigma_t; s) = c^i(s)$ uniformly on $B_\delta$ and the frequencies in (A-41) also converge uniformly on $B_\delta$. Hence, the sparse set $N$ can be chosen uniformly on $B_\delta$.

Suppose first that in condition (ii), there are at least two states of the world, $s$ and $s'$ such that $e_s \neq e_{s'}$ and $c^i(s) < c^i(s')$. Then $u_i(c) - u_i(c - \xi) \geq u_i(\max_{s \in S} e_s) - u_i(\max_{s \in S} e_s - \xi) =: \bar{\xi} > 0$ implies

$$u_i(c^i(s)) - u_i(c^i(s')) > \bar{\xi}. \quad (A-42)$$

Choose $\delta < \bar{\xi} \frac{(1-\beta)}{\mu(u(m'))}$ and observe that for each $\epsilon > 0$, there exists a $\bar{t}(\epsilon)$ such that for all $\sigma \in B_\delta$, any sequence $s \in S^k$, all $s, s', s''$ and $s''' \in S$ and all $t \geq \bar{t}(\epsilon)$, $(t + k + 3) \notin N$,

$$c^i(\sigma_t; s; s''; s; s'''| s^m) - c^i(\sigma_t; s'; s''; s; s'''| s^m) < \epsilon. \quad (A-43)$$

On $\sigma \notin B_\delta$, we have that for any sequence $s \in S^k$, all $s, s', s''$ and $s''' \in S$ and all $t$,

$$c^i(\sigma_t; s; s''; s; s'''| s^m) - c^i(\sigma_t; s'; s''; s; s'''| s^m) < m'. \quad (A-44)$$

Since $\mu$ and $\pi^m$ are Markov for all $n \in \{1...N\}$ and since $\mu(B_\delta) \geq 1 - \delta$, using (A-43) and (A-44), we obtain that for any $s' \in S$, $\lim_{t \to \infty} t \notin N| V(\sigma; s'; s''; s; s'''| s^m) \leq \delta \frac{\beta u(m')}{1-\beta}$.

Furthermore, by Lemma 5.6 and since on $Z(\bar{\sigma}_T)$, each state $\tilde{s}$ a.s. occurs infinitely often on $\sigma$, for each $\sigma$ and $\epsilon$, there is a $\tilde{t}(\epsilon; \sigma)$ such that for all $t \geq \tilde{t}(\epsilon; \sigma)$, $t + 1 \notin N(\sigma)$

$$\left| V(\sigma; s'; s''; s; s'''| s^m) - V(\sigma; s; s''; s; s'''| s^m) \right| < \epsilon \quad (A-45)$$

holds for any $\tilde{s} \in S$ and any $s'', s''' \in S$. In particular, using $\tilde{s} \in \{s; s'\}$ in (A-45) and substituting in (A-40), we obtain:

$$\lim_{t \to \infty} V(\sigma; s') - \lim_{t \to \infty} V(\sigma; s') \geq u_i(c^i(s)) - u_i(c^i(s')) - \frac{\beta^2 u(m')}{1-\beta} > \bar{\xi} - \frac{\beta u(m')}{1-\beta} > 0, \quad (A-46)$$

in contradiction to condition (A-40).

Suppose now that in condition (ii), $c^i(s) = c^i(s') = 0$ for all $s, s' \in S$. Since in equilibrium, the ratio of the MRS of the two agents is 1 and both have correct beliefs, we can reformulate condition (A-37) as follows: for any two states $s$ and $s' \in S$ and any $\epsilon > 0$, there exists a $T_\epsilon$ such that for any
\[ t \geq \max \{ T; T_i \}, t \not\in \mathcal{N}(\sigma) \]
\[ \left| \frac{u'_i(c'(\sigma_i; s))}{u'_j(e_s - c'(\sigma_i; s))} \left( e_s - c'(\sigma_i; s') \right) \right| - 1 < \epsilon, \quad (A-47) \]

where we are using the fact that at each node \((\sigma_i; s)\), the consumption of \(j\) is given by \(e_s - c'(\sigma_i; s)\).

We will show that \(i\) vanishes on \(\sigma\). Indeed, assume to the contrary that for some \(\bar{\epsilon} > 0\), there are infinitely many periods on \(\sigma\) at which \(c'(\sigma_t) > \bar{\epsilon}\). Then, for any \(\epsilon > 0\), we can choose a \(t\) such that if \(c'(\sigma_t) > \bar{\epsilon}\), then \(c'(\sigma_{t+1}) < \epsilon\). Consider the f.o.c. of \(i\) across \(\sigma_t\) and \(\sigma_{t+1}\) and note that since \(u'_i(c'(\sigma_t))\), \(u'_j(e_s - c'(\sigma_t))\), \(e_s - c'(\sigma_{t+1})\), \(\sum_{n=1}^N \mu_{\sigma_t}(\pi_n) \pi_n(s_{t+1} | \sigma_t)\) and the expression in (15) are all uniformly bounded away from 0, whereas \(u'_i(c'(\sigma_{t+1}))_{t \to 0} = \infty\), we can find a period \(t\) such that

\[ \frac{u'_i(c'(\sigma_t))}{\beta u'_i(c'(\sigma_{t+1}))} \sum_{n=1}^N \phi_i(e_n V_{e,\theta}(c')) \mu_{\pi_n}(\pi(s_{t+1} | \sigma_t)) < \frac{\beta u'_j(e_s - c'(\sigma_t))}{\sum_{n=1}^N \mu_{\sigma_t}(\pi_n) \pi_n(s_{t+1} | \sigma_t)} \]

\[ \text{(A-48)} \]

in violation of the first order condition, which holds at any \(\sigma_t\) in equilibrium. Hence, for each \(\bar{\epsilon} > 0\), the set of periods, on which \(i\)'s consumption exceeds \(\bar{\epsilon}\) has to be finite on \(\sigma\). It follows that \(\mathcal{N}(\sigma)\) is finite and, hence, \(\lim_{t \to \infty} c'(\sigma_t) = 0\), i.e., \(i\) vanishes on \(\sigma\).

Combining the two cases, we conclude that \(\mu(B) = 0\), and therefore, \(\mu(\Sigma) = 0\) as well.

We now complete the proof by showing \((i)\) and \((ii)\).

**Claim 1:**

Let \(\sigma, \sigma' \in Z(\bar{\sigma}_T)\). For any \(s \in S\), \(\lim_{t \to \infty, t \not\in (\mathcal{N}(\sigma) \cup \mathcal{N}(\sigma'))} \left| c'(\sigma_t; s) - c'(\sigma'_t; s) \right| = 0\). Furthermore, for any \(s \in S\), \(\lim_{t \to \infty, t \not\in (\mathcal{N}(\sigma) \cup \mathcal{N}(\sigma'))} c'(\sigma_t; s)\) exists.

**Proof of Claim 1:**

Since \(\sigma, \sigma' \in Z(\bar{\sigma}_T)\), by Lemma A.1, for each \(\epsilon > 0\), there is a \(T_\epsilon\) such that for all \(t \geq T_\epsilon, t \not\in (\mathcal{N}(\sigma) \cup \mathcal{N}(\sigma'))\),

\[ \left| \frac{u'_i(c'(\sigma_t; s))}{u'_j(e_s - c'(\sigma_t; s))} \left( e_s - c'(\sigma'_t; s) \right) \right| - 1 < \epsilon. \quad (A-49) \]

Now suppose that there exists a \(\xi > 0\) such that there for some \(T\), \(\left| c'(\sigma_t; s) - c'(\sigma'_t; s) \right| \geq \xi\) for all \(t \geq T, t \not\in (\mathcal{N}(\sigma) \cup \mathcal{N}(\sigma'))\). Let \(t \geq \max \{ T; T_i \}, t \not\in (\mathcal{N}(\sigma) \cup \mathcal{N}(\sigma'))\). W.l.o.g., suppose that \(c'(\sigma_t; s) = c' \geq c'(\sigma'_t; s) + \xi = c' + \xi\). Note that

\[ \max_{c \in [c' + \xi, c]} \frac{u'_i(c)}{u'_j(e_s - c)} \frac{u'_j(e_s - c')}{u'_i(c')} = \frac{u'_i(c' + \xi)}{u'_j(e_s - c') u'_i(c')} \]

\[ \text{(A-50)} \]

41
We will now show that this function is bounded above, by an \( \tilde{\varepsilon} < 1 \). We then obtain:

\[
\lim_{t \to \infty} \left| 1 - \frac{u'_j(c^i(\sigma_t^i; s))}{u'_j(e_s - c^i(\sigma_t^i; s))} - \frac{u'_j(c^i(\sigma_t^i; s) + \xi)}{u'_j(e_s - c^i(\sigma_t^i; s) - \xi)} \right| \geq 1 - \tilde{\varepsilon} > \varepsilon,
\]

(A-51)

in contradiction to condition (A-49), which should hold in equilibrium.

To show that (A-50) is bounded above by some \( \tilde{\varepsilon} < 1 \), first consider the term \( \frac{u'_j(c^i + \xi)}{u'_j(c^i)} \). On the interval \( c' \in [0; e_s - \xi] \), this term reaches its maximum either at \( c' = e_s - \xi \) or at some interior \( c^i_{\max} \). In the former case, the maximum is \( \frac{u'_j(e_s)}{u'_j(e_s - c^i)} < 1 \), and if \( \lim_{c' \to 0} \frac{u'_j(c^i + \xi)}{u'_j(c^i)} = 0 \). Let \( \max_{c' \in [0; e_s - \xi]} \frac{u'_j(c^i + \xi)}{u'_j(c^i)} =: e^i < 1 \).

Now consider the term \( \frac{u'_j(e_s - c^i)}{u'_j(e_s - c^i - \xi)} \). On the interval \( c' \in [0; e_s - \xi] \), this term reaches its maximum either at \( c' = 0 \) or at some interior \( c^i_{\max} \). In the former case, the maximum is \( \frac{u'_j(e_s)}{u'_j(e_s - \xi)} < 1 \), and if \( \lim_{c' \to -\xi} \frac{u'_j(e_s - c^i)}{u'_j(e_s - c^i - \xi)} = 0 \). Let \( \max_{c' \in [0; e_s - \xi]} \frac{u'_j(e_s - c^i)}{u'_j(e_s - c^i - \xi)} =: e^j < 1 \).

It follows that

\[
\frac{u'_j(c' + \xi)}{u'_j(c')} \frac{u'_j(e_s - c^i)}{u'_j(e_s - c^i - \xi)} \leq e^i e^j < 1.
\]

(A-52)

Setting \( \tilde{\varepsilon} := e^i e^j < 1 \) then gives the desired result.

Finally, to show the existence of \( \lim_{t \to \infty, t \notin \mathcal{N}(\sigma)} c^i(\sigma_t; s) \), note that for each \( \sigma \in Z(\bar{\sigma}_T) \), each \( s \in S \) and each \( \epsilon > 0 \), there is a \( T_\epsilon \) such that

\[
\left| \frac{u'_j(c^i(\sigma_t^i; s) + \xi)}{u'_j(e_s - c^i(\sigma_t^i; s) - \xi)} \right| < \epsilon,
\]

(A-53)

for all \( t \geq t_\epsilon, t, t + 1 \notin \mathcal{N}(\sigma) \). Substituting \( (\sigma_{t+1}; s) \) for \( (\sigma_t; s) \) in the argument above, we conclude that \( \lim_{t \to \infty} c^i(\sigma_t; s) - c^i(\sigma_{t+1}; s) = 0 \). Therefore, \( \lim_{t \to \infty, t \notin \mathcal{N}(\sigma)} c^i(\sigma_t; s) \) exists for each path \( \sigma \in Z(\bar{\sigma}_T) \).

Let \( c^i(s) := \lim_{t \to \infty, t \notin \mathcal{N}(\sigma)} c^i(\sigma_t; s) \) for some and, hence, for all \( \sigma \in Z(\bar{\sigma}_T) \).

**Claim 2:**

Let \( \sigma \in Z(\bar{\sigma}_T) \). There is a \( \xi > 0 \) such that for all \( s \) and \( s' \in S \) such that \( e_s \neq e_{s'} \), either \( |c^i(s) - c^i(s')| \geq \xi \), or \( c^i(s) = c^i(s') = 0 \).

**Proof of Claim 2:**

By the assumption of aggregate risk, there are \( s \) and \( s' \in S \) such that \( e_s > e_{s'} \). By definition, for each \( \sigma \in Z(\bar{\sigma}_T) \) and each \( \epsilon > 0 \), there is a \( T_\epsilon \) such that for all \( t \geq T_\epsilon, t \notin \mathcal{N}(\sigma) \)

\[
\left| \frac{u'_j(c^i(\sigma_t^i; s) + \xi)}{u'_j(e_s - c^i(\sigma_t^i; s) - \xi)} \right| < \epsilon.
\]

(A-54)
Suppose first that \( c^i (s') = 0 < c^i (s) \), then, since \( u' (0) = \infty \), the l.h.s. of (A-54) becomes arbitrarily close to 1 arbitrarily often, a contradiction. A similar contradiction obtains if \( c^i (s) = 0 < c^i (s') \). It follows that either \( c^i (s') = c^i (s) = 0 \), or \( c^i (s) > 0 \) and \( c^i (s') > 0 \).

If \( c^i (s') = c^i (s) = 0 \), the statement of Claim 2 is trivially true for \( s \) and \( s' \). Thus, let \( s \) and \( s' \) be such that \( c^i (s) > 0 \) and \( c^i (s') > 0 \). Then, \( T_\epsilon \) can be chosen so that \( \min \{ c^i (s) ; c^i (s') \} > \bar{\xi} \) for some \( \bar{\xi} > 0 \) and for all \( t \geq T_\epsilon , t / \notin \mathcal{N} (\sigma) \).

Assume contrary to the claim that \( c^i (s) = c^i (s') = c \). Consider the expression \( \frac{u_j '(c_s - c)}{u_j '(c_{s'} - c)} \) on the interval \( c \in [0; e_{s'}] \). The maximum of this expression on this interval is obtained either at \( c = 0 \) or at some interior \( c_{\text{max}} \). In the former case, the maximum is \( \frac{u_j '(c_{\text{max}})}{u_j '(e_{s'})} < 1 \), in the latter, the maximum is \( \frac{u_j '(c_{\text{max}})}{u_j '(e_{s'})} = 1 \). Note that since the expression is non-negative, its maximum cannot be at \( c = e_{s'} \), since \( \lim_{c \to e_{s'}} \frac{u_j '(e_{s'} - c)}{u_j '(e_{s'} - c)} = 0 \). We obtain that

\[
\frac{u_j '(e_s - c)}{u_j '(e_{s'} - c)} \leq \max_{c \in [0; e_{s'}]} \frac{u_j '(e_s - c)}{u_j '(e_{s'} - c)} < 1. \tag{A-55}
\]

On the other hand, (A-54) implies:

\[
\lim_{t \to -\infty} \left| \frac{u_j '(c^i (\sigma_t; s))}{u_j '(c^i (\sigma_t; s'))} \frac{u_j '(e_{s'} - c^i (\sigma_t; s'))}{u_j '(e_s - c^i (\sigma_t; s))} - 1 \right| = \lim_{t \to -\infty} \left| \frac{u_j '(e_{s'} - c^i (\sigma_t; s'))}{u_j '(e_s - c^i (\sigma_t; s))} - 1 \right| = \frac{u_j '(e_{s'} - c^i (\sigma_t; s'))}{u_j '(e_s - c^i (\sigma_t; s))} - 1 = 0, \tag{A-56}
\]

a contradiction.

Since \( S \) is finite, we conclude that there is a \( \xi > 0 \) such that \( e_s / \notin e_{s'} \) implies that either \( |c^i (s) - c^i (s')| \geq \xi \), or \( |c^i (s) - c^i (s')| = 0. \]

In the proofs of the following Lemmas 5.8, 5.9 and 5.10, we fix \( i, c^i \) and \( \sigma_t \) and denote by \( y_n := E_{\sigma^n} (V_{i+1} (c^i)) \).

**Proofs of Lemmas 5.8, 5.9 and 5.10:**

Following Osaki and Schlesinger (Unpublished Results, pp. 9-10), we obtain

\[
\frac{\sum_{n=1}^{N} \varphi_i (y_n) \mu_{\sigma^i} (\pi^n)}{\phi_i^{-1} \left( \sum_{n=1}^{N} \varphi_i (y_n) \mu_{\sigma^i} (\pi^n) \right)} = \frac{\phi_i \left( \sum_{n=1}^{N} y_n \mu_{\sigma^i} (\pi^n) - \Phi_A \right)}{\phi_i \left( \sum_{n=1}^{N} y_n \mu_{\sigma^i} (\pi^n) - P_A \right)}, \tag{A-57}
\]

where \( \Phi_A \) is the ambiguity precautionary premium, implicitly defined by:

\[
\sum_{n=1}^{N} \phi_i (y_n) \mu_{\sigma^i} (\pi^n) = \phi_i \left( \sum_{n=1}^{N} y_n \mu_{\sigma^i} (\pi^n) - \Phi_A \right), \tag{A-58}
\]

in analogy to the corresponding definition of Kimball (1990) of the risk precautionary premium, while \( P_A \) is the ambiguity premium, implicitly defined by:

\[
\phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i (y_n) \mu_{\sigma^i} (\pi^n) \right) = \sum_{n=1}^{N} y_n \mu_{\sigma^i} (\pi^n) - P_A, \tag{A-59}
\]

in analogy to the risk premium. For concave functions \( \phi_i, P_A \) is positive, see Pratt (1964).
DAAA (IAAA), is equivalent to $\left(-\frac{\phi''}{\phi'}\right)' \leq 0$, which in turn is equivalent to $\frac{(\phi''')^2 - \phi'' \phi'''}{\phi'}^2 \leq 0$, or

$$-\frac{\phi'''}{\phi'} \left(\frac{\phi''}{\phi'} - \frac{\phi'''}{\phi'''}\right) \leq 0 \quad \text{(A-60)}$$

Since $-\frac{\phi''}{\phi'} > 0$, this implies that for a DAAA (IAAA) function $\phi_i$, $\Phi_A \succ P_A$, see Gollier (2001, p. 238), whenever not all $y_n$ are equal. Since $\phi_i$ is decreasing, we conclude that for $\phi_i$ exhibiting DAAA (IAAA) the expression in (A-57) is greater (smaller) than 1, except when $\Phi_A = P_A = 0$, or when $y_n = y_n'$ for all $n, n' \in \{1\ldots N\}$, in which case (A-57) is equal to 1. For CAAA, $-\frac{\phi''}{\phi'} \left(\frac{\phi''}{\phi'} - \frac{\phi'''}{\phi'''}\right) = 0$ and we obtain $\Phi_A = P_A$ for all values of $(y_n)_{n=1}^{N}$. Hence, the expression in (A-57) exactly equals 1. ■

Proof for Proposition 5.11:

Since $\beta_i = \beta_j$, expression (13) reduces to

$$\lim_{T \to \infty} \frac{1}{T+1} \ln \frac{u_i' \left(\sigma T; s_{T+1}\right)}{u_j' \left(\sigma T; s_{T+1}\right)} = \lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} \ln \left(\frac{\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V_{\sigma+1}^{i} \left(c_t\right)\right)\right] \mu_{\sigma_i} \left(\pi^n\right)}{\sum_{n=1}^{N} \phi_j \left[E_{\pi_n} \left(V_{\sigma+1}^{j} \left(c_t\right)\right)\right] \mu_{\sigma_j} \left(\pi^n\right)}\right)$$

$$+ \lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} \ln \left[\sum_{n=1}^{N} \mu_{\sigma_i} \left(\pi^n\right) \pi^n(\tau_{t+1} | \sigma_t) \right]$$

By lemma 5.8, for a CAAA consumer:

$$\frac{\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V_{\sigma+1}^{i} \left(c_t\right)\right)\right] \mu_{\sigma_i} \left(\pi^n\right)}{\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V_{\sigma+1}^{j} \left(c_t\right)\right)\right] \mu_{\sigma_j} \left(\pi^n\right)} = 1 \quad \text{(A-62)}$$

By lemma 5.9, for an IAAA consumer:

$$\frac{\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V_{\sigma+1}^{i} \left(c_t\right)\right)\right] \mu_{\sigma_i} \left(\pi^n\right)}{\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V_{\sigma+1}^{j} \left(c_t\right)\right)\right] \mu_{\sigma_j} \left(\pi^n\right)} \leq 1 \quad \text{(A-63)}$$

Hence we conclude that, for a CAAA consumer:

$$\lim_{T \to \infty} \frac{1}{T+1} \ln \frac{u_i' \left(\sigma T; s_{T+1}\right)}{u_j' \left(\sigma T; s_{T+1}\right)} = \lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} \ln \left[\frac{\sum_{n=1}^{N} \mu_{\sigma_i} \left(\pi^n\right) \pi^n(\tau_{t+1} | \sigma_t) \sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V_{\sigma+1}^{i} \left(c_t\right)\right)\right] \mu_{\sigma_i} \left(\pi^n\right)}{\sum_{n=1}^{N} \phi_j \left[E_{\pi_n} \left(V_{\sigma+1}^{j} \left(c_t\right)\right)\right] \mu_{\sigma_j} \left(\pi^n\right)}\right]$$

$$\text{(A-64)}$$

while for an IAAA consumer:

$$\lim_{T \to \infty} \frac{1}{T+1} \ln \frac{u_i' \left(\sigma T; s_{T+1}\right)}{u_j' \left(\sigma T; s_{T+1}\right)} \leq \lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} \ln \left[\frac{\sum_{n=1}^{N} \mu_{\sigma_i} \left(\pi^n\right) \pi^n(\tau_{t+1} | \sigma_t) \sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V_{\sigma+1}^{i} \left(c_t\right)\right)\right] \mu_{\sigma_i} \left(\pi^n\right)}{\sum_{n=1}^{N} \phi_j \left[E_{\pi_n} \left(V_{\sigma+1}^{j} \left(c_t\right)\right)\right] \mu_{\sigma_j} \left(\pi^n\right)}\right]$$

$$\text{(A-65)}$$

Note that since

$$\frac{\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V_{\sigma+1}^{i} \left(c_t\right)\right)\right] \mu_{\sigma_i} \left(\pi^n\right) \pi^n(\tau_{t+1} | \sigma_t)}{\sum_{n=1}^{N} \phi_j \left[E_{\pi_n} \left(V_{\sigma+1}^{j} \left(c_t\right)\right)\right] \mu_{\sigma_j} \left(\pi^n\right)}$$

is a probability distribution, we have that its relative entropy with respect to the true probability distrib-
Proof of Lemma A.2:  

Just as in the proof of Lemma 2 of Blume and Easley (2001), for each probability distribution \( \rho_s \), there are uniform bounds \( k_{\rho_s} \) and \( K_{\rho_s} \) such that:

\[
k_{\rho_s} E \left[ 1_{t+1}^s (\sigma) \ln \frac{\rho(s \mid \sigma_t)}{b^i(s \mid \sigma_t)} \mid \sigma_t \right] \leq Var \left[ 1_{t+1}^s (\sigma) \ln \frac{\rho(s \mid \sigma_t)}{b^i(s \mid \sigma_t)} \mid \sigma_t \right] \leq K_{\rho_s} E \left[ 1_{t+1}^s (\sigma) \ln \frac{\rho(s \mid \sigma_t)}{b^i(s \mid \sigma_t)} \mid \sigma_t \right].
\]  

(A-73)

Since there is a finite number of states \( S \), the bounds \( k_{\rho_s} \) and \( K_{\rho_s} \) can be chosen uniformly over all \( \rho_s \). Denote these bounds by \( k_{\rho} \) and \( K_{\rho} \), respectively. Consider first the set of paths \( \tilde{\Sigma} \) as in the statement of
the Lemma. From the bounds derived above, it follows that for any $\sigma \in \hat{\Sigma}$,

$$\lim_{T \to \infty} \sum_{t=0}^{T} \text{Var} \left( \frac{1^{s}_{t+1} (\sigma) \ln \frac{\rho(s | \sigma_t)}{b^{s}(s | \sigma_t)} | \sigma_t}{| \sigma_t} \right) < \infty. \quad (A-74)$$

From Kolmogorov’s inequality it follows that on $\hat{\Sigma}$, $\mu$-a.s.,

$$\lim_{T \to \infty} \sum_{t=0}^{T} \left[ 1^{s}_{t+1} (\sigma) \ln \frac{\rho(s | \sigma_t)}{b^{s}(s | \sigma_t)} - E \left( 1^{s}_{t+1} (\sigma) \ln \frac{\rho(s | \sigma_t)}{b^{s}(s | \sigma_t)} | \sigma_t \right) \right] < \infty. \quad (A-75)$$

Now consider the set of paths $\hat{\Sigma}$ as in the statement of the Lemma. A consequence of the bounds established above is that for any $\sigma \in \hat{\Sigma}$,

$$\lim_{T \to \infty} \sum_{t=0}^{T} \text{Var} \left( \frac{1^{s}_{t+1} \ln \frac{\rho(s | \sigma_t)}{b^{s}(s | \sigma_t)} | \sigma_t}{| \sigma_t} \right) = \infty. \quad (A-76)$$

Denote by $X_T, T \geq 1$,

$$X_T =: \sum_{t=0}^{T-1} \left[ 1^{s}_{t+1} (\sigma) \ln \frac{\rho(s | \sigma_t)}{b^{s}(s | \sigma_t)} - E \left( 1^{s}_{t+1} (\sigma) \ln \frac{\rho(s | \sigma_t)}{b^{s}(s | \sigma_t)} | \sigma_t \right) \right]. \quad (A-77)$$

note that $(X_T)$ is a martingale and can be written as

$$X_T =: \sum_{k=1}^{T} \left[ X_k - E \left(X_k | \sigma_{k-1}\right) \right]. \quad (A-78)$$

Let $\tilde{X}_T =: X^2_T$. $(\tilde{X}_T)_T$ is a submartingale, see Neveu (1972, p. 147) and thus can be written using the Doob’s decomposition as:

$$\tilde{X}_T = M_T + A_T, \quad (A-79)$$

where $(M_T)_T$ is a martingale, whereas $(A_T)_T$ is a predictable ($\sigma_T$-measurable) increasing sequence.

Furthermore,

$$A_T = \sum_{k=1}^{T} \left[ X^2_k - X^2_{k-1} | \sigma_{k-1} \right] = \sum_{k=1}^{T} \text{Var} \left( X_k | \sigma_{k-1} \right) = \sum_{t=0}^{T-1} \text{Var} \left( \frac{1^{s}_{t+1} (\sigma) \ln \frac{\rho(s | \sigma_t)}{b^{s}(s | \sigma_t)} | \sigma_t}{| \sigma_t} \right) \quad (A-80)$$

Since $\sigma \in \hat{\Sigma}$, $\lim_{T \to \infty} A_T = \infty$ and, hence, we can apply Proposition VII-2-4 of Neveu (1972, p. 150), taking the function $f(A_T)$ defined in the proposition to be the identity, to conclude that $\mu$-a.s.,

$$\lim_{T \to \infty} \frac{X^2_{T+1}}{A_{T+1}} = 0, \quad (A-81)$$

$\mu$-a.s. Because of the bound on the variance derived above, we have:

$$\lim_{T \to \infty} \frac{1}{(T+1)K} \sum_{t=0}^{T} \left[ 1^{s}_{t+1} (\sigma) \ln \frac{\rho(s | \sigma_t)}{b^{s}(s | \sigma_t)} - E \left( 1^{s}_{t+1} (\sigma) \ln \frac{\rho(s | \sigma_t)}{b^{s}(s | \sigma_t)} | \sigma_t \right) \right] = 0, \quad (A-82)$$

or

$$\lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} \left[ 1^{s}_{t+1} (\sigma) \ln \frac{\rho(s | \sigma_t)}{b^{s}(s | \sigma_t)} - E \left( 1^{s}_{t+1} (\sigma) \ln \frac{\rho(s | \sigma_t)}{b^{s}(s | \sigma_t)} | \sigma_t \right) \right] = 0. \quad (A-83)$$
It follows that in both cases considered in Lemma A.2,
\[
\lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} 1_{t+1}^* (\sigma) \ln \frac{\rho (s | \sigma_t)}{b (s | \sigma_t)} - E \left( \frac{\rho (s | \sigma_t)}{b (s | \sigma_t)} \right) = 0 \quad (A-84)
\]
and, hence, we can replace the terms of the summation on the r.h.s. of expressions (A-64) and (A-65) by their expectations: i.e., \( \mu \)-a.s.,
\[
\frac{1}{T+1} \sum_{t=0}^{T} \left[ \ln \left( \sum_{n=1}^{N} \mu_{\sigma_t} (\pi^n) \pi^n (s_{t+1} | \sigma_t) \right) \left[ \sum_{n=1}^{N} \phi_t \left( E_n^{\pi} \left( V_{s_t+1}^{\pi} (c^i) \right) \right) \mu_{\sigma_t} (\pi^n) \right] \mu_{\sigma_t} (\pi^n) \right] - \sum_{n=1}^{N} \phi_t \left( E_n^{\pi} \left( V_{s_t+1}^{\pi} (c^i) \right) \right) \mu_{\sigma_t} (\pi^n) \pi^n (s_{t+1} | \sigma_t) \ln \left( \sum_{n=1}^{N} \mu_{\sigma_t} (\pi^n) \pi^n (s_{t+1} | \sigma_t) \right) \]
\]
converges to 0. Hence, a.s. on \( \Sigma \), expression (13) becomes:
\[
\lim_{T \to \infty} \frac{1}{T+1} \ln \frac{u'_i (c^i (\sigma; s_{T+1}))}{u'_j (c^j (\sigma; s_{T+1}))} = \quad (A-85)
\]
and since all of the terms in the square brackets on the r.h.s. are non-positive, we have \( \lim_{T \to \infty} \frac{1}{T+1} \ln \frac{u'_i (c^i (\sigma; s_{T+1}))}{u'_j (c^j (\sigma; s_{T+1}))} \geq 0 \). Hence, \( j \) survives \( \mu \)-a.s. Furthermore,
\[
\lim_{T \to \infty} \frac{1}{T+1} \ln \frac{u'_i (c^i (\sigma; s_{T+1}))}{u'_j (c^j (\sigma; s_{T+1}))} > 0 \quad (A-86)
\]
obtains exactly on those paths, on which \( i \)'s one-step-ahead effective beliefs do not converge to the truth in the sense of (23). On this set of paths, the conditions of Theorem 6 of Blume and Easley (2006) hold and we conclude that \( i \) vanishes a.s.

Proof of Proposition 5.12:
Under the assumptions made, condition (13) reduces to:
\[
\lim_{T \to \infty} \frac{1}{T+1} \ln \frac{u'_i (c^i (\sigma; s_{T+1}))}{u'_j (c^j (\sigma; s_{T+1}))} = \quad (A-87)
\]
We will first show that all of the terms in the summation on the r.h.s. (A-87) are non-negative in expectations. By Lemma 5.4, all of the terms on the r.h.s. are uniformly bounded. We can then apply Lemma A.2 and Theorem 6 of Blume and Easley (2006) to show that \( j \) a.s. vanishes on all paths, on which the expectation of the r.h.s. of (A-87) is strictly positive, i.e., on which \( i \)'s one-step-ahead effective beliefs do not converge towards the truth. On the set of paths, on which \( i \)'s effective beliefs converge towards
the truth, \( i \) a.s. survives.

Denote by \( y_n =: E_{\pi^n} \left( V^i_{\sigma^{n+1}} \left( e^j \right) \right) \). We start by showing that:

\[
E \left[ \ln \sum_{n=1}^{N} \phi_i^\prime \left( y_n \right) \mu_{\sigma^i} \left( \pi^n \right) \pi^n \left( s_{i+1} | \sigma_t \right) \frac{1}{\phi_i^\prime \left( \phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i \left( y_n \right) \mu_{\sigma^i} \left( \pi^n \right) \right) \right) } \right] \geq 0. \tag{A-88}
\]

Note that (A-88) is equivalent to

\[
\prod_{s_{i+1} \in S} \left[ \frac{\sum_{n=1}^{N} \phi_i^\prime \left( y_n \mu_{\sigma^i} \left( \pi^n \right) \pi^n \left( s_{i+1} | \sigma_t \right) \right) \phi_i^\prime \left( \phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i \left( y_n \right) \mu_{\sigma^i} \left( \pi^n \right) \right) \right) }{\phi_i^\prime \left( \phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i \left( y_n \right) \mu_{\sigma^i} \left( \pi^n \right) \right) \right) } \right] \sum_{n=1}^{N} \mu_{\sigma^i} \left( \pi^n \right) \pi^n \left( s_{i+1} | \sigma_t \right) \geq 1 \tag{A-89}
\]

Since the arithmetic mean is larger than the geometric mean

\[
\prod_{s_{i+1} \in S} \left[ \frac{\sum_{n=1}^{N} \phi_i^\prime \left( y_n \mu_{\sigma^i} \left( \pi^n \right) \pi^n \left( s_{i+1} | \sigma_t \right) \right) }{\phi_i^\prime \left( \phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i \left( y_n \right) \mu_{\sigma^i} \left( \pi^n \right) \right) \right) } \right] \sum_{n=1}^{N} \mu_{\sigma^i} \left( \pi^n \right) \pi^n \left( s_{i+1} | \sigma_t \right)
\]

\[
= \prod_{n=1}^{N} \phi_i^\prime \left( y_n \right) = \exp \sum_{n=1}^{N} \mu_{\sigma^i} \left( \pi^n \right) \ln \phi_i^\prime \left( y_n \right) \tag{A-90}
\]

Now note that

\[
\frac{\exp \sum_{n=1}^{N} \mu_{\sigma^i} \left( \pi^n \right) \ln \phi_i^\prime \left( y_n \right) }{\phi_i^\prime \left( \phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i \left( y_n \right) \mu_{\sigma^i} \left( \pi^n \right) \right) \right) } \geq 1 \tag{A-91}
\]

is equivalent to

\[
\sum_{n=1}^{N} \mu_{\sigma^i} \left( \pi^n \right) \ln \phi_i^\prime \left( y_n \right) \geq \ln \phi_i^\prime \left( \phi_i^{-1} \left( \sum_{n=1}^{N} \phi_i \left( y_n \right) \mu_{\sigma^i} \left( \pi^n \right) \right) \right), \tag{A-92}
\]

which can be rewritten as:

\[
\ln \phi_i^\prime \left( \sum_{n=1}^{N} y_n \mu_{\sigma^i} \left( \pi^n \right) - \bar{\Phi}_A \right) \geq \ln \phi_i^\prime \left( \sum_{n=1}^{N} y_n \mu_{\sigma^i} \left( \pi^n \right) - P_A \right), \tag{A-93}
\]

where \( P_A \) is the ambiguity premium as defined in (A-59), whereas \( \bar{\Phi}_A \) is the precautionary premium defined in analogy with (A-58), but with respect to \( \ln \phi_i^\prime \) (rather than \( \phi_i^\prime \)). Note that \( \bar{\Phi}_A \geq P_A \) will obtain iff \( \frac{\sum_{n=1}^{N} \mu_{\sigma^i} \left( \pi^n \right)}{\mu_{\sigma^i} \left( \pi^n \right)} \geq \frac{\sum_{n=1}^{N} \mu_{\sigma^i} \left( \pi^n \right) - P_A}{\mu_{\sigma^i} \left( \pi^n \right) - P_A} \), which is equivalent to \( -\frac{\sum_{n=1}^{N} \mu_{\sigma^i} \left( \pi^n \right)}{\mu_{\sigma^i} \left( \pi^n \right) \phi_i^\prime} \geq -\frac{\sum_{n=1}^{N} \mu_{\sigma^i} \left( \pi^n \right) - P_A}{\sum_{n=1}^{N} \mu_{\sigma^i} \left( \pi^n \right) - P_A}, \) as required in the statement of the proposition. It follows that (A-88) is always satisfied with equality obtaining iff \( y_n = \text{const} \) for all \( n \in \{1...N\} \).

Combining this with Lemma 5.4, we conclude that all of the terms on the right-hand-side of (A-87) are uniformly bounded and non-positive in expectations. As in the proof of Proposition 5.11, applying Lemma A.2 we
obtain that a.s.,
\[
\frac{1}{T+1} \sum_{t=0}^{T} \left[ \ln \left( \sum_{n=1}^{N} \phi_i \left( E_{\pi_i^n(n_{t+1} | \sigma_t)} \right) \right) \mu_{\pi_i^n} \right] - \sum_{s_t \in S} \sum_{n=1}^{N} \mu_{\pi_i^n} \left( s_{t+1} | \sigma_t \right) \ln \left( \sum_{n=1}^{N} \phi_i \left( E_{\pi_i^n(n_{t+1} | \sigma_t)} \right) \right) \mu_{\pi_i^n} \left( s_{t+1} | \sigma_t \right) \left( s_{t+1} | \sigma_t \right)
\]

converges to 0. Hence, a.s. on \( \Sigma \), (A-87) becomes:
\[
\lim_{T \to \infty} \frac{1}{T+1} \ln \left( \frac{u_i'(c(\sigma T; sT+1))}{u_i'(c(\sigma T; sT+1))} \right) = 0
\]

and since all of the terms in the square brackets on the r.h.s. are non-negative, we have \( \lim_{T \to \infty} \frac{1}{T+1} \ln \left( \frac{u_i'(c(\sigma T; sT+1))}{u_i'(c(\sigma T; sT+1))} \right) \leq 0 \). Hence, \( i \) survives a.s. Furthermore,
\[
\lim_{T \to \infty} \frac{1}{T+1} \ln \left( \frac{u_i'(c(\sigma T; sT+1))}{u_i'(c(\sigma T; sT+1))} \right) < 0
\]

obtains exactly on those paths, on which \( i \)'s one-step-ahead effective beliefs do not converge to the truth in the sense of (23). On this set of paths, the conditions of Theorem 6 of Blume and Easley (2006) hold and we conclude that \( j \) vanishes a.s.

**Proof for Example 6.1:**

To show that \( b^i(s^1 | \sigma_t) \) is uniformly bounded away from \( \frac{1}{2} \), note that \( b^i(s^1 | \sigma_t) = \frac{1}{2} \) iff
\[
MRS_i = \frac{c^i(\sigma T; s^1)}{c^i(\sigma T; s^1)} \left( \frac{c^i(\sigma T; s^1)}{c^i(\sigma T; s^1)} \right)^{\frac{\pi}{1-\pi} (1-\pi)} \left( \frac{c^i(\sigma T; s^1)}{c^i(\sigma T; s^1)} \right)^{\frac{1-\pi}{1-\pi} (1-\pi)} = 1,
\]

which obtains only for \( \frac{c^i(\sigma T; s^1)}{c^i(\sigma T; s^1)} = 1 \). Furthermore, \( MRS_i \) is strictly increasing in \( \frac{c^i(\sigma T; s^1)}{c^i(\sigma T; s^1)} \).

As long as \( \frac{c^i(\sigma T; s^1)}{c^i(\sigma T; s^1)} < \frac{e_{i+1}}{e_{i+1}} \), \( MRS_j \) satisfies \( MRS_j = \frac{e_{i+1}-c^i(\sigma T; s^1)}{e_{i+1}-c^i(\sigma T; s^1)} > \frac{e_{i+1}}{e_{i+1}} \). Hence, at \( \frac{c^i(\sigma T; s^1)}{c^i(\sigma T; s^1)} = 1 \), \( MRS_j > \frac{e_{i+1}}{e_{i+1}} > 1 = MRS_i \). Since \( MRS_i \) is continuous and increasing in \( \frac{c^i(\sigma T; s^1)}{c^i(\sigma T; s^1)} \), there exists an \( \bar{e} > 0 \) such that \( MRS_i < \frac{e_{i+1}}{e_{i+1}} \), iff \( \frac{c^i(\sigma T; s^1)}{c^i(\sigma T; s^1)} < 1 + \bar{e} \). It follows that the one-step-ahead effective beliefs of \( i \) satisfy:
\[
\frac{b^i(s^1 | \sigma_t)}{b^i(s^2 | \sigma_t)} \in \frac{(1+\bar{e})^{\pi(1-\pi)} + (1+\bar{e})^{1-\pi} \left( \frac{e_{i+1}}{e_{i+1}} \right)^{\pi(1-\pi)} + (\frac{e_{i+1}}{e_{i+1}})^{1-\pi}}{(1+\bar{e})^{\pi(1+\bar{e})} + (1+\bar{e})^{1-\pi} \left( \frac{e_{i+1}}{e_{i+1}} \right)^{\pi(1-\pi)} + (\frac{e_{i+1}}{e_{i+1}})^{1-\pi}} \]

on all paths \( \sigma \). Let \( \xi = \frac{1+\bar{e}}{(1+\bar{e})^{\pi(1+\bar{e})} + (1+\bar{e})^{1-\pi} \left( \frac{e_{i+1}}{e_{i+1}} \right)^{\pi(1-\pi)} + (\frac{e_{i+1}}{e_{i+1}})^{1-\pi}} - 1 \) and set \( \bar{\xi} = \frac{\xi}{2(2+\xi)} \); it is immediate that \( \frac{b^i(s^1 | \sigma_t)}{b^i(s^2 | \sigma_t)} > 1 + \xi \) implies
\[
b^i(s^1 | \sigma_t) - \frac{1}{2} > \frac{1 + \xi}{2(2+\xi)} - \frac{1}{2} = \frac{\xi}{2(2+\xi)} = \bar{\xi}.
\]
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