Chapter 2

Game Theory

1. Games in strategic form.
2. Dominance and iterated dominance. The Prisoners’ Dilemma.
3. Weak dominance.
6. Games in extensive form.
7. Refinement of Nash Equilibria - Subgame Perfect Equilibria.
8. Repeated Games and the Folk Theorem.

References

2.1 Games in Normal (or Strategic) Form

An \( n \)-person game in strategic form (or, normal form) has 3 essential elements

1. A finite set of players \( I = \{1, 2, \ldots, n\} \).

2. For each player \( i \), a finite set of strategies \( S_i \). Let \( s = (s_1, s_2, \ldots, s_n) \) denote an \( n \)-tuple of strategies, one for each player. This \( n \)-tuple is called a strategy combination or strategy profile. The set \( S = S_1 \times S_2 \times \ldots \times S_n \) denotes the set of \( n \)-tuple of strategies.

3. For each player \( i \), there is a payoff function \( P_i : S \rightarrow R \), which associates with each strategy combination \( (s_1, s_2, \ldots, s_n) \), a payoff \( P_i(s_1, s_2, \ldots, s_n) \) for player \( i \). Since we have one such function for each player \( i \), in all we have \( n \) such functions.

Note: If the typical player is denoted by \( i \), we sometimes denote all other players (her ‘opponents’) by the (vector) \(-i\). Hence, a typical strategy profile is denoted as \((s_i, s_{-i})\).

2.2 Dominance and Iterated Dominance

**Definition 1.** The (pure) strategy \( s_i \) is (strictly) dominated for player \( i \) if there exists \( s'_i \in S_i \) such that \( u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}) \) \( \forall s_{-i} \).

If, in a particular game, some player has a dominated strategy, it is reasonable to expect that the player will not use that strategy.

**Prisoners’ Dilemma**

The Prisoners’ Dilemma game below is an example of a game where a single round of elimination of dominated strategies allows us to solve the game.
Player 2

<table>
<thead>
<tr>
<th>Confess</th>
<th>Not Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>-5,-5</td>
</tr>
<tr>
<td>Player 1</td>
<td></td>
</tr>
<tr>
<td>Not Confess</td>
<td>0,-8</td>
</tr>
<tr>
<td>-8,0</td>
<td>-1,-1</td>
</tr>
</tbody>
</table>

How would you play this game?

In general there may be successive stages of elimination. This method of narrowing down the set of ways of playing the game is described as iterated dominance. If in some game, all strategies except one for each player can be eliminated on the criterion of being dominated (possibly in an iterative manner), the game is said to be dominance solvable.

Player 2

<table>
<thead>
<tr>
<th>Left</th>
<th>Middle</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>4,3</td>
<td>2,7</td>
</tr>
<tr>
<td>Player 1</td>
<td>Middle</td>
<td>5,5</td>
</tr>
<tr>
<td>Bottom</td>
<td>3,5</td>
<td>1,5</td>
</tr>
</tbody>
</table>

We can eliminate dominated strategies iteratively as follows.

1. For player 1, Bottom is dominated by Top. Eliminate Bottom.
2. In the remaining game, for player 2, Right is dominated by Middle. Eliminate Right.
3. In the remaining game, for player 1, Top is dominated by Middle. Eliminate Top.
4. In the remaining game, for player 2, Middle is dominated by Left. Eliminate Middle.

This gives us (Middle,Left) as the unique equilibrium.
2.3 Weak Dominance

Definition 2. The (pure) strategy $s_i$ is weakly dominated for player $i$ if there exists $s'_i \in S_i$ such that $u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}) \ \forall s_{-i}$, with strict inequality holding for some $s_{-i}$.

Here, for player 1, Middle and Top are weakly dominated by Bottom. Eliminate Middle and Top. The equilibria are (Bottom, Left) and (Bottom, Right).

2.4 Nash Equilibrium

However, for many games the above criteria of dominance or weak dominance are unhelpful - none of the strategies of any player might be dominated or weakly dominated.

The following is the central solution concept in game theory.

Definition 3 (Nash Equilibrium in Pure Strategies). A strategy profile $(s^*_i, s^*_{-i})$ is a Nash equilibrium if for each player $i$,

$$u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i}) \ \forall \ s_i \in S_i.$$

Here, for player 1, Middle and Top are weakly dominated by Bottom. Eliminate Middle and Top. The equilibria are (Bottom, Left) and (Bottom, Right).

The only Nash Equilibrium in this game is (Bottom, Right).
A Nash equilibrium is a strategy combination in which each player chooses a best response to the strategies chosen by the other players. In the Prisoners’ Dilemma, the case in which each prisoner confesses is a Nash equilibrium. (If there is a dominant strategy equilibrium, it must be a Nash equilibrium as well).

In general, we can argue that if there is an obvious way to play the game, this must lead to a Nash equilibrium. Of course, there may exist more than one Nash equilibrium in the game, and hence the existence of a Nash equilibrium does not imply that there is an ‘obvious way to play the game’.

How to look for Nash Equilibria in simple games? Consider the following game, known as the “battle of the sexes.”

<table>
<thead>
<tr>
<th></th>
<th>Football</th>
<th>Opera</th>
</tr>
</thead>
<tbody>
<tr>
<td>Football</td>
<td>2,1</td>
<td>-1,-1</td>
</tr>
<tr>
<td>Opera</td>
<td>-1,-1</td>
<td>1,2</td>
</tr>
</tbody>
</table>

In essence, we must examine all strategy combinations, and for each one, check to see if the Nash equilibrium conditions are satisfied. Consider the Battle of the Sexes depicted in the figure above.

a. Start with the strategy combination (Football, Football).

(a1) Look at the payoffs from the husband’s viewpoint. If the wife goes to the football match, is football optimal for him? Yes, because 2 > −1.

(a2) Now look at the payoffs from the wife’s viewpoint. If the husband goes to the football match, is football optimal for her? Yes, because 1 > −1.

Since the answer is ‘yes’ in both (a1) and (a2), (Football, Football) is a Nash equilibrium.

b. Next, consider the strategy combination (Football, Opera).

(b1) Look at the payoffs from the husband’s viewpoint. If the wife goes to the Opera, is football optimal for him? No, because by going to the football he gets -1, and he could do better by going along to the Opera which would fetch 1. For this strategy combination, the Nash equilibrium condition does not hold for the husband.

(b2) We could look at this strategy combination from the wife’s viewpoint,
but given that the Nash condition does not hold in (b1), we need not really bother.

In short, (Football, Opera) is not a Nash equilibrium.

c. Next, consider the strategy combination (Opera, Opera)....

Checking (c1) and (c2), this turns out to be a Nash equilibrium.

d. Next, consider the strategy combination (Opera, Football) This is not a Nash equilibrium.

In sum, there seem to be two Nash equilibria in this game, namely (Opera, Opera) and (Football, Football).

If the game had three strategies for each player, there would be 9 possible strategy combinations for us to check for Nash equilibria.

### 2.5 Nash Equilibrium in Mixed strategies

Some games do not seem to admit Nash equilibria in pure strategies. Consider the game below called “matching-pennies.”

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Heads</th>
<th>Tails</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heads</td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
<tr>
<td>Tails</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
</tbody>
</table>

Notice that this game seems to have no Nash equilibria, at least in the sense that they have been described thus far. But, in fact it does have a Nash equilibrium in mixed strategies.

The first stage in the argument is to enlarge the strategy space by constructing probability distributions over the strategy set $S_i$.

**Definition 4** (Mixed Strategy). A mixed strategy $s_i$ is a probability distribution over the set of (pure) strategies.

In the matching pennies game, a pure strategy might be Heads. A mixed strategy could be Heads with probability $1/3$, and Tails with probability $2/3$. Notice that a pure-strategy is only a special case of a mixed strategy.
A Nash equilibrium can now be defined in the usual way but using mixed strategies instead of pure strategies.

**Definition 5 (Nash Equilibrium).** A mixed-strategy profile \((\sigma^*_i, \sigma^{-i})\) is a Nash equilibrium if for each player \(i\),

\[
u_i(\sigma^*_i, \sigma^{-i}) \geq u_i(s_i, \sigma^{-i}) \quad \forall s_i \in S_i.
\]

The essential property of a mixed strategy Nash Equilibrium in a 2 player game is that each player’s chosen probability distribution must make the other player indifferent between the strategies he is randomizing over. In a \(n\) player game, the joint distribution implied by the choices of each player in every combination of \((n - 1)\) players must be such that the \(n\)-th player receives the same expected payoff from each of the strategies he plays with positive probability.

Once we include mixed strategy equilibria in the set of Nash Equilibria, we have the following theorem.

**Theorem 1 (Existence).** Every finite-player, finite-strategy game has at least one Nash equilibrium.

Clearly, if a game has no equilibrium in pure strategies, the use of mixed-strategies is very useful. However, even in games that do have one or more pure strategy Nash equilibria, there might be yet more equilibria in mixed-strategies. For instance, we could find a mixed-strategy Nash Equilibrium in the Battle of the Sexes.

### 2.6 Games in Extensive Form

The extensive form is particularly useful when the interaction is principally dynamic. It provides a clear description of, say, the order in which the players move, what their choices are, the information that each player has at each stage and so on. The extensive form is often represented by a game tree.

The description involves the following elements.

1. A finite set of players \(I = \{1, 2, \ldots, n\}\). In addition, there may be an additional player to capture the uncertainty, called Nature (denoted by \(N\)).
2. A game tree consists of a set of nodes with a binary precedence relationship. Think of it as a configuration of nodes and branches. A node (more accurately, a decision node) represents a point at which a player (or ‘nature’) must choose an action. The choice of an action takes that player down a branch to a successor node. The idea of an initial node and terminal node(s) is obvious in this context. A game tree is a configuration of nodes and branches running from the initial node to the terminal nodes, with the restriction that there be no closed loops in the tree.

3. One player or (nature) is assigned to each node. This is just a way of specifying which player must choose (take an action) at that node.

4. For each node, there is a finite set $A$ of available actions, which lead to the immediate successor nodes of that node.

5. Each player’s nodes are partitioned into information sets, which measures the fineness of the information available to that player when s/he chooses an action. If two nodes lie in the same information set, the player knows that s/he is at one of those two nodes but does not know which one.

6. An assignment of payoffs, one for each player, at each terminal node.

7. A probability distribution over nature’s moves.

The notion of a strategy is fairly straightforward in a normal form game. However, for an extensive form game, it is a little bit more complicated. To understand what a typical element of the strategy set $S_i$ for player $i$ is, let $h$ be a typical information set for player $i$, and $A(h)$ the set of actions available at that information set. A (pure) strategy for player $i$ specifies which action she must take at each of her information sets. The set of all strategies for that player is given as $S_i = \Pi_h A(h)$.

**Note:** It is very important to understand the distinction between actions and strategies for an extensive form game. A strategy is a complete plan of actions.
2.7 Actions and Strategies

2.7.1 Game 1

In game 1, player 1 has a total of 4 actions: A, B, C, D. However, player 1 moves at three different nodes - at each of these nodes 1 has 2 possible actions. Thus the total number of strategies for player 1 is $2 \times 2 \times 2 = 8$. For example, one of the strategies of player 1 is: play A initially, and then if player 2 plays L, then play C and if player 2 plays R, then play D. Such a strategy is written as ACD. The eight strategies of player 1 are:

1. A C C
2. A C D
3. A D C
4. A D D
5. B C C
6. B C D
7. B D C
8. B D D

Player 2 has 2 actions - L and R, and 4 strategies:
1. L L
2. L R
3. R L
4. R R

2.7.2 Game2

In game 2, on the other hand, player 1 moves only at two different information sets (each node is also a trivial information set). Thus 1 has only 4 strategies: AC, AD, BC, BD. Player 2 only moves at one information set - thus for 2, actions and strategies coincide. Player 2 has only 2 actions as well as 2 strategies: L and R.

2.8 Analyzing Extensive Form Games

Now that we can write down the strategies for players, how do we identify the Nash equilibria of such games? We do so by converting extensive form games to normal form games. To every extensive form game there is a
corresponding strategic form game. But a given strategic form game can, in general, correspond to several different extensive form games.

The normal form for game 2 is as follows:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td>0,0</td>
<td>2,2</td>
</tr>
<tr>
<td>AD</td>
<td>1,1</td>
<td>0,0</td>
</tr>
<tr>
<td>BC</td>
<td>1,0</td>
<td>0,1</td>
</tr>
<tr>
<td>BD</td>
<td>1,0</td>
<td>0,1</td>
</tr>
</tbody>
</table>

Player 1

From this, it is easy to see that there are 2 pure strategy Nash equilibria: (AC, R) and (AD, L).

2.9 Equilibrium Refinement

Weak Dominance We may choose to eliminate equilibria that involve the use of weakly dominated strategies.

Subgame perfect equilibria (SPE), and the issue of credibility A subgame is a game consisting of a node which is a singleton, that node’s successors and the payoffs at the associated end-nodes.

A strategy combination is a subgame perfect equilibrium (SPE) if it is a Nash Equilibrium (NE) for the entire game and the implied strategies for any subgame are a NE for that subgame. We will discuss some illustrative examples in the lecture. Here is a simple example:

Consider the following extensive form game.
The normal form is given by

<table>
<thead>
<tr>
<th></th>
<th>tt</th>
<th>nt</th>
<th>tn</th>
<th>nn</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>1,1</td>
<td>2,0</td>
<td>1,1</td>
<td>2,0</td>
</tr>
<tr>
<td>N</td>
<td>0,2</td>
<td>0,2</td>
<td>3,3</td>
<td>3,3</td>
</tr>
</tbody>
</table>

Thus there are 3 pure strategy Nash Equilibria: (T,tt), (N,nn), (N,tn).

However, in the subgame on the left hand side, the (trivial) Nash equilibrium (in this subgame only one player plays - so 2’s optimal strategy in the subgame is trivially the Nash equilibrium for the subgame) is ‘t’. In the subgame on the right hand side, the Nash equilibrium is ‘n’. Thus the only Nash equilibrium that induces Nash equilibria in all subgames is (N,tn) - this is therefore the only subgame perfect Nash equilibrium.

The issue of subgame perfection is closely linked to those of credible threats and of credible promises. Very crudely speaking, consider a Nash equilibrium which is not subgame perfect. That implies there must be a subgame such that the strategy over that remaining subgame is not a Nash equilibrium for that subgame. So if by chance we end up at that subgame, we do not expect that the players will find it profitable to stick to that ‘portion’ of the strategy: if not, that portion of the strategy is not credible. The issue is best discussed through some examples such as entry deterrence, credibility of government policy etc.
Other Refinements  There are other kinds of refinements that result in, say, sequential equilibria, trembling-hand perfection, etc. but constraints of time will prevent us from exploring these in any detail. Those interested in these are advised to read more extensively in these areas.

2.10 Repeated Games and the Folk Theorem

Suppose a particular game such as the ‘Prisoners’ Dilemma’ is played a large number of times. Can we say something about the behaviour of players in such ‘supergames’ that is not obvious in the analysis of the one-shot game?

To anticipate the argument, we will try and establish that if the game is repeated a large number of times, we cannot rule out some outcomes that are clearly unlikely in case the game was played just once.

First, we need to have an appropriate notion of payoffs in the supergame, or more accurately, the relationship between the payoff in the supergame and in the one-shot constituent game (now called the stage game). The average payoff over the supergame is some aggregate measure of the payoffs from the stage games, with later payoffs possibly discounted for the lag with which they will become available.

The ‘folk theorems’ for repeated games are usually some variant of a simple idea, namely that, if the players are sufficiently patient, then any feasible, individually rational payoff combination can be supported as a Nash equilibrium (the two adjectives need some explanation). One conclusion that emerges from this is that outcomes that were ruled out in the one-shot game (eg. cooperation in the Prisoners’ Dilemma) can be ‘sustained’ in the associated repeated game. The intuition for this might run as follows: if players deviate from say, an agreement to cooperate they could be punished by the other(s) in subsequent rounds which would cause loss of utility in every subsequent period. (How hard they can be punished depends on what their reservation utility is; how costly this future punishment seems to them depends on their discount factor). The possibility of reduction of utility in later rounds engenders cooperation. This idea is quite useful in understanding why cooperation is sustained in the real world. This has considerable application in models of say, lender-debtor interaction, seller cartels, entry deterrence, etc.