A Transshipment Game *

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Abstract

In situations where a seller has surplus stock and another seller is stocked out, it may be desirable to transfer surplus stock from the former to the latter. We examine how the possibility of such transshipments between two independent locations affects the optimal inventory orders at each location. If each location aims to maximize its own profits – we call this local decision-making – their inventory choices will not, in general, maximize joint profits. We find transshipment prices that induce the locations to choose inventory levels consistent with joint-profit maximization.

Keywords: transshipments, newsvendor model, Nash equilibrium

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1 Introduction

Bosch, based in Germany, is one of the world’s largest producers of power tools and electric parts for automobiles. Bosch products are distributed in Norway through five independently-owned distributors. Each distributor has an exclusive territory, an arrangement that restricts their ability to sell directly to dealers in rival territories. Given that the demand for automobile parts is stochastic, and that maintaining inventory is costly, the optimal level of inventory entails some possibility of stock out – situations in which demand exceeds the chosen inventory level. A distributor who is stocked out can replenish stock from Bosch in Germany but direct delivery takes about three weeks to Norway. Given this long lag, localized fluctuations in demand can be met through transferring stock overnight from a distributor who has surplus stock to another who is stocked out. Such transfers – or transshipments – are desirable whenever the associated costs of transferring goods is not too high. If the distributors choose inventory levels to maximize their profits, the possibility of transshipments affects their ex-ante inventory choices. This paper examines optimal inventory choice in the presence of transshipments.

Conventional models of transshipment do not always capture the story above very accurately. These typically assume that inventory orders at each location are somehow coordinated by some central agency. For instance, it may be that the various locations are all owned by a single ‘parent-firm’, whose objective is to maximize some measure of aggregate performance, such as sum of profit or sales across locations. The work of Krishnan and Rao (1965), Karmarkar and Patel (1977), Robinson (1990), and Archibald et al (1997), for instance, is very much in this spirit. In effect, the transshipments then take place within a firm, and can be thought of as intra-firm transshipments.

In reality, however, we find that transshipments often happen in contexts where the locations are not all owned by one firm, nor are their inventory orders coordinated by any
explicit mechanism. As in the case of the Norwegian distributors of Bosch, each location is often an independently-owned firm that chooses its inventory level to maximize its own profits. Transshipments now represent voluntary trades between firms and are, in that sense, *inter-firm transshipments*. The absence of any explicit mechanism for coordinating inventory levels, assumed in conventional models, is significant. This paper examines inventory choice with transshipments in such decentralized environments, and looks at the efficiency of the outcomes that emerge.

There is crucial difference between intra-firm transshipments and inter-firm transshipments. If all locations are owned by a single principal, transshipment prices do not matter: the price charged by one location for transshipping goods to another is a purely internal transfer price and does not affect aggregate profit. On the other hand, in the case of inter-firm transshipments, transshipment prices affect profits of each firm directly. How are these transshipment prices set? It is natural to model transshipment prices as being determined by direct negotiation between the two firms involved in the transaction. Even then, the precise outcome of such negotiation is sensitive to various assumptions. For instance, how is bargaining power distributed between the firms? Can transshipment prices be determined or influenced by the central supplier?

Localized decision-making is inefficient in the sense that coordination of inventory orders can improve average profitability. The intuition for this result is easy to see. The choice of inventory levels involves an externality: large inventory carried by one firm makes it easier for the other firms to draw upon this inventory in cases of stock-outs. On the other hand, small inventories carried by one firm makes it easier for the other firm to dispose of their excess inventory in case of left over stocks. Given this externality, inventory choice involves strategic interaction, and the Nash equilibria of the resulting game – inventory levels such that each firm reacts optimally to the other’s inventory level – is of interest. We find that, in general, joint profits are not maximized at the unique Nash equilibrium, and coordinated
inventory choice could increase profitability.

Moreover, the inventory choice of each location depends on transshipment prices. High transshipment prices make it profitable for each firm to carry more inventory. For low transshipment prices we find that order quantities are inoptimally low (relative to the coordinated solution); and for high transshipment prices they are excessively high. We find the intermediate prices that decentralize the solution that maximizes joint profits. This is in the spirit of recent work in Supply Chain Management, like Chen (1996), Cachon and Zipkin (1999), and Lee and Whang (1999), which designs incentive schemes that would induce decentralized decision makers to achieve the centralized optimal solution.

Many industries in which transshipment is performed, such as apparel, sporting goods and toys, are characterized by long lead times, short selling seasons and high demand uncertainty. The classic newsvendor model applies in this context. The newsvendor model is the basis of most existing transshipment literature. For a survey of the newsvendor problem and extensions, see Porteus (1990). A newsvendor must choose an inventory level of newspapers prior to knowing the true level of demand for them. Stockouts are costly because they result in lost sales, and surplus stock is costly because the salvage value is lower than the cost of procuring inventory. Given the relative costs of stockouts and surpluses, and given the demand distribution, the optimal inventory level can be determined. In essence, ours is a model with two newsvendors – the case with only two newsvendors is very tractable here, and yet sufficient to capture the essential elements of the transshipment problem. The two newsvendors must choose their inventory levels simultaneously, and before they know the true level of demand. Ex post, if one newsvendor ends up with surplus stock and the second one is stocked out, surplus stock can be transshipped from the former to the latter. Transshipments thus alleviate the problem of localized demand shocks and, ex-ante, this possibility affects optimal inventory choices at each location. We look at Nash equilibria of the resulting game.
Other newsvendor extensions in situations of strategic interaction have previously been considered as well, but in different contexts. Parlar (1988), Karjalainen (1992), and Lippman and McCardle (1997) consider models with more than one newsvendor, but with an important difference from our model. In their models, if a newsvendor is stocked out, it is the customer who moves to a rival newsvendor so that demand is transferable between locations. In our story, which is more plausible in the presence of exclusive territory arrangements, customers are immobile but goods are mobile between locations at a cost. It is further complicated by the issue of the price charged for a transshipment between locations. It turns out that this difference has significant implications for the outcome.

Van Mieghem (1999) studies the value of subcontracting and finds contract structures that achieve the coordinated profit. However, the transfer is one-directional – the manufacturer purchases from the subcontractor, but not vice versa. Brown (1999) addresses a problem, which has similar structure to ours, of two manufacturers who reserve capacity from a common source.

The earliest work on the transshipment problem seems to be that of Krishnan and Rao (1965) which assumes single period order-up-to policies and equal costs at each location. Robinson (1990) extends Krishnan and Rao to the multiperiod case. Tagaras (1989) also considers the multiperiod case and defines a set of assumptions that lead to ‘complete pooling.’ Complete pooling means that if one location has excess stock while another location is short, the number of units transshipped will be the minimum of the excess and the shortage. Additionally, no transshipments will occur if both locations are short or if both have excess stock. Herer and Rashit (1999) address the case of fixed and joint replenishment costs, as well as providing a number of analytical properties of the solution. Some papers, including Karmarkar and Patel (1977) and Karmarkar (1987), assume that transshipments occur before demand is realized. Most, however, assume that transshipments occur after demand is realized but before it is satisfied. Much of the research in this area has focused
on repairable items. See for instance Lee (1987) and Axsäter (1990). Finally, many papers, including Jönsson and Silver (1987), Chang and Lin (1991), Diks and de Kok (1996), and Evers (1996), assume that transshipments occur routinely instead of when a stockout is imminent. All of the above research assumes that transshipments are coordinated centrally.

The paper is organized as follows. The next section sets up the basic framework that allows transshipment between two locations. The classic newsvendor problem can be seen as a special case within this framework in Section 3. We then examine how the possibility of transshipments between firms affects inventory orders in different environments. In the first environment, inventory levels at each location are centrally determined by a ‘parent’ firm to maximize the sum of profits at two locations. In the second environment, each location chooses its inventory for exogenously given transshipment prices. Lastly we look at how alternative assumptions about negotiation power may result in alternative configurations of transshipment prices and compare equilibrium outcomes at these. To illustrate the model, we give numerical examples for both the normal and uniform distributions, and compare these to the corresponding results of the case of no interaction (the classical newsvendor model) and the case of central coordination.

2 The model

The basic framework relies on Krishnan and Rao (1965) and Robinson (1990). Consider two retail firms at distinct locations, indexed \( i, j = 1, 2; i \neq j \). Firm \( i \) purchases inventory \( Q_i \) from a central supplier at a fixed unit cost \( c_i > 0 \). This is the cost inclusive of delivery to location \( i \), so that \( c_1 \neq c_2 \) may reflect purely differences in delivery costs. When placing the inventory orders, the firms do not know the level of demand \( D_i \) that will be realized at the two locations, but the joint distribution over demand realizations \( \{ D_1 \times D_2 \} \) is common knowledge. For analytical convenience the probability distribution over demand is assumed
to be twice differentiable and strictly increasing on its support.

Firm $i$ obtains revenue $r_i > c_i$ for every unit sold, and if it cannot meet the entire demand, it is penalized $p_i \geq 0$ for every unit of unmet demand. The latter can be viewed as the reputational cost of turning away customers. We define $v_i = r_i + p_i$ as the marginal value of additional retail sales at location $i$. Unsold inventory has salvage value $s_i < c_i$.

The sequence of events is as follows. First, firms place their orders simultaneously. Then, demand realizations are observed. Order quantities and demand realizations are observable. If inventory at one location exceeds realized demand, it may transship surplus inventory to the other location. The other location would be interested in receiving transshipments only if its demand realization exceeds its inventory order.

Actual transshipments depend also on the physical cost of transshipment and the price charged for transshipment. We assume that transshipment prices are independent of the demand realizations and inventory levels at the two locations. For instance, they do not vary with the magnitude of surplus stock at one location or the size of the shortage at the other. Let $c_{ij}$ be the price charged by location $i$ for each unit transshipped to location $j$, and $\tau_{ij}$ be the associated unit cost of relocating the good from $i$ to $j$. We assume that cost $\tau_{ij}$ is incurred by location $i$, so that it obtains a surplus of $c_{ij} - \tau_{ij}$ for each unit transshipped to $j$. It would find it profitable to transship whenever this surplus exceeds the salvage value $s_i$ of excess stock. Location $j$ would find it profitable to accept transshipments whenever it is stocked out and the transshipment price is less than the marginal value of additional sales at that location, that is, $c_{ij} < v_j$. Thus, transshipments from location $i$ to location $j$ are mutually profitable whenever there is excess demand at location $j$, excess supply at location $i$ and $c_{ij} \in [s_i + \tau_{ij}, v_j]$. To avoid trivialities, we assume that $s_i + \tau_{ij} < v_j$, for $i = 1, 2$. In addition, we assume that $c_i < c_j + \tau_{ji}$, $s_i < s_j + \tau_{ji}$, and $v_i < v_j + \tau_{ji}$. These ensure that it is not beneficial to always buy through another location, and that transshipment occurs only when we have excess stock at one location and, simultaneously, excess demand at
Define $x^+ = \max\{x, 0\}$. Then for $c_{ij} \in [s_i + \tau_{ij}, v_j]$, the optimal level of transshipment from location $i$ to $j$, $T_{ij}$, is given by the lower of two magnitudes, the excess demand $(D_j - Q_j)^+$ at location $j$, and the excess supply $(Q_i - D_i)^+$ at location $i$. That is,

$$T_{ij} = \min [(D_j - Q_j)^+, (Q_i - D_i)^+] .$$

Note that any correlation structure is captured in $T_{ij}$. Given transshipments, retail sales at location $i$ are,

$$R_i = \min (D_i, Q_i) + T_{ji},$$

while unsold stock at location $i$ is

$$U_i = (Q_i - D_i - T_{ij})^+,\quad$$

and unmet demand is

$$Z_i = (D_i - Q_i - T_{ji})^+.\quad$$

These expressions depend on $D_i, D_j, Q_i$ and $Q_j$. Expected profit at location $i$ is given by

$$\pi_i(Q_i, Q_j) = E \{ r_i R_i + (c_{ij} - \tau_{ij}) T_{ij} - c_{ji} T_{ji} + s_i U_i - p_i Z_i \} - c_i Q_i , \quad (1)$$

where the expectation is over demand realizations.

### 3 The newsvendor problem

We first look at the classic newsvendor problem. This involves no transshipments at all but identifies a useful benchmark.
We use the above structure, setting \( T_{ij} = T_{ji} = 0 \), and dropping subscripts. Expected profit for the newsvendor is

\[
\pi^n(Q) = E\{rR + sU - pZ\} - cQ
\]

(2)

\[
= E\{r \min(D, Q) + s(Q - D)^+ - p(D - Q)^+\} - cQ,
\]

where the superscript \( n \) denotes newsvendor. Given the inventory order \( Q \) and distribution over demand, let \( \Pr(D > Q) \) denote the probability of stock outs. The marginal profitability of increasing the inventory level is

\[
\frac{\partial \pi^n}{\partial Q} = r \Pr(D > Q) + s \Pr(D < Q) + p \Pr(D > Q) - c.
\]

(3)

The optimal inventory choice trades off the expected marginal benefit with its marginal cost \( c \). If the distribution over demand is continuous and strictly increasing, there exists a unique order quantity \( Q^n \) that maximizes the newsvendor’s profit. Setting expression (3) to zero and rearranging, we obtain the familiar solution for the newsvendor problem:

\[
\Pr(D < Q) = \frac{v - c}{v - s}.
\]

(4)

4 Transshipments with central coordination

We now consider the case in which inventory decisions for each location are centrally coordinated to maximize aggregate profits (what we called intra-firm transshipments). This is the two-location case analyzed by Robinson (1990). By following Rudi and Zheng (1996), our solution is somewhat more compact and facilitates comparisons to the other cases.

A parent firm chooses inventory levels \((Q_1, Q_2)\) to maximize the sum of profits across locations. Given demand realizations, transshipments are carried out whenever stock out in one location is accompanied by excess stock at the other. The expected value of total profits for the two locations, denoted as \( \pi^t \), is

\[
\pi^t(Q_1, Q_2) = E \left\{ r_1R_1 + r_2R_2 - \tau_{21}T_{21} - \tau_{12}T_{12} + s_1U_1 + s_2U_2 - p_1Z_1 - p_2Z_2 \right\}
\]
$$-c_1 Q_1 - c_2 Q_2.$$  \hspace{1cm} (5)

Calculating the derivatives as before, we get that the expected value of a marginal unit of inventory at location \(i\) is

$$\frac{\partial \pi_t}{\partial Q_i} = (r_i + p_i) \Pr (D_i > Q_i + (Q_j - D_j)^+)$$

$$+ (r_j + p_j) \Pr (D_j > Q_j + Q_i - D_i, Q_i > D_i)$$

$$+ \tau_{ji} \Pr (Q_i < D_i < Q_i + Q_j - D_j) - \tau_{ij} \Pr (Q_i + Q_j - D_j < D_i < Q_i)$$

$$+ s_i \Pr (Q_i > D_i + (D_j - Q_j)^+)$$

$$+ s_j \Pr (Q_j > D_j + D_i - Q_i, D_i > Q_i) - c_i.$$  

Collecting terms, we have

$$\frac{\partial \pi_t}{\partial Q_i} = v_i (1 - \Pr (D_i < Q_i) - \Pr (Q_i < D_i < Q_i + Q_j - D_j))$$

$$+ (\tau_{ji} + s_j) \Pr (Q_i < D_i < Q_i + Q_j - D_j)$$

$$+ (v_j - \tau_{ij}) \Pr (Q_j + Q_i - D_j < D_i < Q_i)$$

$$+ s_i (\Pr (D_i < Q_i) - \Pr (Q_i + Q_j - D_j < D_i < Q_i)) - c_i.$$  \hspace{1cm} (6)

To understand (6), note that, first, additional inventory at \(i\) will result in incremental sales at location \(i\), yielding \(v_i\) per unit, except when (a) there is excess inventory at location \(i\) (formally, \(D_i < Q_i\)), or (b) effective excess supply at location \(j\) could have enabled transshipments from \(j\) to \(i\) (which happens whenever \(Q_i < D_i < Q_i + Q_j - D_j\)). In case (b), the marginal unit of \(Q_i\) is still valuable in that it saves transshipment costs \(\tau_{ji}\) and releases an additional unit at \(j\), with salvage value \(s_j\): the opportunity value of the marginal unit of \(Q_i\) is \(\tau_{ji} + s_j\). Second, if there is no stock out at location \(i\) but demand exceeds effective supply at location \(j\) (formally, we have \(Q_j + Q_i - D_j < D_i < Q_i\)), the marginal unit can be transshipped to \(j\), yielding \(v_j - \tau_{ij}\). Lastly, if demand at location \(j\) does not exceed effective supply, and there is no stock out at \(i\), the marginal unit is worth only its
salvage value $s_i$. We subtract the marginal procurement cost $c_i$ from the above to get the net marginal benefit.

To simplify the notation, for given $(Q_1, Q_2)$, we define events and associated probability functions as specified in Table 1. $E^1_i$ is the event that location $i$ has excess inventory. $E^2_i$ is the event that location $i$ has excess inventory, but the excess is not sufficient to fully cover the shortage at location $j$. Lastly, $E^3_i$ is the event that location $i$ is stocked out, but the shortage is less than the excess inventory at location $j$. These are illustrated graphically in Figure 1. If the joint distribution over demand is continuously differentiable, then these probability functions will be continuous as well. That, combined with our parameter restrictions, ensures that we have an interior solution. The profit-maximizing inventory choice can be found by setting the expected net marginal benefit to zero. Robinson (1990)

<table>
<thead>
<tr>
<th>Event</th>
<th>Description</th>
<th>Probability</th>
<th>Transshipments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^1_i$</td>
<td>$D_i &lt; Q_i$</td>
<td>$\alpha_i(Q_i)$</td>
<td>$T_{ji} = 0$</td>
</tr>
<tr>
<td>$E^2_i$</td>
<td>$Q_i + Q_j - D_j &lt; D_i &lt; Q_i$</td>
<td>$\beta_i(Q_i, Q_j)$</td>
<td>$T_{ji} = 0, T_{ij} = Q_i - D_i$</td>
</tr>
<tr>
<td>$E^3_i$</td>
<td>$Q_i &lt; D_i &lt; Q_i + Q_j - D_j$</td>
<td>$\gamma_i(Q_i, Q_j)$</td>
<td>$T_{ji} = D_i - Q_i, T_{ij} = 0$</td>
</tr>
</tbody>
</table>

Figure 1: Graphical illustration of events $E^2$ and $E^3$
shows that expected profit function is concave in \((Q_1, Q_2)\), so the first order conditions are sufficient for optimality.

Rearranging (6), the conditions characterizing the optimal inventory order are

\[
\alpha_i(Q_i) - \beta_i(Q_i, Q_j) \left( \frac{v_j - s_i - \tau_{ij}}{v_i - s_i} \right) + \gamma_i(Q_i, Q_j) \left( \frac{v_i - s_j - \tau_{ji}}{v_i - s_i} \right) = \frac{v_i - c_i}{v_i - s_i},
\]

(7)

for \(i = 1, 2\). Let \((Q_1', Q_2')\) denote the solution to this problem. Comparing (4) and (7), the latter is simply an adjustment of the simple newsvendor solution. The second term on the left hand side adjusts \(Q_i\) up due to the possibility of transshipment from \(i\) to \(j\); and the third term adjusts \(Q_i\) down due to the possibility of transshipment from \(j\) to \(i\). We denote the aggregate profit at this solution as \(\pi'(Q_1', Q_2')\).

5 Local inventory choice given transshipment prices

Suppose, next, that each firm chooses its inventory level in order to maximize its own profits. This is the case of inter-firm transshipments, where we say that inventory decisions are made locally. Expected profit at each location now depends not only on order quantities, but also on transshipment prices \((c_{12}, c_{21})\). In this section, we solve for the optimal inventory levels assuming exogenously-given transshipment prices. The next section examines how these choices vary with transshipment prices.

Location \(i\) chooses \(Q_i\) taking as given the other location’s inventory choice \(Q_j\) and transshipment prices \((c_{12}, c_{21})\), such that \(c_{ij} \in [s_i + \tau_{ij}, v_j]\). Expected profit at each decentralized location \(i\) is denoted as \(\pi_i^d\). We have:

\[
\pi_i^d(Q_i, Q_j, c_{ij}, c_{ji}) = E\{r_i R_i - c_{ji} T_{ji} + (c_{ij} - \tau_{ij}) T_{ij} + s_i U_i - p_i Z_i\} - c_i Q_i.
\]

(8)

The expected marginal profit of an additional unit at location \(i\) is

\[
\frac{\partial \pi_i^d}{\partial Q_i} = v_i \left( 1 - \Pr(D_i < Q_i) - \Pr(Q_i < D_i < Q_i + Q_j - D_j) \right)
\]
\[ +c_{ji} \Pr (Q_i < D_i < Q_i + Q_j - D_j) \]
\[ + (c_{ij} - \tau_{ij}) \Pr (Q_i + Q_j - D_j < D_i < Q_i) \]
\[ + s_i (\Pr (D_i < Q_i) - \Pr (Q_i + Q_j - D_j < D_i < Q_i)) - c_i. \] (9)

The intuition here follows closely the previous case, with one crucial difference. As before, the marginal unit of \( Q_i \) yields \( v_i \) except in events \( E_1^1 \) or \( E_3^1 \). However, now, in the latter event the marginal value of \( Q_i \) is the actual transshipment price \( c_{ji} \) charged by \( j \). Contrast this with the joint profit maximization in the previous section, where the marginal value was the opportunity cost, namely \( \tau_{ji} + s_j \). Similarly, in event \( E_2^2 \), when location \( i \) transshipped to \( j \), it now obtains \( c_{ij} - \tau_{ij} \). The profit-maximizing inventory choice can be found by setting the expected net marginal benefit to zero. This defines a reaction function \( Q_i(Q_j) \) for location \( i \)'s optimal inventory, given \( Q_j \) and \((c_{12}, c_{21})\). The Nash equilibrium is given by \((Q_1^d, Q_2^d)\) such that equation (10) holds for \( i = 1, 2 \).

\[ \alpha_i(Q_i) - \beta_i(Q_i, Q_j) \left( \frac{c_{ij} - s_i - \tau_{ij}}{v_i - s_i} \right) + \gamma_i(Q_i, Q_j) \left( \frac{v_i - c_{ji}}{v_i - s_i} \right) = \frac{v_i - c_i}{v_i - s_i} \] (10)

We have

**Proposition 1** Let \( c_{ij} \in [s_i + \tau_{ij}, v_j] \) for \( i = 1, 2 \). There exists a unique Nash equilibrium, \((Q_1^d, Q_2^d)\).

**Proof** To establish the existence of a unique Nash equilibrium it is sufficient to show that the reaction functions are monotonic, and the absolute value of the slope is less than 1 (see Fudenberg and Tirole (1991)). Note that the functions \( \alpha_i(Q_i), \beta_i(Q_i, Q_j) \), and \( \gamma_i(Q_i, Q_j) \) are continuous and differentiable in \( Q_i \) and \( Q_j \). Let \( f_x \) denote the probability density function associated with a random variable \( x \). We define the following marginal probabilities.

\[ b_{ij}^1 = \Pr (D_i < Q_i) f_{D_i + D_j | D_i < Q_i} (Q_i + Q_j) \]
\[ b_{ij}^2 = \Pr (D_i + D_j > Q_i + Q_j) f_{D_i | D_i + D_j > Q_i + Q_j} (Q_i) \]

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\[ g_{ij}^{1} = \Pr (D_i > Q_i) f_{D_j + D_i > Q_i} (Q_i + Q_j) \]
\[ g_{ij}^{2} = \Pr (D_j + D_i < Q_i + Q_j) f_{D_i | D_j + D_i < Q_i + Q_j} (Q_i) \]

and

\[ a_i = f_{D_i} (Q_i) . \]

Implicit differentiation of equation (10) and rearrangement yields a characterization of the reaction function:

\[
\frac{\partial Q_i}{\partial Q_j} = - \frac{(c_{ij} - s_i - \tau_{ij}) b_{ij}^1 + (c_{ji} - s_j - \tau_{ji}) g_{ij}^2}{(v_i - s_i) a_i + (c_{ij} - s_i - \tau_{ij}) (b_{ij}^1 - b_{ij}^2) + (c_{ji} - s_j - \tau_{ji}) (g_{ij}^1 - g_{ij}^2)}
\]

(11)

Given our parameter restrictions, it is easy to check that the slope of the reaction function is non-positive, and less than one in absolute value.

\[ \square \]

Let \((\pi^d_1(Q^d_1, Q^d_2), \pi^d_2(Q^d_1, Q^d_2))\) denote the pair of maximized values of profits at this equilibrium.

From (10), we find that the asymptotic behavior of \(Q_i(Q_j)\) as \(Q_j \to \infty\) is given by solving

\[ \alpha_i(Q_i) = \left( \frac{c_{ji} - c_i}{c_{ji} - s_i} \right)^{+} \]

with respect to \(Q_i\).

### 6 Transshipment prices

We now turn to transshipment prices. We consider predetermined transshipment prices generally, rather than assuming any specific mechanism for determining these. One approach is to assume that transshipment prices are set by the transshipping firm, i.e., the firm with surplus stock. Alternatively, one could assume that prices are set by the firm that receives transshipment, i.e., the firm that is stocked out. These possibilities, discussed as Cases 1 and 2 below, are extreme cases of a more general scenario, in which transshipment prices are set by negotiation between the two firms. In this more general scenario
$c_{ij} \in [s_i + \tau_{ij}, v_j]$, with the relative bargaining power of the two firms determining where in the interval it lies. In other words, our model applies to any negotiated transshipment price in the relevant interval, and hence is not dependent on the process of negotiation.

We could also allow for a priori asymmetry between the firms. Suppose one firm, say firm 1, has all the bargaining power in the price negotiation, regardless of whether it has surplus stock or is stocked out. We then expect transshipment prices to reflect this asymmetry. The outcome is discussed as Case 3.

An alternative approach would be to assume that transshipment prices are set by some other procedure. Suppose prices are set by a external agency, such as a common supplier, and locations then choose only inventory levels in response. This poses the question: can we find transshipments prices that will maximize joint profits, or equivalently, can the coordinated solution be induced through decentralized prices? Indeed, one could ask this question relative to any other measure of performance (aggregate sales, revenue, etc.) that the upstream supplier wants to optimize. For a discussion of such ‘vertical contractual relations’, see Katz (1989). Equally, we could view the external agency as an industry association that specifies transshipment prices to maximize aggregate profits through collusion. If appropriate transshipment prices exist, they could support a collusive outcome in a repeated game context.

We begin with a general result and then consider each of the above possibilities in turn.

**Proposition 2** Firm i’s optimal choice of inventory level $Q_i^d$ is increasing in $c_{ij} \in [s_i + \tau_{ij}, v_j]$ and decreasing in $c_{ji} \in [s_j + \tau_{ji}, v_i]$ for $i = 1, 2$.

**Proof** This follows directly from the conditions for Nash equilibrium given in (10).
Case 1

Suppose transshipment prices are set by the transshipping firm, so that \( c_{ij} \) is set by firm \( i \). Note that firm \( i \)'s profit is non-decreasing in \( c_{ij} \) and is strictly increasing in those states where \( T_{ij} \) is positive. Hence, the expected profit is increasing in \( c_{ij} \), making it optimal for firm \( i \) to choose \( c_{ij} = v_j \). Substituting this in equation (10), we obtain for \( i = 1, 2 \)

\[
\alpha_i(Q_i) - \beta_i(Q_i, Q_j) \left( \frac{v_j - s_i - \tau_{ij}}{v_i - s_i} \right) = \frac{v_i - c_i}{v_i - s_i}.
\]

The solution to the set of equations (13) defines the unique Nash equilibrium \((Q_1^d, Q_2^d)\) for this case.¹ This solution yields the following structural results.

**Proposition 3** For \( c_{ij} = v_j \) for \( i = 1, 2 \) we have that \( Q_i^d > Q_i^n \) and \( Q_i^d > Q_i^t \).

**Proof** By comparing equations (4), (7) with equations (13).

From the latter inequality and the fact that the expected profit function of the total system is concave, it follows that joint-profits are not maximized at this equilibrium.

Case 2

Suppose, to the contrary, that transshipment prices are set by the firm that receives transshipments, or that \( c_{ij} \) is set by firm \( j \) and \( c_{ji} \) is set by firm \( i \). Using arguments similar to those for case 1, we argue that firm \( j \) will choose \( c_{ij} = s_i + \tau_{ij} \). Substituting this in equation (10), we obtain for \( i = 1, 2 \)

\[
\alpha_i(Q_i) + \gamma_i(Q_i, Q_j) \left( \frac{v_i - s_j - \tau_{ji}}{v_i - s_i} \right) = \frac{v_i - c_i}{v_i - s_i}.
\]

The solution to the set of equations (14) defines the Nash equilibrium for this case. Similarly to case 1, we find the following structural results.

¹Indeed, given the described sequence of events, this equilibrium is subgame perfect.
Proposition 4 For $c_{ij} = s_i + \tau_{ij}$ for $i = 1, 2$ we have that $Q_i^d < Q_i^a$ and $Q_i^d < Q_i^t$.

Proof By comparing equations (4), (7) with equations (14).

Once again, joint profits are not maximized at this equilibrium.

Case 3

Suppose all the bargaining power in price setting lies with firm 1. Using arguments similar to cases 1 and 2, we can see that firm 1 will choose $c_{12} = v_2$ and $c_{21} = s_2 + \tau_{21}$. Substituting this in equation (10) gives

$$\alpha_1(Q_1) - \beta_1(Q_1, Q_2) \left( \frac{v_2 - s_1 - \tau_{12}}{v_1 - s_1} \right) + \gamma_1(Q_1, Q_2) \left( \frac{v_1 - s_2 - \tau_{21}}{v_1 - s_1} \right) = \frac{v_1 - c_1}{v_1 - s_1} \quad (15)$$

and

$$\alpha_2(Q_2) = \frac{v_2 - c_2}{v_2 - s_2}. \quad (16)$$

At this equilibrium, the weaker firm, namely firm 2, gains nothing from transshipments. Not surprisingly, its optimal inventory choice is independent of $Q_1$ and just replicates the newsvendor solution (see equation (4)).

Coordinating transshipment prices

We move on to address the following question: Does there exist a pair of transshipment prices that optimize the total profit? If so, this can in many practical situations be implemented by a central authority as an incentive design to make each location behave in order to optimize aggregate profit.

Proposition 5 There exists a unique set of transshipment prices $c_{ij}$ for $i = 1, 2$ that yields the joint optimal solution, given by

$$\hat{c}_{ij} = \frac{v_j \beta_i \beta_j + (s_j + \tau_{ji} - v_i) \beta_j \gamma_i - (s_i + \tau_{ij}) \gamma_i \gamma_j}{\beta_i \beta_j - \gamma_i \gamma_j}, \quad (17)$$
where each of the probabilities in the above expression are evaluated at \( Q_i = Q^*_i \).

**Proof** It is sufficient to look for transshipment prices which induce the firms to choose optimal order quantities, as defined in the case with central coordination. Existence follows from propositions 3 and 4 and continuity. Uniqueness follows from proposition 2. The expressions given by (17) are then found by equating the left hand sides of equations (7) and (10).

7 Two examples

We compute explicit solutions for the case where demand at the two locations is distributed independently, or that the correlation coefficient of demand across locations is zero. We first consider the case where demand at each location is distributed normally, and then look at the case where it is distributed uniformly over a compact interval. For the normal case, we also examine non-zero correlations.

In each case the other parameter values are also assumed to be symmetric. Each firm can procure inventory at cost \( c_i = 20 \), sell it at unit price \( r_i = 40 \). Salvage value \( s_i = 10 \) and penalty for lost sales \( p_i = 0 \). Transshipment costs, \( \tau_{ij} = \tau_{ji} = 2 \).

Note that even though we use independent demand distributions and identical parameters in these examples, the optimality conditions and conditions for Nash equilibria allow for generalization of both.
7.1 Normal distribution

Suppose demand realizations at the two locations are independent and distributed normally, with mean 100 and standard deviation 50:

\[ D_i \sim N(100, 50). \]

The optimal order quantities and average profits are reported in Table 2 below.

<table>
<thead>
<tr>
<th>Environment</th>
<th>Optimal inventory</th>
<th>Avg profits per firm, ( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Transshipment</td>
<td>122.5</td>
<td>1530</td>
</tr>
<tr>
<td>Central coordination Robinson</td>
<td>117.1</td>
<td>1676</td>
</tr>
<tr>
<td>Local decisions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c_{ij} = c_{ji} = 12 )</td>
<td>107.0</td>
<td>1660</td>
</tr>
<tr>
<td>( c_{ij} = c_{ji} = 18 )</td>
<td>112.3</td>
<td>1672</td>
</tr>
<tr>
<td>( c_{ij} = c_{ji} = 20 )</td>
<td>114.1</td>
<td>1674</td>
</tr>
<tr>
<td>( c_{ij} = c_{ji} = 22 )</td>
<td>115.9</td>
<td>1675</td>
</tr>
<tr>
<td>( \hat{c_{ij}} = \hat{c_{ji}} = 23.3 )</td>
<td>117.1</td>
<td>1676</td>
</tr>
<tr>
<td>( c_{ij} = c_{ji} = 26 )</td>
<td>119.4</td>
<td>1675</td>
</tr>
<tr>
<td>( c_{ij} = c_{ji} = 35 )</td>
<td>127.0</td>
<td>1661</td>
</tr>
<tr>
<td>( c_{ij} = c_{ji} = 40 )</td>
<td>130.9</td>
<td>1648</td>
</tr>
</tbody>
</table>

As these numbers suggest, transshipments improve profitability considerably over the case where transshipments are not permitted. The highest increment in profitability is obtained if inventory orders are centrally coordinated. The equilibrium order quantity is increasing in the transshipment price: if transshipment prices are too low, each firm is tempted to order too low a quantity relative to the centrally coordinated optimum. There exists an coordinating transshipment price that decentralizes the joint-profit maximizing outcome: if the parent firm could impose \( c_{ij} = 23.3 \), it would result in inventory orders that maximize aggregate profits.

\(^2\)We truncate the distribution at 0 and redistribute this proportionally to the positive part of the distribution to rule out negative realizations of the demand variable. Note that this yields a slightly higher mean and slightly lower variance.
Figures 2, 3 and 4 show response functions and equilibria graphically for the cases $c_{ij} = 12$, $c_{ij} = 26$ and $c_{ij} = 40$ respectively.

For the case of the lowest feasible transshipment price, $c_{ij} = 12$ (corresponding to case 2), we see that for small $Q_i$, $Q_j(Q_i)$ is rather flat. This is because location $i$ will need its supply and therefore location $j$ can not expect to receive any transshipped units. Also there is no benefit for location $j$ in having excess stock to transship in case of a stockout at location $i$ due to the fact that the transshipment price equals the sum of the salvage value $s_j$ and the transshipment cost $\tau_{ji}$. (Note that $Q_j(Q_i = 0) = Q_j^n$, the newsvendor quantity.) As $Q_i$ increases, more cheap units will become available, and it is profitable to reduce $Q_j$. As $Q_i$ gets very large, location $j$ can with high probability cover its need by cheap transshipped units, and should not order any initial units.

The case of $c_{ij} = 26$ is more realistic, in that a location will gain both from transshipping excess stock and profit from units received through transshipment. This is reflected in the response function in that the slope of $Q_j(Q_i)$ does not change that dramatically. Note that as $c_{ij}$ and $c_{ji}$ increase, location $i$ wants more stock both in the hope that location $j$ will buy more and to prevent it needing to buy units from location $j$. 
Figure 3: $c_{ij} = 26$ for $i = 1, 2$

Figure 4: $c_{ij} = 40$ for $i = 1, 2$
For \( c_{ij} = 40 \), \( Q_j(Q_i) \) has slope close to \(-1\) for small \( Q_i \). This is because a unit increase in \( Q_i \) (almost certainly) leads to a reduced demand for transshipment from location \( j \) at full margin. For large \( Q_i \), it converges to the newsvendor solution since receiving transshipment does not improve profit.

Thus far, we have considered only the case of independent demand distributions. As one might expect, however, non-zero correlation changes the optimal solution. Figure 5 illustrates the effect of the correlation coefficient, \( \rho \), has on the optimal transshipment price \( \hat{c}_{ij} \) and the resulting optimal order quantities \( Q_t^i \). Note that as \( \rho \) approaches 1, the resulting order quantity approaches the newsvendor solution \( Q^n \).

Table 3 illustrates Case 3 of the previous section, assuming all the bargaining power lies with firm 1. In this situation, we have \( c_{12} = 40, c_{21} = 12 \). As expected, firm 2 does not gain through transshipments and so chooses the inventory level that corresponds to the

<table>
<thead>
<tr>
<th>Environment</th>
<th>Optimal Inventory</th>
<th>Inventory profits</th>
</tr>
</thead>
<tbody>
<tr>
<td>Local decisions</td>
<td>( Q^d )</td>
<td>( \pi^d )</td>
</tr>
<tr>
<td>Asymmetric transshipment</td>
<td>Firm 1</td>
<td>112.0</td>
</tr>
<tr>
<td>Asymmetric transshipment</td>
<td>Firm 2</td>
<td>122.5</td>
</tr>
<tr>
<td>prices ( c_{12} = 40, c_{21} = 12 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
newsvendor problem. Firm 1 chooses a lower inventory and obtains higher profits.

7.2 Uniform Distribution

If demand is distributed uniformly over a compact interval, it is possible to provide explicit solutions. If

\[ D_i \sim U[0, 1] \]

we can specify the probability of events \( E^1_i, E^2_i \) and \( E^3_i \) as below.

\[
\alpha_i(Q_i) = \begin{cases} 
Q_i & \text{if } Q_i \leq 1 \\
1 & \text{otherwise}
\end{cases}
\]

\[
\beta_i(Q_i, Q_j) = \begin{cases} 
0.5Q_i^2 + Q_i(1 - Q_i - Q_j) & \text{if } Q_j \leq 1 - Q_i \\
0.5(1 - Q_j)^2 & \text{if } 1 - Q_i < Q_j \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
\gamma_i(Q_i, Q_j) = \begin{cases} 
0.5Q_j^2 & \text{if } Q_j \leq 1 - Q_i \\
0.5(1 - Q_i)(2Q_j + Q_i - 1) & \text{if } 1 - Q_i < Q_j \leq 1 \\
0.5(1 - Q_i)(2Q_j + Q_i - 1) - 0.5(Q_j - 1)^2 & \text{if } 1 < Q_j \leq 2 - Q_i \\
0 & \text{otherwise}
\end{cases}
\]

This allows us to solve explicitly for the optimal inventory choice in alternative environments. To facilitate comparison with the previous case, we choose a uniform distribution with the same mean, namely 100. We have \( D_i \sim U[0, 200] \) which has a standard deviation of 57.7. Of course, the probability of the various events will be a straightforward transformation of the expressions above. The newsvendor solution in our example is given by \( Q_i^n = \frac{2}{3} \times 200 = 133.3 \). The optimal inventory for the case with central coordination is given by the feasible root of \( 28Q^2 - 57Q + 24 = 0 \) adjusted for higher mean; we get \( Q^c_i = Q^c_j = 119.0 \). With local decision-making the quantities depend on transshipment prices as illustrated by Tables 4 and 5.

Optimal inventory choices and expected profits are as follows:

As in the previous example, transshipments improve profitability, and order quantities are increasing in the transshipment price. Here \( \hat{c}_{ij} = \hat{c}_{ji} = 21.5 \) induces the coordinated solution.
Table 4: Inventory choice when $D_i \sim U(0, 200)$

<table>
<thead>
<tr>
<th>Environment</th>
<th>Optimal inventory $Q^d$</th>
<th>Avg. profits per firm, $\pi^d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Transshipment newsvendor</td>
<td>133.4</td>
<td>1334</td>
</tr>
<tr>
<td>Central Coordination Robinson</td>
<td>119.0</td>
<td>1529</td>
</tr>
<tr>
<td>Local decisions with symmetric</td>
<td>107.0</td>
<td>1511</td>
</tr>
<tr>
<td>transshipment prices coordinating</td>
<td>114.4</td>
<td>1527</td>
</tr>
<tr>
<td>$c_{ij} = c_{ji} = 12$</td>
<td>119.6</td>
<td>1529</td>
</tr>
<tr>
<td>$c_{ij} = c_{ji} = 18$</td>
<td>119.0</td>
<td>1529</td>
</tr>
<tr>
<td>$c_{ij} = c_{ji} = 20$</td>
<td>118.4</td>
<td>1529</td>
</tr>
<tr>
<td>$c_{ij} = c_{ji} = 22$</td>
<td>124.6</td>
<td>1526</td>
</tr>
<tr>
<td>$c_{ij} = c_{ji} = 26$</td>
<td>129.6</td>
<td>1516</td>
</tr>
<tr>
<td>$c_{ij} = c_{ji} = 30$</td>
<td>135.8</td>
<td>1498</td>
</tr>
<tr>
<td>$c_{ij} = c_{ji} = 35$</td>
<td>141.4</td>
<td>1475</td>
</tr>
</tbody>
</table>

Table 5 simulates Case 3, assuming all the bargaining power lies with firm 1. Once again, firm 2 chooses the newsvendor inventory level, while firm 1 choose a lower inventory and makes higher profits.

Table 5: Asymmetric transshipment prices with uniform distribution

<table>
<thead>
<tr>
<th>Environment</th>
<th>Optimal Inventory $Q^d$</th>
<th>profits $\pi^d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asymmetric transshipment</td>
<td>Firm 1</td>
<td>105.8</td>
</tr>
<tr>
<td>prices $c_{12} = 40, c_{21} = 12$</td>
<td>Firm 2</td>
<td>133.4</td>
</tr>
</tbody>
</table>

8 Extensions

The analysis allows some natural extensions. If the locations are supplied by a common supplier, we could incorporate its objectives into the analysis. Let $c$ be the supplier’s unit cost. For given $c_i = c_j$, the supplier’s profit will be maximized by maximizing the sum $Q_i + Q_j$, which following our analysis can be done by setting $c_{ij} = v_j$. By including $c_i$ as decision variables, the supplier will trade off the unit margin $c_i - c$ with the volume $Q_i$, which is decreasing in $c_i$. 

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The extension of this analysis to more than two locations is less than straightforward. With three or more firms, the pattern of transshipment along possible arcs is sensitive to the configuration of transshipment prices. For instance, a location that is stocked out might obtain surplus stock first from the location with the lowest transshipment price, then from the next one, and so on. Further, the determination of transshipment prices may be more complicated. With choice of a suitable allocation rule and transshipment-price determination process, we expect a Nash equilibrium can be found, but discussion of that is beyond the scope of this paper.

Ours is a single-period model. In a multi-period context, strategic interaction of the sort studied here is complicated by the fact that strategies depend on the entire history of previous choices. As the ‘folk-theorem’ in game-theory suggests, there typically is a multiplicity of Nash equilibria. Ad hoc restrictions on the strategy set can narrow down the equilibrium set but this does not add much to the results described in this paper.

9 Conclusions

In this paper, we take a new approach to the well-studied transshipment problem that fits well with scenarios found in many industries. In particular, local decision makers optimize their own performance, rather than a single central decision maker optimizing overall performance. We show how this affects the optimal inventory orders at each location. We find that if each location aims to maximize its own profits, their inventory choices will not, in general, maximize joint profits. To the extent that inventory choice varies with transshipment prices, we find transshipment prices that induce the locations to choose inventory levels consistent with joint-profits maximization.
References


