Chapter 7

Choice under Uncertainty

1. Expected Utility Theory.
2. Risk Aversion.
3. Applications: demand for insurance, portfolio choice

7.1 Expected Utility Theory

Formally a lottery involves a probability distribution over a set of ‘prizes’. Let $X$ be the set of prizes, with typical elements $x, y$. We assume that the set is finite, so that there must be a best prize (call it $b$), and a worst prize (call it $w$). Lotteries will be denoted by the symbols $f, g, h$.

Consider lottery $f$ that yields one of two prizes: either $x_1$ with probability $p$ or $x_2$ with probability $(1 - p)$. We use the following notation to describe lottery $f$ as:

$$\langle p \circ x_1 \ominus (1 - p) \circ x_2 \rangle$$

We denote a lottery that give prize $x$ ‘for sure’ as $\delta_x$.

An individuals has preferences over lotteries. To model choice under uncertainty, we assume that these preferences have some structure: that
is, they satisfy some axioms.

**A1. Rationality:** Completeness and transitivity.

**A2. Independence:** Consider 3 lotteries \( f, g, h \) and suppose \( f \sim g \). Then

\[
\langle \alpha \circ f \bigoplus (1 - \alpha) \circ h \rangle \sim \langle \alpha \circ g \bigoplus (1 - \alpha) \circ h \rangle
\]

**A3. Continuity:** For all \( x \) there exists some number (call it \( \alpha_x \)) such that

\[
\delta_x \sim \langle \alpha_x \circ \delta_b \bigoplus (1 - \alpha_x) \circ \delta_w \rangle
\]

Let us call this \( \alpha_x = u(x) \). Notice that \( u(b) = 1, u(w) = 0 \).

**A4. Monotonicity:**

\[
\langle p \circ \delta_b \bigoplus (1 - p) \circ \delta_w \rangle \succ \langle q \circ \delta_b \bigoplus (1 - q) \circ \delta_w \rangle \quad \text{if and only if } p > q.
\]

**The Representation Theorem (informal)**

Given the four axioms, and given the manner in which we have constructed \( u(\cdot) \), one lottery \( f \) is a good as another \( g \) if, and only if, \( EU \) of lottery \( f \geq EU \) of lottery \( g \).

**Proof:** Suppose lottery \( f \) yields one of two prizes: either \( x_1 \) with probability \( f_1 \), or \( x_2 \) with probability \( f_2 \). Likewise, suppose \( g \) yields \( y_1 \) with probability \( g_1 \), or \( y_2 \) with probability \( g_2 \).

**Step 1:** \( f \) is defined as \( f_1 \circ x_1 \bigoplus f_2 \circ x_2 \), which is the same as

\[
f_1 \circ \delta_{x_1} \bigoplus f_2 \circ \delta_{x_2}.
\]  

(7.1)

Then, by continuity, we can find some number \( u(x_1) \) such that

\[
\delta_{x_1} \sim u(x_1) \circ \delta_b \bigoplus (1 - u(x_1)) \circ \delta_w.
\]  

(7.2)

Likewise,

\[
\delta_{x_2} \sim u(x_2) \circ \delta_b \bigoplus (1 - u(x_2)) \circ \delta_w.
\]  

(7.3)
Relations (7.1), (7.2) and (7.3) when combined repeatedly with the independence axiom, imply

\[ f \sim \left[ f_1 u(x_1) + f_2 u(x_2) \right] \circ \delta_b \bigoplus \left[ f_1 (1 - u(x_1)) + f_2 (1 - u(x_2)) \right] \circ \delta_w. \]

(7.4)

Repeating Step 1 for lottery \( g \), we get,

\[ g \sim \left[ g_1 u(y_1) + g_2 u(y_2) \right] \circ \delta_b \bigoplus \left[ g_1 (1 - u(y_1)) + g_2 (1 - u(y_2)) \right] \circ \delta_w. \]

(7.5)

**Step 2.** Notice that (7.4) amounts to

\[ f \sim \left[ EU(f) \circ \delta_b \bigoplus (1 - EU(f)) \circ \delta_w \right] \]

and (7.5) amounts to

\[ g \sim \left[ EU(g) \circ \delta_b \bigoplus (1 - EU(g)) \circ \delta_w \right]. \]

**Step 3.** The result follows from the monotonicity axiom.

Acceptable transformation of utility functions. If \( u(\cdot) \) is a valid utility function, so is \( au(\cdot) + c \), where \( a \) and \( c \) are constants and \( a > 0 \).

### 7.2 Risk Aversion

We now specialize the above analysis to the case where the prize is money. Then, for any lottery we can define its expected value. If an individual prefers [getting the expected value of the gamble] to the [gamble], the individual is said to be risk averse. If the preference holds the other way around, the individual is said to be risk-loving. If the individual is indifferent between a gamble and its expected value, she is said to be risk-neutral. Risk-averse preferences imply that the utility function \( u(w) \) is concave.
Measures of Risk Aversion

Consider a utility function \( u(w) \), where \( w \) denotes wealth. Assume that \( u \) is twice-differentiable. How can we measure the risk aversion associated with the utility function?

(a) Concavity of the utility function.

Problem: the measure is not invariant to scale changes.

(b) Arrow-Pratt measure of absolute risk-aversion.

\[
a(w) = -\frac{u''(w)}{u'(w)}.
\]

(c) Arrow-Pratt measure of relative risk-aversion.

\[
r(w) = -\frac{wu''(w)}{u'(w)}.
\]

Some commonly-used utility functions

1. Consider the utility function \( u(w) = aw + b \). This is linear in wealth, and captures risk-neutral preferences. Note that \( u'' = 0 \).

2. Consider \( u(w) = -e^{-\rho w} \). Note that

\[
u'(w) = \rho e^{-\rho w},
\]

and

\[
u''(w) = -\rho^2 e^{-\rho w},
\]

so that

\[
a(w) = -\frac{u''(w)}{u'(w)} = -\frac{-\rho^2 e^{-\rho w}}{\rho e^{-\rho w}} = \rho.
\]

This utility function displays constant absolute risk aversion, with \( \rho \) as the coefficient of absolute risk aversion. Hence the above utility functions is called the Constant Absolute Risk Aversion (CARA) utility function.
3. Consider, next,

\[ u(w) = \frac{w^{1-\lambda} - 1}{1 - \lambda}. \]

Here

\[ u'(w) = w^{-\lambda}, \]

and

\[ u''(w) = -\lambda w^{-\lambda-1}, \]

so that

\[ r(w) = -\frac{wu''(w)}{u'(w)} = -\frac{-\lambda w^{-\lambda}}{w^{-\lambda}} = \lambda. \]

This utility function displays constant relative risk aversion, with \( \lambda \) as the coefficient of relative risk aversion. The above utility function is called the Constant Relative Risk Aversion Utility (CRRA) utility function.

4. Consider, next the utility function

\[ u(w) = \ln w. \]

Here

\[ u'(w) = \frac{1}{w}, \]

and

\[ u''(w) = -\frac{1}{w^2}, \]

so that

\[ r(w) = -\frac{wu''(w)}{u'(w)} = 1. \]

This is the special case where the constant coefficient of relative risk aversion equals 1.

In fact, we can show that this is a special limiting case of the CRRA function above. Consider

\[ \lim_{\lambda \to 1} \frac{w^{1-\lambda} - 1}{1 - \lambda}. \]
We can use L'Hôpital’s rule to find an expression for this limiting case.

L'Hôpital’s rule: Consider the limit of a fractions \( f(x)/g(x) \) as \( x \to q \).
If \( f(q) = g(q) = 0 \), the fraction \( f(x)/g(x) \) is not defined at \( x = q \).
L'opital’s rule says that, if \( f(x) \) and \( g(x) \) are differentiable, the limit of the ratio is given by the ratio of the derivatives.

\[
\lim_{x \to q} \frac{f(x)}{g(x)} = \frac{f'(q)}{g'(q)}.
\]

Hence, in the expression

\[
\lim_{\lambda \to 1} \frac{w^{1-\lambda} - 1}{1 - \lambda},
\]
we differentiate the expressions in the numerator and the denominator with respect to \( \lambda \).

\[
\frac{d[w^{1-\lambda} - 1]}{d\lambda} = -w^{1-\lambda} \ln w.
\]

\[
\frac{d[1 - \lambda]}{d\lambda} = -1.
\]
Evaluating the ratio of these derivatives at \( \lambda = 1 \) yields \( \ln w \).

### 7.3 Applications of Expected Utility Theory

#### 7.3.1 Example: Demand for Insurance

Consider a risk averse individual with net worth \( w \). With probability \( p > 0 \) she faces a risk (a 'hurricane') that would reduce her wealth to \( w - L \). Suppose she can insure herself against this risk by paying premium \( \pi \) for every unit of insurance she buys. How much insurance will she buy if insurance markets are competitive?

Suppose she buys \( x \) units of cover: this means she must pay \( \pi x \) as premium but will be repaid \( x \) in the even of loss. Thus
• if no hurricane, her wealth is \( w - \pi x \);

• if hurricane causes loss, her wealth is \( w - L - \pi x + x \).

She must choose \( x \) to maximize her expected utility

\[
EU(x) = pu(w - \pi x - L + x) + (1 - p)u(w - \pi x),
\]

with first-order condition

\[
p(1 - \pi)u'(w - \pi x^* - L + x^*) - (1 - p)\pi u'(w - \pi x^*) = 0,
\]

or

\[
\frac{u'(w - \pi x^* - L + x^*)}{u'(w - \pi x^*)} = \frac{(1 - p)\pi}{p(1 - \pi)}
\]

The second order condition is guaranteed by the assumed risk aversion, which implies that \( u \) is concave.

If the insurance markets are competitive, so that \( \pi \), the price of insurance is driven down to its actuarially fair value \( p \), we must have \( \pi = p \). If so, the first-order condition amounts to

\[
u'(w - \pi x^* - L + x^*) = u'(w - \pi x^*).
\]

Suppose \( u \) is strictly concave. If so, the slope of the function \( u' \) decreases continuously. It follows that Given the strict concavity of \( u(\cdot) \), the last relation suggests

\[
w - \pi x^* - L + x^* = w - \pi x^*
\]

or that

\[
x^* = L.
\]

In words, if insurance was available at actuarially fair rates, a strictly risk-averse individual would like to buy complete insurance.
7.3.2 Example: Demand for risky financial assets

Suppose an investor must allocate her initial wealth $w_0$ across two financial assets. The first asset is a safe asset that returns $R_f$ units for every unit invested. The second asset is risky, and its returns is given by a random variable $\tilde{z}$ with distribution function $F(z)$.

The investor is strictly risk averse with utility function $u(w)$ where $w$ is the final wealth. Suppose she invests an amount $\alpha$ in the risky asset and the rest $w - \alpha$ in the safe asset. The uncertain payoff to this portfolio is

$$\alpha \tilde{z} + (w - \alpha)R_f.$$ 

What are the possible values of $\alpha$ that the investor can choose? If her initial investment in the risky asset is constrained by her initial wealth, we have $\alpha \leq w_0$. If she can borrow, say at rate $R_f$, to invest in the risky asset, this upper bound can be exceeded? Can $\alpha$ take negative values? If we allow short-selling, negative values of $\alpha$ are permissible.

However, to keep things simple let us constrain $\alpha \in [0, w_0]$. What is the optimal choice of $\alpha$ in this interval? The answer depends on the distribution function $F(z)$ and on the utility function. The individual chooses $\alpha$ to maximizes

$$\int u(R_f w + \alpha (z - R_f))dF(z).$$

The first-order condition for an interior maximum, $\alpha^*$, is

$$\int (z - R_f)u'(R_f w + \alpha^*(z - R_f))dF(z) = 0.$$ 

7.4 Violations of Expected Utility Theory: Allais Paradox

Suppose we are offered the choice between the following lottery tickets. Ticket A gives a 11% chance of winning £1m. Ticket B gives us a slightly lower probability, 10%, of winning but the prize is £5m. Which of these two opportunities would you prefer?
Ticket C gives us 1 million for sure (it isn’t a lottery ticket, but a cheque), and lottery D gives us a an 89% of winning £1m, 1% chance of getting nothing, and 10% chance of getting £5m. Which would you prefer?

<table>
<thead>
<tr>
<th></th>
<th>$E_1$ (10%)</th>
<th>$E_2$ (1%)</th>
<th>$E_3$ (89%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>£1m</td>
<td>£1m</td>
<td>£0m</td>
</tr>
<tr>
<td>B</td>
<td>£5m</td>
<td>£0m</td>
<td>£0m</td>
</tr>
<tr>
<td>C</td>
<td>£1m</td>
<td>£1m</td>
<td>£1m</td>
</tr>
<tr>
<td>D</td>
<td>£5m</td>
<td>£0m</td>
<td>£1m</td>
</tr>
</tbody>
</table>

Experimental evidence points to a marked preference for B over A, and of C over D in two separate pair-wise comparisons. This pattern of preference seems systematic; it can be replicated experimentally.

This evidence is said to constitute a violation of the independence axiom in the following manner. Compare lotteries A and B — they yield identical prizes in event $E_3$. If we believe in the Independence axiom, preferences between A and B should depend on only those events in which the prizes under A and B differ, namely $E_1$ and $E_2$.

Likewise, preferences between C and D cannot depend on what happens under $E_3$, and once again, should depend only on $E_1$ and $E_2$.

And, (here lies the crux), if the choice between A and B depends on $E_1$ and $E_2$, and if the choice between C and D also depends on $E_1$ and $E_2$, we should expect that somebody who prefers A to B should also prefer C to D.
Exercises

1. Consider a risk-averse investor with wealth $w_0$ who must allocate her wealth between two assets. The safe asset returns $R$ for every unit invested while the risky asset returns $H > R$ with probability $p$ and $L < R$ with probability $1 - p$. Suppose her utility function is given by a differentiable utility function $u(.).

   (a) Suppose first that $pH + (1 - p)L < R$. What is the optimal portfolio choice? Why?

   (b) Suppose now that $pH + (1 - p)L > R$ and $u(w) = \ln w$. What is the optimal portfolio choice for this investor?

   (c) Suppose next that $pH + (1 - p)L > R$ and $u(w) = -e^{-aw}$. What is the optimal portfolio choice for this investor?

2. Suppose an investor’s final-period wealth is distributed normally with mean $\mu$ and variance $\sigma^2$. Let the utility function be given by $u(w) = -e^{-rw}$. Show that the utility if such an individual can be described by the mean and variance alone (i.e., this is a case of mean-variance utility).
Solutions

1. If the investor invests $\alpha$ in the risky asset and the rest $w_0 - \alpha$ in the riskless asset, her final wealth is

$$w = \begin{cases} 
\alpha H + (w_0 - \alpha)R \equiv w_{\text{high}}, & \text{if the payoff to risky asset is high;} \\
\alpha L + (w_0 - \alpha)R \equiv w_{\text{low}}, & \text{if the payoff is low.}
\end{cases}$$

with expected utility

$$pu(w_{\text{high}}) + (1 - p)u(w_{\text{low}}).$$

The choice problem is to choose $\alpha$ to maximise expected utility. The first order condition for an interior maximum is

$$pu'(w_{\text{high}})\frac{\partial w_{\text{high}}}{\partial \alpha} + (1 - p)u'(w_{\text{low}})\frac{\partial w_{\text{low}}}{\partial \alpha} = 0.$$ 

In this case it reduces to

$$p(H - R)u'[\alpha H + (w_0 - \alpha)R] + (1 - p)(L - R)u'[\alpha L + (w_0 - \alpha)R] = 0.$$

We consider the three cases in the question.

(a) If $pH + (1 - p)L < R$, the optimal investment is zero. The expected return to the risky asset is less than that to the safe asset.

(b) Suppose $pH + (1 - p)L > R$ and $u(w) = \ln w$. In this case, $u'(x) = 1/x$. The first order condition writes as

$$\frac{p(H - R)}{\alpha H + (w_0 - \alpha)R} = \frac{(1 - p)(R - L)}{\alpha L + (w_0 - \alpha)R}.$$ 

On simplification this reduces to

$$\alpha^* = \left( \frac{R[pH + (1 - p)L - R]}{(H - R)(R - L)} \right) w_0 \equiv \phi w_0,$$ 

Note that the investor holds a constant fraction $\phi$ of her initial wealth $w_0$ in the risky asset.
(c) Suppose now that \( pH + (1 - p)L > R \) and \( u(w) = -e^{-aw} \). In this case, \( u'(x) = ae^{-aw} \). The first order condition for an interior maximum is

\[
p(H - R)ae^{-aw_{\text{high}}} + (1 - p)(L - R)ae^{-aw_{\text{low}}} = 0,
\]

or that

\[
e^{a(w_{\text{high}} - w_{\text{low}})} = \frac{p(H - R)}{(1 - p)(R - L)} = k, \text{ say.}
\]

Taking logs on both sides (and noting that the restriction \( pH + (1 - p)L > R \) implies \( k > 1 \) so \( \ln k > 0 \),

\[
a(w_{\text{high}} - w_{\text{low}}) = \ln k
\]

or

\[
[aH + (w_0 - \alpha)R] - [\alpha L + (w_0 - \alpha)R] = a(H - L) = \frac{1}{a} \ln k,
\]

or that

\[
\alpha^* = \frac{1}{a(H - L)} \ln k = \frac{1}{a(H - L)} \ln \left[ \frac{p(H - R)}{(1 - p)(R - L)} \right]
\]

Note that the investment in the risky asset is independent of initial wealth \( w_0 \).

2. As \( w \) is distributed normally, its density function is

\[
f(w) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(w - \mu)^2}{2\sigma^2} \right].
\]

If \( u(w) = -\exp(-rw) \), then expected utility is

\[
\int u(w)f(w)dw = \int -\exp(-rw) \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(w - \mu)^2}{2\sigma^2} \right] dw
\]

\[
= -\int \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -rw - \frac{(w - \mu)^2}{2\sigma^2} \right] dw
\]

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Consider the expression in the square brackets

\[-rw - \frac{(w - \mu)^2}{2\sigma^2} = \frac{-1}{2\sigma^2}(2r\sigma^2w + w^2 - 2\mu w + \mu^2)\]

\[= \frac{-1}{2\sigma^2}[w^2 + 2(r\sigma^2 - \mu)w + \mu^2]\]

Adding and subtracting \((r\sigma^2 - \mu)^2\) to ‘complete the square’ inside the square brackets

\[-rw - \frac{(w - \mu)^2}{2\sigma^2}\]

\[= \frac{-1}{2\sigma^2}[w^2 + 2(r\sigma^2 - \mu)w + \mu^2 + (r\sigma^2 - \mu)^2 - (r\sigma^2 - \mu)^2 + \mu^2]\]

\[= \frac{-1}{2\sigma^2}[(w + (r\sigma^2 - \mu))^2 - (r\sigma^2 - \mu)^2 + \mu^2]\]

\[= \frac{1}{2}(r^2\sigma^2 - \mu r) - \frac{1}{2\sigma^2}[w - (\mu - r\sigma^2)]^2\]

Using this

\[\int u(w) f(w) dw = -\int \frac{1}{\sigma\sqrt{2\pi}} \exp \left[ -rw - \frac{(w - \mu)^2}{2\sigma^2} \right] dw\]

\[= -\exp(\frac{1}{2} r^2\sigma^2 - \mu r) \int \frac{1}{\sigma\sqrt{2\pi}} \exp \left[ -[w - (\mu - r\sigma^2)]^2 \right] dw\]

\[= -\exp(-r(\mu - \frac{1}{2} r\sigma^2)).\]

as the value of the integral, being the integral of a normal distribution with mean \((\mu - r\sigma^2)\) and variance \(\sigma^2\), equals 1.

Recall that a monotonically increasing transformation of a utility function is a valid utility function, and note that \(-\exp(-rx)\) is a monotonically increasing transformation of \(x\): to say that the utility of the risky payoff depends on \(-\exp(-rx)\) is the same as saying it depends on \(x\). Thus, for this case, utility depends on

\[\mu - \frac{1}{2} r\sigma^2.\]
Higher mean increases utility, while higher variance decreases it at the rate $\frac{1}{2}r$. 