MSc Economics
Economic Theory and Applications I
Microeconomics

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Chapter 1

Theory of Choice: An Axiomatic Approach

Hal Varian, *Microeconomic Analysis*, 3rd ed. (1992), Ch 7.1

1.1 Consumer Choice

- Economics is a study of Choices - made by individuals, firms, and government that govern the allocation of scarce resources.

- E.g.1: Why do consumers choose to buy a car rather than a holiday?

- E.g.2: How do workers allocate time between work and leisure?

- Consumers are faced with possible consumption bundles in some set $X \in \mathbb{R}^k_+$, assumed closed and convex. They have preferences on the consumption bundles in $X$, which we wish to model.
1.2 Preference Relations

- Building block: **binary relations**. A binary relation defined on a set \( X \) is a set of ordered pairs in \( X \).

- E.g. \( x \) is taller than \( y \); \( x \) is as tall as \( y \).

- Given two bundles then we have a binary relation for preferences "\( x \) is at least as good as \( y \)"', denoted by \( x \ R \ y \) or \( x \succeq y \). This is called the **weak preference relation**.

- Can then define two induced relations: **strict preference relation**, "\( x \) is strictly preferred to \( y \)"’, denoted by \( x \ P \ y \) or \( x \succ y \), and **indifference relation**, "\( x \) is considered indifferent to \( y \)"’, denoted by \( x \sim y \). These are "induced" as,

\[
\begin{align*}
x \succ y & \iff x \succeq y \text{ and not } y \succeq x \\
x \sim y & \iff x \succeq y \text{ and } y \succeq x
\end{align*}
\]

1.3 Ordering of Bundles

In order to order bundles based on the preference relation \( \succeq \), we require some axioms for \( \succeq \),

1. **Reflexive** - \( \forall x \in X, x \succeq x \) (trivial)

2. **Complete** - \( \forall x, y \in X, x \succeq y, y \succeq x \) or both, i.e. any two bundles can be compared.

3. **Transitive** - \( \forall x, y, z \in X, \text{ if } x \succeq y, y \succeq z \text{ then } x \succeq z \). This is necessary for any discussion of preference maximisation, i.e. if not transitive, then there might be set of bundles \( X \) with no best elements.

If all three axioms are satisfied, then the preference relation is called **rational**.

Other often assumed properties are:

4. **Continuity** - \( \forall y \in X, \text{ the sets } \{ x : x \succeq y \} \text{ and } \{ x : y \succeq x \} \text{ are closed.} \)

\[\text{It follows that } \{ x : x \succ y \} \text{ and } \{ x : y \succ x \} \text{ are open sets.}\]

\[\text{See Section 1.6 Appendix for the definitions.}\]
\{x : x \succeq y\} is a closed set as if \{x^i\} is a sequence of consumption bundles that are all at least as good as a bundle \(y\), and if this sequence converges to some bundle \(x^*\), then \(x^*\) must also be at least as good as \(y\), i.e. no discontinuity in the \(\succeq\) relation. Note if \(\{x : x \succeq y\}\) was an open set, then there may be sequences \(\{x^i\}\) that converges to a point on the boundary which is not in the set, i.e. the preference suddenly flips to \(x < y\). \(\{x : x \succ y\}\) is an open set as the sequences can converges to a point on the boundary where \(x \succeq y\). But it also means that if \(x \succ y\), and \(z\) is a bundle close enough to \(x\), then \(z\) must also be strictly preferred to \(y\). This would not be the case in a closed set if \(x\) was on the boundary.

5. **Weak monotonicity** - If \(x \succeq y\) then \(x \succeq y\), i.e. “at least as much of everything is at least as good.”

6. **Strong monotonicity** - If \(x \succeq y\) and \(x \neq y\), then \(x \succ y\), i.e. “at least as much of every good, and strictly more of some good, is strictly better.”

7. **Local Non-satiation** - Given \(x \in X\) and \(\varepsilon > 0\), \(\exists\) some bundle \(y \in X\) with \(|x - y| < \varepsilon\) such that \(y \succ x\), i.e. one can always do a little bit better. Note strong monotonicity implies local nonsatiation but not the other way around. This rules out “thick” indifference curve.

8. **Convexity** - \(\forall x, y, z \in X\), if \(x \succeq z\) and \(y \succeq z\), then \(tx + (1 - t)y \succeq z\) \(\forall 0 \leq t \leq 1\), i.e. an agent prefers averages to extremes.

9. **Strict Convexity** - \(\forall x, y, z \in X\), if \(x \succeq z\) and \(y \succeq z\), then \(tx + (1 - t)y \succ z\) \(\forall 0 < t < 1\).

Given a preference ordering of bundles, one can then draw an **indifference curve** as a set of all bundles that are indifferent to each other. If the preference is convex there can be “flat spots”, but not if it is **strictly convex**. The set of all bundles on and above an indifference curve is called the **upper contour set**.
### 1.4 Utility Function

A numerical representation of preferences by a function $u : X \rightarrow \mathbb{R}$ such that

$$x \succ y \iff u(x) > u(y)$$

It can be shown that if the preference ordering is complete, reflexive, transitive and continuous, then it can be represented by a continuous utility function.\(^2\) It is important to note that the function is ordinal, i.e. if $u(x)$ represents some preferences $\succeq$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a positive monotonic function\(^3\), then $f(u(x))$ will represent exactly the same preference, i.e. $f(u(x)) > f(u(y)) \Leftrightarrow x \succ y$.

Example of the use of utility function: The marginal rate of substitution - how much should one consume less (more) of a good if he consumes one more (less) of another good, to keep the two consumption bundles indifferent? Let $u(x_1, ..., x_2)$ be a utility function. Then on an indiffERENCE curve,

$$du(x) = \frac{\partial u(x)}{\partial x_i} dx_i + \frac{\partial u(x)}{\partial x_j} dx_j = 0$$

Hence

$$\frac{dx_j}{dx_i} = -\frac{\frac{\partial u(x)}{\partial x_j}}{\frac{\partial u(x)}{\partial x_i}}$$

i.e. the ratio of the marginal utility. Note that MRS is independent of the ordinal utility function chosen, i.e. it is a property of the underlying preference ordering. To see this, if $v(x) = g(u(x))$ also represents the same preference ordering (i.e. $g(.)$ is a positive monotonic function), then

$$\frac{dx_j}{dx_i} = \frac{\frac{\partial v(x)}{\partial x_j}}{\frac{\partial v(x)}{\partial x_i}} = \frac{g'(u) \frac{\partial u(x)}{\partial x_j}}{g'(u) \frac{\partial u(x)}{\partial x_i}} = -\frac{\frac{\partial u(x)}{\partial x_j}}{\frac{\partial u(x)}{\partial x_i}}$$

On the other hand to state for example “$x$ is twice as preferred to $y$”, one requires a cardinal utility function.

\(^2\)Varian p.97 shows a weaker version of this with strong monotonicity as an additional condition.

\(^3\)A function $g : \mathbb{R} \rightarrow \mathbb{R}$ is a positive monotonic transformation if $g$ is a strictly increasing function, i.e. $x > y \Rightarrow g(x) > g(y)$. 

1.5 Example: Lexicographic Preference Relation


For simplicity, assume that a set $X = \mathbb{R}_+^2$. Define $x \succeq y$ if either “$x_1 > y_1$” or “$x_1 = y_1$ and $x_2 \geq y_2$”. This is known as the lexicographic preference relation. The name derives from the way words are organised in a dictionary. This ordering is reflexive, complete, transitive, strongly monotone, and strictly convex (check). Yet there is no utility function that represents this preference ordering. The intuitive proof is as follows. With this preference ordering, no two distinct bundles are indifferent; i.e. indifference sets are singletons. Thus we have in $X = \mathbb{R}_+^2$ two dimensions of distinct indifference sets. Yet each of these indifference sets must be assigned, in an order-preserving way, a different utility number from the one-dimensional real line. This is a mathematical impossibility. Thus no numerical representation exists for this preference relation. So what has gone wrong? The reason why no utility function exists for this ordering is the fact that the preferences are not continuous. To see this, consider the sequence of bundles $x^n = (1/n, 0)$. Comparing this to $y = (0, 1)$, for every $n$ we have $x^n \succeq y$. But $\lim_{n \to \infty} x^n = (0, 0) \prec y$, i.e. the preference ranking flips over at the limit point. Thus this is an example that demonstrates the discontinuity characteristic of the lexicographic preference ordering.
1.6 Appendix

[Hal Varian, Ch 26.4]

- An open ball of radius \( e \in \mathbb{R}_+ \) at \( x \in \mathbb{R}^n \) is defined as \( B_e(x) = \{ y \in \mathbb{R}^n : |y - x| < e \} \).

- Then a set of points \( A \) is an open set if \( \forall x \in A, \exists \) some \( B_e(x) \in A \). Thus the boundary cannot be included. This is equivalent to stating that \( \forall x \in A, x \) is in the interior of \( A \).

- The compliment of a set \( A \) in \( \mathbb{R}^n \) is \( \mathbb{R}^n \setminus A = \{ y \in \mathbb{R}^n : y \notin A \} \).

- Then a set \( A \) is a closed set if \( \mathbb{R}^n \setminus A \) is an open set. Note this definition implies that the closed set includes the \( \{-\infty, \infty\} \) points.

- A set \( A \) is bounded if \( \exists x \in A \) and some \( e \in \mathbb{R}_+ \) such that \( A \subset B_e(x) \). It can of course be either open or closed.

- \( A \) is then compact if it is both closed and bounded, i.e. it no longer contains the \( \{-\infty, \infty\} \) points.

- A sequence \( \{x^i\} \) is said to converge to \( x^* \) if \( \forall e \in \mathbb{R}_+ \), \( \exists \) an integer \( m \) such that \( \forall i > m, x^i \in B_e(x) \). Thus \( \lim_{i \to \infty} x^i = x^* \).

- Using this idea one can have another definition for a closed set: a set \( A \) is a closed set if every convergent sequence in \( A \) converges to a point in \( A \). Note in an open set, one can have sequences converging to a point on the boundary, i.e. \( \lim_{i \to \infty} x^i \notin A \).

- A function \( f(x) \) is continuous at \( x^* \) if for every sequence \( \{x^i\} \) that converges to \( x^* \), we have the sequence \( \{f(x^i)\} \) that converges to \( \{f(x^*)\} \).

- A function that is continuous at every point in its domain is called a continuous function.
Chapter 2

Consumer Theory


2.1 Utility Maximisation Problem

Basic hypothesis: a rational consumer will always choose a most preferred bundle from the set of affordable alternatives, i.e. bundles that satisfy the consumer’s budget constraint. If $m$ is the income of the consumer, and $p = (p_1, ..., p_k)$ is the vector of prices of goods $1, ..., k$, then the set of affordable bundles $x$ is given by the budget constraint $p \cdot x \leq m$.

Then using the utility function representation of the consumer’s preference orderings, the problem of choosing the most preferred bundle can be written as,

$$\max_{x \geq 0} u(x) \text{ such that } p \cdot x \leq m$$

We must first check if a solution exists, and if it does, whether it is unique.

**Proposition 2.1** The solution exists if $u(.)$ is continuous, and the feasible consumption bundle set $X$ is compact.$^2$

---

$^1$ $p \cdot x$ is an inner product of two vectors,

$$p \cdot x = p_1 x_1 + p_2 x_2 + ... + p_k x_k$$

$^2$ This uses the **Weierstrass’s Theorem**: let $f : X \rightarrow \mathbb{R}$ be a continuously function whose domain is a
More informally we require the prices to be strictly positive and the income not to be unbounded above for the feasible set \( X \) to be compact. We know from Chapter 1 that \( u(\cdot) \) is continuous for rational and continuous preferences. The solution is also unique if we assume the preference to be strictly convex. Denote this solution by \( x^* \). If preferences satisfy local non-satiation, then we cannot have \( p.x^* < m \) as if this is so, there must be some bundle \( x \) close to \( x^* \) which is preferred to \( x^* \). Thus under this assumption we must have \( p.x^* = m \), i.e. the budget constraint “binds”.

So given \( (p, m) \), (UMP) can be solved to yield a unique utility maximising consumption bundle \( x^*(p, m) \). This is called the “Marshallian” or “ordinary” or “Walrasian” or “constant income” demand function. The maximised utility attained with this optimal bundle is then given by \( u(x^*(p, m)) \equiv v(p, m) \). The latter function, which gives the maximised value of utility at \( (p, m) \) (i.e. an optimal value function) is called the indirect utility function.

Properties \( v(p, m) \) is,

1. Homogeneous of degree zero in \( (p, m) \). (Intuitive)
2. Non-increasing in \( p \) and non-decreasing in \( m \).
3. Continuous at all \( p \gg 0, m > 0 \).
4. Quasi-convex in \( p \), i.e. \( \{ p : v(p, m) \leq k \} \) is a convex set \( \forall k \). Thus the price indifference curves are convex (see Fig 2.1).

Note that with regards to Property 2,

- An example of where \( v(p', m) = v(p, m) \) despite \( p' > p \) is when \( u = ax_1 + bx_2 \). Then when \( \frac{a}{b} < \frac{p_1}{p_2} \), the consumer’s optimal bundle will be \( \left(0, \frac{m}{p_2}\right) \), which is unaffected by an increase in \( p_1 \).

compact subset \( X \subset \mathbb{R}^n \). Then there exist points \( x_{\min}, x_{\max} \in X \) such that \( \forall x \in X, f(x_{\min}) \leq f(x) \leq f(x_{\max}) \); that is, \( x_{\min} \in X \) is the global min of \( f \) in \( X \) and \( x_{\max} \in X \) is the global max of \( f \) in \( X \). See for example Simon and Blume (1994), *Mathematics for Economists*, Ch 30.

\footnote{A function \( f : \mathbb{R}^n_+ \rightarrow \mathbb{R} \) is \textit{homogeneous of degree} \( k \) if \( f(tx) = t^k f(x) \) \( \forall t > 0 \).}

\footnote{See Section 2.9 Appendix.}
If preferences satisfy the local non-satiation assumption, then $v(p, m)$ will be strictly increasing in $m$. This gives a one-to-one relationship between $v(p, m)$ and $m$ for given prices $p$. Then we can invert the function and solve for $m$ as a function of the level of utility, i.e. the minimal amount of income necessary to achieve utility $u$ at prices $p$. This inverse of the indirect utility function is the expenditure function investigated in Section 2.2.

Remark 2.1 Given $(p, m)$, the optimal bundle is chosen such that the MRS of the goods equals the relative prices.

Proof. Intuitive. Or more technically, by solving the Lagrangian optimisation,

$$\max_{x \geq 0, \lambda} L = u(x) - \lambda(p \cdot x - m) \quad (2.1)$$

the first-order conditions are

$$\frac{\partial u(x^*)}{\partial x_i} - \lambda p_i = 0 \quad \text{for } i = 1, ..., k$$

Then it follows that

$$\frac{\partial u(x^*)/\partial x_i}{\partial u(x^*)/\partial x_j} = \frac{p_i}{p_j} \quad \text{for } i, j = 1, ..., k$$
Proposition 2.2 (Roy’s Identity) For $p \gg 0$ and $m > 0$, 

$$x_i(p, m) = -\frac{\partial v(p, m) / \partial p_i}{\partial v(p, m) / \partial m}, \quad \forall i = 1, \ldots, k$$

Proof. Apply Envelope Theorem\(^5\) to (2.1) at $x^*$, i.e. $u(x^*) \equiv v(p, m)$,

$$\frac{\partial v(p, m)}{\partial p_i} \bigg|_{x(p, m) \text{ constant}} = -\lambda x_i(p, m)$$

$$\frac{\partial v(p, m)}{\partial m} \bigg|_{x(p, m) \text{ constant}} = \lambda$$

Thus the ratio of the two gives $x_i(p, m)$. ■

2.2 Expenditure Minimisation Problem

We can look at the same problem in a different way. Suppose instead of having $(p, m)$ given and finding the optimal consumption bundle, we are given $(p, U)$ i.e. the price vector and the minimum utility level that we want attained. Then we can find the minimum cost required to attain this,

$$\min_{x \geq 0} \ p \cdot x \text{ such that } u(x) \geq U$$

(EMP)

Provided that the level of utility $U$ is achievable, this problem too has a solution. And once again if the preferences are strictly convex, the solution is unique. This optimal consumption bundle, this time denoted $h^*(p, U)$, is called the “Hicksian” or “compensated” demand function. The minimum cost required to attain $U$ is calculated by $p \cdot h^*(p, U) \equiv e(p, U)$ is again an optimal value function, and is called the expenditure function.

Properties $e(p, U)$ is,

1. Homogeneous of degree one in $p$, i.e. $h^*(p, U)$ is h.g.d.0 in $p$. (Intuitive)

2. Non-decreasing in $p$ (why not strictly increasing?) and strictly increasing in $U$.

3. Continuous at all $p \gg 0$.


\(^5\)See Section 2.9 Appendix.
Proof of Property 4. We need to prove that \( e(p'', U) \geq t e(p, U) + (1-t) e(p', U) \) for \( 0 \leq t \leq 1 \), where \( p'' = tp + (1-t)p' \). As \( h^*(p, U) \) and \( h^*(p', U) \) are the expenditure minimising bundles at prices \( p \) and \( p' \) respectively, we have,

\[
\begin{align*}
  p.h^*(p'', U) & \geq p.h^*(p, U) \\
  p'.h^*(p'', U) & \geq p'.h^*(p', U)
\end{align*}
\]

Multiply the former by \( t \) and the latter by \( 1-t \), and then summing up we get,

\[
tp.h^*(p'', U) + (1-t)p'.h^*(p'', U) \geq te(p, U) + (1-t)e(p', U)
\]

But the left-hand side is \( \{tp + (1-t)p'\}h^*(p'', U) = p''.h^*(p'', U) = e(p'', U) \). Thus the concavity is proved. ■

The intuition here is that as \( p \) doubles, by retaining the same consumption bundle the expenditure will double; however one can possibly do better by choosing a more appropriate consumption bundle at the new \( p \). Thus the expenditure may increase less than linearly.

Proposition 2.3 (Shephard’s Lemma) For \( p \gg 0 \),

\[
h_i(p, U) = \frac{\partial e(p, U)}{\partial p_i}, \quad \forall i = 1, ..., k
\]

Proof. The Lagrangian optimisation for the EMP is,

\[
\min_{h \geq 0, \lambda} L = p.h + \lambda(U - u(h)) \quad (2.2)
\]

Applying Envelope Theorem to (2.2) at \( h^* \), i.e. \( p.h^* \equiv e(p, U) \),

\[
\frac{\partial e(p, U)}{\partial p_i} = \left. \frac{\partial L(h, \lambda)}{\partial p_i} \right|_{h(p, U) \text{ constant}} = h_i(p, m)
\]

■
2.3 Duality Theory

Assuming unique solutions to the UMP and EMP, \( x(p, m) \) and \( h(p, U) \) are continuously differentiable, and \( v(p, m) \) and \( e(p, U) \) are twice differentiable, the following identities are true, demonstrating the duality property of UMP and EMP,

1. \( e(p, v(p, m)) \equiv m \), i.e. the minimum expenditure necessary to reach utility \( v(p, m) \), which is in turn the maximum utility attained at \( p \) and \( m \), is \( m \).

2. \( v(p, e(p, U)) \equiv U \), i.e. the maximum utility from income \( e(p, U) \) is \( U \).

3. \( x_i(p, m) \equiv h_i(p, v(p, m)) \), the Marshallian demand at income \( m \) is the same as the Hicksian demand at utility \( v(p, m) \).

4. \( h_i(p, U) \equiv x_i(p, e(p, U)) \), the Hicksian demand at utility \( U \) is the same as the Marshallian demand at income \( e(p, U) \).

2.4 Slutsky Decomposition

Proposition 2.4 The Slutsky equation is given by,

\[
\frac{\partial x_j(p, m)}{\partial p_i} = \frac{\partial h_j(p, U)}{\partial p_i} - \frac{\partial x_j(p, m)}{\partial m} \frac{\partial x_i(p, m)}{\partial m} (2.3)
\]

Proof. Use identity 4 for good \( j \) and differentiate both sides by \( p_i \),

\[
\frac{\partial h_j(p, U)}{\partial p_i} = \frac{\partial x_j(p, m)}{\partial p_i} + \frac{\partial x_j(p, m)}{\partial m} \frac{\partial e(p, U)}{\partial p_i}
\]

But by Shephard’s Lemma \( \frac{\partial e(p, U)}{\partial p_i} = h_i(p, U) \), which at the optimal point equals \( x_i(p, m) \). Substitution and rearranging yields (2.3).

This is another result of the Duality Theory. This holds as the optimal outcome of the UMP is the same as that of the EMP. Note that as both the UMP and the EMP assume given \( p \), and hence constant \( m \), the Slutsky equation above only holds for constant \( m \) (i.e. not \( m = p \cdot w \), where \( w \) is an endowment vector). For an example of Slutsky decomposition with price-dependent income, see Section 2.7.
The Slutsky equation permits an intuitive interpretation. Effectively, it allows a notional decomposition of the effect of a change in price on Marshallian demand. An increase in the price of any commodity, say in \( p_i \), has two effects. One, it changes the relative price between various commodities, and two, it decreases the overall level of real income for any consumer who purchases a positive quantity of the \( i^{th} \) commodity. The first leads to the substitution effect keeping the utility constant (represented by the first term in (2.3)), and the second leads to the income effect (captured by the second term). The minus sign for the income effect reflects the decrease in real wealth as the price increases.

Now consider the matrix of substitution terms \( \frac{\partial h_j}{\partial p_i} \). This is symmetric since, using Shephard’s Lemma again,

\[
\frac{\partial h_j}{\partial p_i} = \frac{\partial^2 e}{\partial p_i \partial p_j} = \frac{\partial^2 e}{\partial p_j \partial p_i} = \frac{\partial h_i}{\partial p_j}
\]

The matrix is in fact negative semi-definite\(^6\) due to the concavity of the expenditure function.\(^7\) This implies that the compensated own-price effect is non-positive, i.e.

\[
\frac{\partial h_i}{\partial p_i} = \frac{\partial^2 e}{\partial p_i^2} \leq 0
\]

since negative semi-definite matrices have non-positive diagonal terms. These are properties of “unobservable” Hicksian demands. However by using Slutsky equation we can state that the matrix \( \begin{bmatrix} \frac{\partial x_i}{\partial p_i} + \frac{\partial x_i}{\partial m} \cdot \frac{\partial z_i}{\partial m} \end{bmatrix} \) is also symmetric and negative semi-definite. This is now a testable prediction. This seemingly arbitrary matrix in fact becomes useful in considering the integrability problem. This is to say that given observed demand functions, can we find the original utility function or the expenditure function (i.e. reversing the Roy’s Identity or Shepherd’s Lemma)? Or more fundamentally how do we even know if a solution exists? It turns out that the integrability condition that ensures the existence of an expenditure

\(^6\)A square matrix \( A \) is (see Varian Ch 26.2)

1. Positive definite if \( x'Ax > 0 \ \forall x \neq 0 \);
2. Negative definite if \( x'Ax < 0 \ \forall x \neq 0 \);
3. Positive semi-definite if \( x'Ax \geq 0 \ \forall x \);
4. Negative semi-definite if \( x'Ax \leq 0 \ \forall x \).

\(^7\)See p.55 Theorem 1.15 of Jehle and Reny 2nd ed.
total effect = substitution effect + income effect

function that is consistent with the observed demand functions, is that this substitution matrix is symmetric and negative semi-definite. (See Varian Ch 8.5 if interested.)

The Slutsky equation is also useful in determining the relationship between different kind of goods. We know that the own-price substitution effect $\frac{\partial x_i}{\partial p_i}$ is non-positive. Then for a normal good (i.e. $\frac{\partial x_i}{\partial m} > 0$), $\frac{\partial x_i}{\partial p_i} < 0$ unambiguously, and hence it is an ordinary good. On the other hand for an inferior good (i.e. $\frac{\partial x_i}{\partial m} < 0$), the sign of $\frac{\partial x_i}{\partial p_i}$ is now ambiguous. However to have $\frac{\partial x_i}{\partial p_i} > 0$ (i.e. a Giffen good) $\frac{\partial x_i}{\partial m}$ must be negative, and hence a Giffen good must be an inferior good.8

2.5 Welfare Measurement

Question: What is the point of Duality Theory?

Answer: For welfare analysis. For welfare measurements such as compensating and equivalent variations (see below) an estimation of $e(p, U)$ is required, but this is (and in general EMP is) unobserved. However UMP is, and using identity 4 above, one can estimate the “unobservable” Hicksian demand from “observable” Marshallian demands.

8Somewhat relating to this, note that for a necessary good $0 \leq \frac{\partial x_i}{\partial m} < \frac{x_i}{m}$, i.e. the marginal increase as one’s income increases is less than the average consumption (thus spends the extra income on something else), while for a luxury good $\frac{x_i}{m} \leq \frac{\partial x_i}{\partial m}$. 
2.5.1 Money Metric Utility Functions

Consider a situation where a consumer has a choice between receiving a goods bundle $x$ or some income $m$. Given current prices $p$, the question is how much income, which we will write $m(p, x)$, would he need to be indifferent between the two. The answer is the minimum cost required to buy a bundle $z$ that is on the same indifference curve as $x$. Thus this is the exactly the same as the EMP above with the minimum utility level $U$ given by $u(x)$, and therefore the solution is

$$m(p, x) \equiv e(p, u(x))$$

This basically gives a monetary value to the utility of holding $x$, and is thus called the money metric utility function.

Alternatively one could ask a question how much income $\mu(p; q, m)$ one would need at prices $p$, to be as well off as having income $m$ at prices $q$. It is not too difficult to see that the solution is given by

$$\mu(p; q, m) \equiv e(p, v(q, m))$$

This is called the money metric indirect utility function.

2.5.2 Compensating and Equivalent Variations

What the policy makers are interested is the welfare changes due to policy changes. For this some measures of welfare change from $(p^0, m^0)$ to $(p^1, m^1)$ are required. One way of doing this is to use compensating variation and equivalent variation which are defined by,

$$v(p^1, m^1 - CV) = v(p^0, m^0)$$

$$v(p^0, m^0 + EV) = v(p^1, m^1)$$
Marriage can bring you as much joy as £60,000 a year, claim economists using a mathematical formula which takes into account income, personal traits and happiness levels.

It's a Sunday morning, and you are just surfacing from sleep. You turn over in bed, and put your arm around your loving, faithful partner. Life is good.

Rewind.

It's a Sunday morning, and you are just surfacing from sleep. You turn over in bed, and put your arm around your pile of £50 notes. There are 1,200 of them.

Life - apparently - is just as good.

A study by two economists claims to have found that, contrary to generations of wisdom, money can actually make you happy.

A lasting marriage brings as much happiness as having an extra £60,000 added to your pay packet, Professor Andrew Oswald of the University of Warwick and David Blanchflower of Dartmouth College in the US say.

Similarly, losing a job causes £40,000-worth of unhappiness.

The study of 100,000 people randomly sampled across the UK and US also compared satisfaction and mental well-being rates in other countries. It found there had been a decline in the number of people married (72% in the early 1970s, 55% by the late 90s). But married people said they were much happier than the unmarrieds.

Getting divorced, separated, or widowed made people much more unhappy than losing their jobs.

The happiest people were women, the highly educated, married couples, and those whose parents have not divorced, the report says. Women who co-habit are happier than those who live alone, but are not as happy as those who are married.

Happiness and satisfaction with life tend to be shaped like a U or J, it says, with high levels in youth and old age, but a drop in the 30s.

Note the signs assume that one is always better off in the after-change state. For example then if a change in agricultural policy leads to a fall in price and incomes for the farmers, the amount of compensation required can be estimated using CV at new prices. On the other hand checking which of the possible policies make the consumers better-off can be better analysed using EV at current prices. These estimations can be done by inverting above definitions,

\[
CV = e(p^1, v(p^1, m^1)) - e(p^1, v(p^0, m^0)) = m^1 - e(p^1, v(p^0, m^0))
\]

\[
EV = e(p^0, v(p^1, m^1)) - e(p^0, v(p^0, m^0)) = e(p^0, v(p^1, m^1)) - m^0
\]

2.5.3 Consumer Surplus as an Approximation

The classic tool for measuring welfare change is Marshallian consumer surplus. If \(x(p)\) is the demand for some good as a function of its price, then the CS associated with
a price movement from \( p^0 \) to \( p^1 \) is,

\[
CS = \int_{p^0}^{p^1} x(p)dp
\]

This was first proposed by Marshall\(^9\) who used the area to the left of the market demand curve as a welfare measure in the special case where wealth effects are absent. Let us investigate this a little further.

Consider the case where \( m^0 = m^1 = m \), and assume also that only the price of good 1 changes from \( p^0 \) to \( p^1 \). Then using Shephard’s Lemma the two above variations can be written as,

\[
CV = m - e(p^1, u^0) = e(p^0, u^0) - e(p^1, u^0) = \int_{p^1}^{p^0} h(p, u^0)dp
\]

\[
EV = e(p^0, u^1) - m = e(p^0, u^1) - e(p^1, u^1) = \int_{p^1}^{p^0} h(p, u^1)dp
\]

where \( u^i = v(p, m) \) for \( i \in \{0, 1\} \). Thus the CV is the integral of the Hicksian demand curve associated with the initial level of utility, and the EV is that associated with the final level of utility. Hence the correct measure of welfare is an integral of the Hicksian demand curve rather than the Marshallian. However Marshallian consumer surplus can still be used as an approximation. We know from the Slutsky equation for own-price change that

\[
\frac{\partial h_i(p, U)}{\partial p_i} = \frac{\partial x_i(p, m)}{\partial p_i} + \frac{\partial x_i(p, m)}{\partial m} x_i(p, m)
\]

Thus if the good in question is a normal good (i.e. \( \frac{\partial x_i}{\partial m} > 0 \)), the slope of the Hicksian demand curve \( \frac{\partial h_i}{\partial p_i} \) will be less negative than that of the Marshallian demand curve \( \frac{\partial x_i}{\partial p_i} \). This implies that as shown in Figure 2.3, the Hicksian inverse demand curve \( \frac{\partial p_i}{\partial h_i} \) will be steeper than the Marshallian inverse demand curve \( \frac{\partial p_i}{\partial x_i} \). It follows then that the areas to the left of the Hicksian demand curves will bound the area to the left of the Marshallian demand curve, and that for this normal good case,

\[
EV > CS > CV
\]

The relationship reverses for an inferior good. Finally in the case of a quasilinear utility function where there is no income effect for good 1, we have \( h(p, u^0) = x(p, m) = h(p, u^1) \)

---

Consumer surplus

$$p_0, p_1, x(p, m), h(p, u)$$

Figure 2.3: Bounds on Consumer Surplus

and hence

$$EV = CS = CV$$

In this case then the CS is an exact measure of welfare change.

### 2.6 Aggregation Issue

Given $H$ consumers with income $m = (m^1, ..., m^H)$, and the price vector $p$ for the $k$ consumption goods, we can now calculate the individual demands $x^h(p, m^h) = (x^h_1(p, m^h), ..., x^h_k(p, m^h))$ for $h = 1, ..., H$. Then we can define the aggregate demand function by

$$X(p, m) = \sum_{h=1}^{H} x^h(p, m^h)$$

The question is, do any of the properties described above for individual workers, such as Roy’s Identity or Slutsky’s equation, carry through this aggregation? If that is the case then the aggregate behaviour can be treated as it were generated by a single “representative” consumer. It turns out though that unfortunately, the aggregate demand function will in general possess no interesting properties other than homogeneity and continuity. This has the implications that there is a problem for having micro underpinning on macro aggregate theories, and also that it is hard to test consumer theories.
For example consider Roy’s Identity. In aggregate form what we would like is,

\[ X_i(p, m) = -\frac{\partial V/\partial p_i}{\partial V/\partial M}, \quad \forall i = 1, \ldots, k \]

where \( V(p, m) = \sum_{h=1}^{H} v^h(p, m^h) \) and \( M = \sum_{h=1}^{H} m^h \). However when substituting these we get,

\[ X_i(p, m) = -\sum_{h=1}^{H} \frac{\partial v^h/\partial p_i}{\partial v^h/\partial M} \neq -\sum_{h=1}^{H} \frac{\partial v^h/\partial p_i}{\partial v^h/\partial m^h} = \sum_{h=1}^{H} x_i^h(p, m^h) \]

clearly implying that Roy’s Identity does not survive aggregation in general.

Fortunately there is a necessary and sufficient condition for successful aggregation, which is that all individual indirect utility function is of the Gorman form,

\[ v^h(p, m^h) = a^h(p) + b(p)m^h \]

Then

\[ V(p, M) = \sum_{h=1}^{H} a^h(p) + b(p)M \]

Try Roy’s Identity again then,

\[ -\frac{\partial V/\partial p_i}{\partial V/\partial M} = -\sum_{h=1}^{H} \frac{\partial a^h/\partial p_i}{b(p)} + \frac{\partial b^h/\partial p_i}{b(p)} M = -\sum_{h=1}^{H} \frac{\partial v^h/\partial p_i}{\partial v^h/\partial m^h} \]

and hence \( X_i(p, m) = \sum_{h=1}^{H} x_i^h(p, m^h) \) as desired. The point is that as can be seen from the consumers’ demands \( x_i^h(p, m^h) = \frac{1}{a(p)} \frac{\partial a^h}{\partial p_i} + \frac{1}{b(p)} \frac{\partial b^h}{\partial p_i} m^h \), all consumers have the same marginal propensity to consume good \( i \), \( \frac{\partial x_i^h(p, m^h)}{\partial m^h} = \frac{1}{a(p)} \frac{\partial b}{\partial p_i} \), which is also independent of \( m^h \). Hence the aggregate demand function is independent of the distribution of income but only the total income matters. Two examples of Gorman form utility functions are quasilinear, i.e. \( v(p, m) = v(p) + m \), and homothetic, i.e. \( v(p, m) = v(p)m \) (see Section 2.9 Appendix).
2.7 Application: Neoclassical Model of Labour Supply

Suppose consumption is financed out of two incomes: a labour income from $H$ hours worked at wage rate $w$, and a fixed non-labour income $y$. Total income is thus $m = y + wH$, so that the budget constraint is now $p.x = y + wH$. Workers of course have an upper limit on how many hours they can work, and this is denoted by $L$, i.e. $0 \leq H \leq L$.

Now $l \equiv L - H$ can be thought of as the total leisure time, consumed at price $w$. The vector $x = (x_1, x_2, ..., x_k)$ on the other hand refers to consumption of goods other than leisure, at prices $p$. Noting that the worker’s endowment income is now $y + wL$, his optimisation problem is,

$$\max_{l,x} u(l, x) \text{ subject to } wl + p.x = y + wL \text{ and } 0 \leq l \leq L$$

The solution is the Marshallian demand functions $l^*(w, p, y + wL)$ and $x^*(w, p, y + wL)$. A dual EMP at utility level $U$ leads to Hicksian solutions $h^*_l(w, p, U)$ and $h^*_x(w, p, U)$.

Consider then the Slutsky equations for consumption of leisure. When the price $p_i$ of a consumption good $i$ increases,

$$\frac{\partial l^*}{\partial p_i} = \frac{\partial h^*_l}{\partial p_i} - \frac{\partial l^*}{\partial m}x_i$$

This is as before as the endowment income $y + wL$ is independent of $p$. The endowment income is, however, dependent on the price of leisure $w$: an increase in $w$ increases the worker’s income. Hence for $l^*(w, p, y + wL)$ with $m = y + wL$,

$$\frac{\partial l^*}{\partial w} = \frac{\partial l^*}{\partial w} \bigg|_{m \text{ constant}} + \frac{\partial l^*}{\partial m} \frac{\partial m}{\partial w}$$

$$= \left( \frac{\partial h^*_l}{\partial w} - \frac{\partial l^*}{\partial m}l \right) + \frac{\partial l^*}{\partial m}L$$

$$= \frac{\partial h^*_l}{\partial w} + \frac{\partial l^*}{\partial m}H$$

This means that the effect of a rise in the wage on the amount of leisure taken is a sum of a negative term $\left( \frac{\partial h^*_l}{\partial w} \right)$ and a positive term $\left( \frac{\partial l^*}{\partial m}H \right)$, assuming that leisure is a normal good. Thus the total effect is now ambiguous. This is because on one hand a rise in the wage rate makes leisure more expensive (lost opportunity cost of earning an income), while on the other hand an increase in one’s income may increase the demand for leisure. This can lead to a Backward-bending labour supply curve.
2.8 Example

Consider the UMP

\[
\max_{x_1, x_2} \quad u(x_1, x_2) = \sqrt{x_1 x_2} \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 \leq m
\]

Form the Lagrangian (knowing that the budget constraint will bind),

\[
\max_{x_1, x_2, \lambda} \quad L(x_1, x_2, \lambda) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - \lambda (p_1 x_1 + p_2 x_2 - m)
\]

The FOCs are

\[
\begin{align*}
L_1 &= \frac{1}{2} x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - \frac{1}{2} x_2^{\frac{1}{2}} x_1^{\frac{1}{2}} - \lambda p_1 = 0 \\
L_2 &= \frac{1}{2} x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - \frac{1}{2} x_2^{\frac{1}{2}} x_1^{\frac{1}{2}} - \lambda p_2 = 0 \\
L_{\lambda} &= -p_1 x_1^{\frac{1}{2}} - p_2 x_2^{\frac{1}{2}} + m = 0
\end{align*}
\]

We know that for Cobb-Douglas function with a linear constraint, the SOC is satisfied.

Now these lead to the Marshallian demands

\[
(x_1^*, x_2^*) = \left( \frac{1}{2} \frac{m}{p_1}, \frac{1}{2} \frac{m}{p_2} \right)
\]

Then the indirect utility function is therefore,

\[
v(p_1, p_2, m) = \left( \frac{1}{2} \frac{m}{p_1} \right)^{\frac{1}{2}} \left( \frac{1}{2} \frac{m}{p_2} \right)^{\frac{1}{2}} = \frac{1}{2} \frac{m}{\sqrt{p_1 p_2}}
\]

Check Roy’s Identity:

\[
\begin{align*}
-\frac{\partial v}{\partial p_1} &= -\frac{1}{2} \frac{mp_1}{p_1^{\frac{3}{2}} p_2^{\frac{1}{2}}} = \frac{1}{2} \frac{m}{p_1} = x_1^*
\end{align*}
\]

Now try EMP,

\[
\min_{h_1, h_2} \quad p_1 h_1 + p_2 h_2 \quad \text{subject to} \quad \sqrt{h_1 h_2} \geq U
\]

The Lagrangian is this time,

\[
\min_{h_1, h_2, \lambda} \quad L(h_1, h_2, \lambda) = p_1 h_1 + p_2 h_2 + \lambda \left( U - h_1^{\frac{1}{2}} h_2^{\frac{1}{2}} \right)
\]
The FOCs are
\[ L_1 = p_1 - \lambda \frac{1}{2} h_1^* \frac{1}{2} h_2^* = 0 \]
\[ L_2 = p_2 - \lambda \frac{1}{2} h_1^* \frac{1}{2} h_2^* = 0 \]
\[ L_\lambda = U - h_1^* h_2^* = 0 \]

Which lead to the Hicksian demands
\[ (h_1^*, h_2^*) = \left( U \sqrt{\frac{p_2}{p_1}}, U \sqrt{\frac{p_1}{p_2}} \right) \]

Then the expenditure function is
\[ e(p_1, p_2, U) = p_1 U \sqrt{\frac{p_2}{p_1}} + p_2 U \sqrt{\frac{p_1}{p_2}} = 2U \sqrt{p_1 p_2} \]

Check Shephard’s Lemma:
\[ \frac{\partial e}{\partial p_1} = U \sqrt{\frac{p_2}{p_1}} = h_1^* \]

We can further check the Duality identities,
\[ m \equiv e(p_1, p_2, v(p_1, p_2, m)) = 2v(p_1, p_2, m) \sqrt{p_1 p_2} \iff v(p_1, p_2, m) = \frac{1}{2} \frac{m}{\sqrt{p_1 p_2}} \]
\[ U \equiv v(p_1, p_2, e(p_1, p_2, U)) = \frac{1}{2} e(p_1, p_2, U) \sqrt{p_1 p_2} \iff e(p_1, p_2, U) = 2U \sqrt{p_1 p_2} \]
\[ x_1^* \equiv h_1^*(p_1, p_2, v(p_1, p_2, m)) = \frac{1}{2} \frac{m}{\sqrt{p_1 p_2}} \sqrt{\frac{p_2}{p_1}} = \frac{1}{2} \frac{m}{p_1} \]
\[ h_1^* \equiv x_1^*(p_1, p_2, e(p_1, p_2, U)) = \frac{1}{2} \frac{2U \sqrt{p_1 p_2}}{p_1} = U \sqrt{\frac{p_2}{p_1}} \]

Now check Slutsky’s equation. For own-price effect on demand,
\[ \frac{\partial x_1^*}{\partial p_1} = -\frac{1}{2} \frac{m}{p_1^2} \]
\[ \frac{\partial h_1^*}{\partial p_1} = -\frac{1}{2} \frac{p_1^2}{p_2^2} U = -\frac{1}{2} \frac{p_1^2}{p_2^2} \times \frac{1}{2} \frac{m}{\sqrt{p_1 p_2}} = -\frac{m}{4} \frac{p_1}{p_1^2} \]
\[ \frac{\partial x_1^*}{\partial m} = \frac{1}{2} \frac{1}{p_1} \times \frac{1}{2} \frac{m}{p_1} = \frac{1}{4} \frac{m}{p_1^2} \]
Thus

\[
\frac{\partial h_1^*}{\partial p_1} - \frac{\partial x_1^*}{\partial m} x_1^* = -\frac{1}{4} \frac{m}{p^2} - \frac{1}{4} \frac{m}{p_1^2} = -\frac{1}{2} \frac{m}{p_1^2} = \frac{\partial x_1^*}{\partial p_1}
\]

For cross-price effect on demand,

\[
\frac{\partial x_1^*}{\partial p_2} = 0
\]

\[
\frac{\partial h_1^*}{\partial p_2} = \frac{1}{2} p_1^{\frac{1}{2}} p_2^{\frac{1}{2}} U = \frac{1}{2} p_1^{\frac{1}{2}} p_2^{\frac{1}{2}} \times \frac{1}{2} \frac{m}{\sqrt{p_1 p_2}} = \frac{1}{4} \frac{m}{p_1 p_2}
\]

\[
\frac{\partial x_1^*}{\partial m} x_2^* = \frac{1}{2} \frac{1}{p_1} \times \frac{1}{2} \frac{m}{p_2} = \frac{1}{4} \frac{m}{p_1 p_2}
\]

Thus

\[
\frac{\partial h_1^*}{\partial p_2} - \frac{\partial x_1^*}{\partial m} x_2^* = \frac{1}{4} \frac{m}{p_1 p_2} - \frac{1}{4} \frac{m}{p_1 p_2} = 0 = \frac{\partial x_1^*}{\partial p_2}
\]
2.9 Appendix

2.9.1 Quasi-Concavity and Quasi-Convexity

We will demonstrate the usefulness of the concepts of quasi-concavity and quasi-convexity using Cobb-Douglas utility functions. We saw in Chapter 1 that rational and continuous preference orderings can be represented by an ordinal utility function \( u(x) \). This means that any positive monotonic transformation of \( u(x) \) will represent the same preference orderings. So consider a Cobb-Douglas utility function \( u(x, y) = x^{1/2} y^{1/2} \). This is a nice concave function on the non-negative quadrant, as shown in Figure 2.4. Now consider the monotonic transformation \( g(u) = u^4 \). The new utility function \( u(x, y) = x^2 y^2 \) still represents the original preference orderings, but now is no longer concave (though not convex either), as seen in Figure 2.5. What this is demonstrating is that concavity and convexity of a function are cardinal and not ordinal properties. This is where properties of quasi-concavity and quasi-convexity come in useful:

Definition 2.1 A function \( f \) defined on a convex subset \( X \subset \mathbb{R}^n \) is quasi-concave if \( \forall a \in \mathbb{R} \) the upper level set

\[
C_a^+ \equiv \{ x \in X : f(x) \geq a \}
\]

is a convex set. Similarly, \( f \) is quasi-convex if \( \forall a \in \mathbb{R} \) the lower level set

\[
C_a^- \equiv \{ x \in X : f(x) \leq a \}
\]

is a convex set.

In our Cobb-Douglas examples the upper level sets are basically the sets of \((x, y)\) such that \(u(x, y)\) is greater than equal to a certain utility level, i.e. the upper contour set of the indifference curves. As can be seen from Figures 2.4 and 2.5 that these upper level sets are convex in both cases, i.e. the quasi-concavity property is preserved by a positive monotonic transformation. Thus unlike concavity and convexity, quasi-concavity and quasi-convexity are ordinal properties. Note that it is easily seen that if \( f(x) \) is concave then it is also quasi-concave, and similarly convexity implies quasi-convexity.
Figure 2.4: $u(x,y) = x^{1/2}y^{1/2}$

Figure 2.5: $u(x,y) = x^2y^2$
2.9.2 Envelope Theorem

Theorem 2.1 (Envelope Theorem for Unconstrained Optimisation) Consider the unconstrained maximisation problem of a function \( u = f(x,y;\phi) \) with respect to variables \((x,y)\) at a parameter value \(\phi\),

\[
\max_{x,y} \ u = f(x,y;\phi)
\]

Given the solution \((x^*,y^*)\) at \(\phi\), let \(v(\phi) = f(x^*(\phi),y^*(\phi);\phi)\) denote the maximum-value function. Then

\[
\frac{dv(\phi)}{d\phi} = \frac{\partial f}{\partial \phi}
\]

i.e. \((x,y)\) can be treated as constants when differentiating the maximum-value function with respect to \(\phi\).

\[\text{Proof.}\] First note that the first-order conditions of the maximisation problem are,

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0
\]

Now totally differentiate \(f(x,y;\phi)\) with respect to the parameter \(\phi\) at the optimal point \((x^*(\phi),y^*(\phi))\),

\[
\frac{dv(\phi)}{d\phi} = \frac{\partial f}{\partial x} \frac{\partial x^*(\phi)}{\partial \phi} + \frac{\partial f}{\partial y} \frac{\partial y^*(\phi)}{\partial \phi} + \frac{\partial f}{\partial \phi}
\]

But the first two terms disappear using the first-order conditions. ■

Theorem 2.2 (Envelope Theorem for Constrained Optimisation) Consider the constrained maximisation problem of a function \( u = f(x,y;\phi) \) with respect to variables \((x,y)\) at a parameter value \(\phi\),

\[
\max_{x,y} \ u = f(x,y;\phi) \quad \text{subject to} \quad g(x,y;\phi) = 0
\]

Given the solution \((x^*,y^*)\) at \(\phi\), let \(v(\phi) = f(x^*(\phi),y^*(\phi);\phi)\) denote the maximum-value function. Then

\[
\frac{dv(\phi)}{d\phi} = \frac{\partial L}{\partial \phi}
\]

where \(L(x,y,\lambda;\phi)\) is the Lagrangian function of the problem with the multiplier \(\lambda\).
Proof. First consider the Lagrangian optimisation,

\[ \max_{x, y, \lambda} L = f(x, y; \phi) - \lambda g(x, y; \phi) \]

The first-order conditions are,

\[
\begin{align*}
L_x &= \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0 \\
L_y &= \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0 \\
L_\lambda &= -g(x^*, y^*; \phi) = 0
\end{align*}
\]

Now totally differentiate \( f(x, y; \phi) \) with respect to the parameter \( \phi \) at the optimal point \( (x^*(\phi), y^*(\phi)) \),

\[
\frac{dv(\phi)}{d\phi} = \frac{\partial f}{\partial x} \frac{\partial x^*(\phi)}{\partial \phi} + \frac{\partial f}{\partial y} \frac{\partial y^*(\phi)}{\partial \phi} + \frac{\partial f}{\partial \phi}
\]

Using the first two first-order conditions this becomes,

\[
\frac{dv(\phi)}{d\phi} = \lambda \frac{\partial g}{\partial x} \frac{\partial x^*(\phi)}{\partial \phi} + \lambda \frac{\partial g}{\partial y} \frac{\partial y^*(\phi)}{\partial \phi} + \frac{\partial f}{\partial \phi}
\]

But differentiating the third first-order condition yields

\[
\frac{\partial g}{\partial x} \frac{\partial x^*(\phi)}{\partial \phi} + \frac{\partial g}{\partial y} \frac{\partial y^*(\phi)}{\partial \phi} + \frac{\partial g}{\partial \phi} = 0
\]

Substituting this back we get

\[
\frac{dv(\phi)}{d\phi} = \frac{\partial f}{\partial \phi} - \lambda \frac{\partial g}{\partial \phi} = \frac{\partial L}{\partial \phi}
\]

keeping \((x, y)\) constant. □

2.9.3 Quasilinear Utility Function

A quasilinear utility function has the following form,

\[
U(x_0, x_1, ..., x_k) = x_0 + u(x_1, ..., x_k)
\]

i.e. it is linear in one or more (in this case good 0) of the goods. Investigate this for a two-good case:

\[
\max_{x_0, x_1} U(x_0, x_1) = x_0 + u(x_1) \text{ subject to } x_0 + p_1 x_1 \leq m
\]
where $p_0$ is normalised to 1. By substitution $U(x_0, x_1) = u(x_1) + m - p_1 x_1$, which has the first-order condition $u'(x_1) = p_1$. This is independent of $m$; i.e. given relative prices the consumer will consume the same amount of $x_1$ no matter how much his income is. An example may be one’s demand for pencils: how much would your demand change as your income changes? It is most likely that any increases in income would go into consumption of other goods. Diagrammatically this implies that the indifference curves are parallel in the direction of $x_0$, i.e. the distances between any two indifference curves are the same at all points of the curves.

Note that, as in the first-order condition the demand of good 1 is only a function of the price of good 1, we can write the demand function as $x_1(p_1)$. The demand for good 0 is then $x_0 = m - p_1 x_1(p_1)$ from the budget constraint. Substituting this back into the utility function then yields the indirect utility function

$$V(p_1, m) = m - p_1 x_1(p_1) + u(x_1(p_1)) = v(p_1) + m$$

where $v(p_1) = u(x_1(p_1)) - p_1 x_1(p_1)$.

### 2.9.4 Homogeneous and Homothetic Utility Functions

**Definition 2.2** A function $u(x)$ is said to be **homogeneous of degree** $k$ if

$$u(tx) = t^k u(x) \quad \forall t > 0$$

Homogeneous functions have many useful properties. Consider the two input case:

**Properties** If $u(x_1, x_2)$ is a once differentiable homogeneous function (i.e. of any degree $k$),

1. The partial derivatives $u_1$ and $u_2$ are homogeneous of degree $k - 1$.
2. The slope of the tangent line to the level sets (i.e. the slope of the indifference curve if $u(x)$ is a utility function) is constant along each ray from the origin.
3. (Euler’s Theorem) For all $(x_1, x_2)$,

$$x_1 \frac{\partial u}{\partial x_1}(x) + x_2 \frac{\partial u}{\partial x_2}(x) = ku(x)$$
For proofs see Simon and Blume Ch 20.1. Property 2 implies that if $u(x)$ is a utility function, then the MRS is constant along rays from the origin, and that the income elasticity of demand is identically 1.

One problem is though that homogeneity is not an ordinal property. For example $u(x_1, x_2) = x_1x_2$ is clearly a homogeneous function, but even the simplest monotonic transformation $g(u(x)) = u(x) + 1$ makes the resulting function $v(x_1, x_2) = x_1x_2 + 1$ non-homogeneous. Instead we can define another set of functions,

**Definition 2.3** A function $v(x)$ is said to be **homothetic** if it is a monotone transformation of a homogeneous function, that is, if there is a monotonic transformation $g(z)$ and a homogeneous function $u(x)$ such that $v(x) = g(u(x))$ $\forall x$ in the domain.

Clearly then a monotonic transformation of a homothetic function is homothetic, and hence the property is now ordinal. As we also know that the MRS is an ordinal concept (note marginal utility is not), homothetic functions also have property that the slopes of the indifference curves are constant along each ray from the origin. In fact the converse is also true, i.e. if the slopes of the indifference curves are constant along each ray from the origin, then the function is homothetic (but not necessarily homogeneous). Thus we have another definition for homothetic functions,

**Definition 2.4** A utility function $v(x)$ is **homothetic** if the MRS is homogeneous of degree zero.

Now if the MRS is constant no matter what the income level is, it implies that the consumer will spend the same proportion of his income on each goods. This means that his demand for each good will be linear to his income (i.e. double the income, double the demand). Moreover if we define homotheticity by restricting the homogeneity of the underlying function $u(x)$ to degree 1 (as Varian does), then we know that doubling consumption of each good doubles the level of utility. Thus utility will also be linear to his income. Hence one can write the indirect utility function in the form $v(p, m) = v(p)m$ where $v(p) = v(p, 1)$. 

Chapter 3

Producer Theory


3.1 Technology

A **production plan** is a list of net outputs of various goods, represented by a vector $y \in \mathbb{R}^n$ where $y_j$ is negative (positive) if the $j^{th}$ good serves as a net input (output). The set of all technologically feasible production plans is the firm’s **production possibility set**, which is denoted by $Y \in \mathbb{R}^n$.

Sometimes production possibilities are written in a manner which seems more intuitive, at least for plans which involve only one output and one or more inputs. In this case the output can be written as a scaler $y$ and the input as a vector $x$, with the production plan being $(y, -x)$. In this case we can define an **input requirement set**, as the set of all input bundles that produce at least $y$,

$$V(y) = \{ x \in \mathbb{R}^n_+ : (y, -x) \in Y \}$$

The **isoquant** $Q(y)$ is the set of all input bundles that produce exactly $y$,

$$Q(y) = \{ x \in \mathbb{R}^n_+ : x \in V(y) \text{ and } x \notin V(y') \text{ for } y' > y \}$$

The examples of isoquants: Cobb-Douglas $Q(y) = \{ (x_1, x_2) \in \mathbb{R}^2_+ : y = x_1^{a} x_2^{1-a} \}$; Leontief $Q(y) = \{ (x_1, x_2) \in \mathbb{R}^2_+ : y = \min\{ax_1, bx_2\} \}$. 
3.2 Possible Restrictions on Technology Set

1. **Monotonicity** - \( \{ x \in V(y) \text{ and } x' \geq x \} \) implies \( x' \in V(y) \), i.e. can always produce the same amount of output by larger inputs.

2. **Convexity** - \( V(y) \) is a convex set, i.e. if \( x, x' \in V(y) \) then \( tx + (1-t)x' \in V(y) \) \( \forall 0 \leq t \leq 1 \). This is equivalent to saying that the associated production function is quasi-concave, as \( V(y) = \{ x : f(x) \geq y \} \) which is just the upper contour set of \( f(x) \).

3. **Regularity** - \( V(y) \) is a closed, non-empty set \( \forall y \geq 0 \). Non-empty set means that there is some conceivable way of producing any given level of output.

3.3 Technical Rate of Substitution

*Technical rate of substitution* describes how much more of input \( i \) one must have to maintain the same output if input \( j \) is decreased by 1. This is just the slope of the isoquant surface. For example for two inputs, the definition of an isoquant is \( f(x_1, x_2) \equiv y \) and thus,

\[
dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0
\]

or

\[
\frac{dx_2}{dx_1} = -\frac{\partial f/\partial x_1}{\partial f/\partial x_2}
\]

Thus TRS is the ratio of the marginal productivities.

3.4 Returns to Scale

A technology is said to have,

1. **Constant returns to scale** if \( f(tx) = tf(x) \) \( \forall t \geq 0 \).

2. **Increasing returns to scale** if \( f(tx) > tf(x) \) \( \forall t > 1 \).

3. **Decreasing returns to scale** if \( f(tx) < tf(x) \) \( \forall t > 1 \).
Note the difference in the restrictions on $t$. Note also for IRS the PPF is convex (or PPS is concave) and thus there is no competitive producer theory for IRS production technologies. Alternative definitions for CRS are: \( \forall y \in Y, ty \in Y \forall t \geq 0, \) or \( \forall x \in V(y), tx \in V(ty) \forall t \geq 0. \)

Example (Cobb-Douglas): \( y = x_1^a x_2^b \Rightarrow f(tx_1, tx_2) = (tx_1)^a (tx_2)^b = t^{a+b} x_1^a x_2^b = t^{a+b} f(x_1, x_2). \) So CRS when \( a + b = 1, \) IRS when \( > \) and DRS when \( <. \)

### 3.5 Profit Maximisation Problem

Here we consider a competitive, or a price-taker firm. Let \((y, -x)\) denote a production plan, with the associated vector of output prices \(p\) and input prices \(w\). In general prices may depend on the production plan, so we have \(p(y, -x)\) and \(w(y, -x)\). However if the markets are perfectly competitive, i.e. the representative firm is too small to affect the prices of inputs or outputs through its production plans, then \(p(y, -x) = p\) and \(w(y, -x) = w.\)

The profits for a given production plan are the difference between revenue and cost, i.e. \(py - wx.\) The profit maximisation problem is to maximise this subject to the technological constraints, i.e. given \((p, w),\)

\[
\max_{y, x} \pi(y, -x) = py - wx \text{ subject to } (y, -x) \in Y \tag{PMP}
\]

Note that for the one-output good case the constraint becomes simply \(f(x) = y.\) Now for an interior solution to exist, typically it is sufficient that the technology be strictly convex, regular, and \(p, w > 0.\) The solution \((y^*(p, w), -x^*(p, w))\) is the output supply and factor demand functions. The optimal value function is the profit function \(\pi(y^*(p, w), -x^*(p, w)) = \pi(p, w),\) i.e. the maximum profit that the firm can make at prices \((p, w).\) We now investigate the properties of these functions.

#### 3.5.1 Demand and Supply Functions

1. Both factor demand and supply functions are homogeneous of degree 0, i.e. \(x_i^*(tp, tw) = x_i^*(p, w), y_j^*(tp, tw) = y_j^*(p, w) \forall t > 0.\) (Intuitive)
2. The factor demand curve is downward-sloping, i.e. \( \frac{\partial x_i}{\partial w_i} < 0 \) \( \forall i \).

3. The change in a firm’s demand for input \( i \) when the price of input \( j \) changes equals the change in the firm’s demand for input \( j \) when the price of input \( i \) changes, i.e. \( \frac{\partial x_i}{\partial w_j} = \frac{\partial x_j}{\partial w_i} \) \( \forall i \neq j \).

**Proof of Properties 2 & 3.** Let us consider a one-good output, two-good input case, with the price of the output good normalised to \( p = 1 \). Then for the production function \( f(x_1, x_2) \), the first-order conditions are,

\[
\frac{\partial f(x_1(w_1, w_2), x_2(w_1, w_2))}{\partial x_1} = w_1 \\
\frac{\partial f(x_1(w_1, w_2), x_2(w_1, w_2))}{\partial x_2} = w_2 
\]

i.e. marginal productivity must equal marginal cost at all input prices \( (w_1, w_2) \). So when \( (w_1, w_2) \) change \( (x_1, x_2) \) must change to keep these equalities, i.e.,

\[
f_{11} \frac{\partial x_1}{\partial w_1} + f_{12} \frac{\partial x_2}{\partial w_1} = 1, \quad f_{21} \frac{\partial x_1}{\partial w_1} + f_{22} \frac{\partial x_2}{\partial w_1} = 0 \\
f_{11} \frac{\partial x_1}{\partial w_2} + f_{12} \frac{\partial x_2}{\partial w_2} = 0, \quad f_{21} \frac{\partial x_1}{\partial w_2} + f_{22} \frac{\partial x_2}{\partial w_2} = 1
\]

or in a matrix form,

\[
\begin{pmatrix}
  f_{11} & f_{12} \\
  f_{21} & f_{22}
\end{pmatrix}
\begin{pmatrix}
  \frac{\partial x_1}{\partial w_1} \\
  \frac{\partial x_2}{\partial w_1} \\
  \frac{\partial x_1}{\partial w_2} \\
  \frac{\partial x_2}{\partial w_2}
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix}
\]

The first matrix is the Hessian matrix which we know for a regular maximum (i.e. ruling out zero second derivative) to be symmetric negative definite. Therefore it is non-singular and can be inverted,

\[
\begin{pmatrix}
  \frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} \\
  \frac{\partial x_2}{\partial w_1} & \frac{\partial x_2}{\partial w_2}
\end{pmatrix} =
\begin{pmatrix}
  f_{11} & f_{12} \\
  f_{21} & f_{22}
\end{pmatrix}^{-1}
\]

The matrix on the left-hand side is the **substitution matrix**, i.e. it describes how the firm substitutes one input for another as the factor prices change, which according to this result is simply the inverse of the Hessian matrix. Now as the inverse of a symmetric negative definite matrix is also symmetric negative definite, we have the results that the diagonal elements \( \frac{\partial x_1}{\partial w_1}, \frac{\partial x_2}{\partial w_2} < 0 \), and that \( \frac{\partial x_1}{\partial w_2} = \frac{\partial x_2}{\partial w_1} \).
3.5.2 Profit Function

\( \pi(p, w) \) possesses some important properties that follow directly from its definitions (i.e. no assumptions about convexity, monotonicity or other sorts of regularity is necessary),

1. Non-decreasing in \( p \), non-increasing in \( w \). Thus if \( p_j' \geq p_j \) for all outputs and \( w_i' \leq w_i \) for all inputs, then \( \pi(p', w') \geq \pi(p, w) \).

2. Homogeneous of degree 1 in \((p, w)\) for \((p, w) \not\geq 0\), i.e. unchanged in real terms.

3. Convex in \((p, w)\) for \((p, w) > 0\), i.e. if \( p_3 = tp_1 + (1-t)p_2 \) and \( w_3 = tw_1 + (1-t)w_2 \) for \( 0 \leq t \leq 1 \), then \( \pi(p_3, w_3) \leq t\pi(p_1, w_1) + (1-t)\pi(p_2, w_2) \).

**Proof.** By definition of profit maximisation,

\[
\begin{align*}
p_1y_1 - w_1x_1 &\geq tp_1y_1 - tw_1x_1 \geq tp_1y_3 - tw_1x_3 \\
p_2y_2 - w_2x_2 &\geq (1-t)p_2y_2 - (1-t)w_2x_2 \geq (1-t)p_2y_3 - (1-t)w_2x_3
\end{align*}
\]

Add the two,

\[
t\pi(p_1, w_1) + (1-t)\pi(p_2, w_2) \geq \{tp_1 + (1-t)p_2\}y_3 - \{tw_1 + (1-t)w_2\}x_3 = \pi(p_3, w_3)
\]

i.e. always can do at least as well with a possibility of pursuing the optimal input/output choices at the extreme prices. (See Fig 3.1)  

4. Continuous in \((p, w)\) when \( \pi(p, w) \) is well-defined and \( (p, w) \not\geq 0 \).

Now similar to Roy’s Identity in UMP, we have the following,

**Proposition 3.1 (Hotelling’s Lemma)** Suppose \( \pi(p, w) \) is differentiable in \((p, w)\). Let \((y^*, -x^*)\) be the optimal production plan at that price. Then,

\[
y_j = \frac{\partial \pi}{\partial p_j} \text{ and } -x_i = \frac{\partial \pi}{\partial w_i} \quad \forall i, j = 1, 2, ...
\]
Figure 3.1: Single output good profit function

**Proof.** We prove this for a one-output good case using the Envelope Theorem for unconstrained optimisation. By substitution the PMP becomes simply

$$\max_x \pi(x) = p.f(x) - w.x$$

for which we obtain the factor demands $x^*(p, w)$ and the profit function $\pi(p, w)$. Now to investigate what happens to the optimal-value function $\pi(p, w)$ as the parameters $(p, w)$ change, the Envelope Theorem states that we can treat the choice variables $x^*(p, w)$ as constants, and hence,

$$\frac{\partial \pi}{\partial p} = f(x^*) = y^*$$

$$\frac{\partial \pi}{\partial w_i} = -x_i^*$$

Intuitively this is stating that a unit increase in $p_j$ (while keeping all other prices constant - hence the partial derivative) has two effects: a direct effect where profit increases by $y_j$, and an indirect effect where the firm re-chooses the optimal production plan $(y^*, -x^*)$. However at the optimal point for an infinitesimal change in $p_j$ the latter effect is zero, and thus the change in profit will equal $y_j$. A similar argument can be made for the effect of a change in $w_i$.

This relationship between $\pi(p, w)$ and $(y, -x)$ allows us to state, using the already stated properties of the profit function, the following properties for the supply and factor
demand functions,

1. $\pi(p, w)$ is homogeneous of degree 1 in $(p, w)$ implies that output supply and factor demand functions are h.d. 0.

2. As the matrix of the second-order derivatives (the Hessian matrix) of a convex function is positive semi-definite, and $\pi(p, w)$ is a convex function in $(p, w)$, it follows that the matrices of price derivatives of the supply and factor demand functions are positive semi-definite. Moreover Hotelling’s Lemma implies that this matrix is symmetric. For example for a two-good case,

\[
\begin{pmatrix}
\frac{\partial^2 \pi}{\partial p_1^2} & \frac{\partial^2 \pi}{\partial p_1 \partial p_2} \\
\frac{\partial^2 \pi}{\partial p_2 \partial p_1} & \frac{\partial^2 \pi}{\partial p_2^2}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\
\frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2}
\end{pmatrix}
\]

which is the substitution matrix. The symmetry $\frac{\partial y_j}{\partial p_k} \forall j, k$ provides the strongest prediction of PMP. The positive semi-definiteness implies that the diagonal elements are non-negative, i.e. an increase in own price will always weakly increases $y$ or weakly decrease $x$.

3.5.3 Example

Consider a Cobb-Douglas production function $y = L^a K^b$ where $a + b < 1$. Then the PMP is,

\[
\max_{L,K} \quad py - wL - rK \quad \text{subject to} \quad y = L^a K^b
\]

By substitution this is,

\[
\max_{L,K} \quad pL^a K^b - wL - rK
\]

and therefore the FOCs are,

\[
\frac{\partial}{\partial L} : \quad a p L^{a-1} K^{b} - w = 0 \\
\frac{\partial}{\partial K} : \quad b p L^{a} K^{b-1} - r = 0
\]
Solving this simultaneous equations yields the factor demand functions,

\[ L^* = \left( \frac{a}{w} \right)^{1-b} \left( \frac{b}{r} \right)^b p \]

\[ K^* = \left( \frac{a}{w} \right)^a \left( \frac{b}{r} \right)^{1-a} p \]

Check the properties of these factor demands. First of all they should be decreasing in their own prices,

\[ \frac{\partial L^*}{\partial w} = -\left( \frac{1 - b}{1 - a - b} \right) \frac{L^*}{w} \]

\[ \frac{\partial K^*}{\partial r} = -\left( \frac{1 - a}{1 - a - b} \right) \frac{K^*}{r} \]

which for \( a + b < 1 \) are indeed negative. Secondly the cross-price effects must be equal,

\[ \frac{\partial L^*}{\partial r} = -\frac{1}{1 - a - b} \left\{ \left( \frac{a}{w} \right)^{1-b} \left( \frac{b}{r} \right)^{1-a} p \right\} = \frac{\partial K^*}{\partial w} \]

The profit-maximising output supply function is given by,

\[ y^* = L^* a K^* b \]

\[ = \left\{ \left( \frac{a}{w} \right)^a \left( \frac{b}{r} \right)^b p^{a+b} \right\} \]

Check that this is weakly increasing in its own-price,

\[ \frac{\partial y^*}{\partial p} = \frac{a + b}{1 - a - b} \frac{y^*}{p} > 0 \]

Finally the profit function is (after a lot of fiddly maths),

\[ \pi(p, w, r) = py^* - wL^* - rK^* \]

\[ = (1 - a - b) \left\{ \left( \frac{a}{w} \right)^a \left( \frac{b}{r} \right)^b p \right\} \]

Note that this is only strictly positive for \( a + b < 1 \), i.e. PMP is only feasible for DRS production functions.
Now check Hotelling’s Lemma,

\[
\frac{\partial \pi}{\partial p} = \left\{ \left( \frac{a}{w} \right)^a \left( \frac{b}{r} \right)^b \right\}^{\frac{1}{1-a-b}} = y^* \\
\frac{\partial \pi}{\partial w} = -\left\{ \left( \frac{a}{w} \right)^{1-b} \left( \frac{b}{r} \right)^b \right\}^{\frac{1}{1-a-b}} = -L^* \\
\frac{\partial \pi}{\partial r} = -\left\{ \left( \frac{a}{w} \right)^a \left( \frac{b}{r} \right)^{1-a} \right\}^{\frac{1}{1-a-b}} = -K^*
\]

as required.

3.6 Cost Minimisation Problem

The problem with PMP is that it is only applicable if (a) the firm operates competitively (i.e. the firms are price-takers), and (b) the production possibility set is convex. CMP on the other hand permits analyses of problems such as that of a natural monopoly operating in competitive input markets, by modelling the behaviour of the firm as that of minimising the cost of producing a given output. The firm’s optimisation problem is given by,

\[
\min_x \ w \cdot x \ \text{subject to} \ x \in V(y) \ \text{(CMP)}
\]

Once again we will consider a one-output good case for simplicity, in which case the constraint is \( f(x) = y \) for a given output \( y \). This can be used as usual by using a Lagrangian optimisation,

\[
\min_{x, \lambda} \ L(x, \lambda) = w \cdot x + \lambda(y - f(x)) \quad (3.1)
\]

The FOCs are

\[
w_i - \lambda \frac{\partial f(x^*)}{\partial x_i} = 0 \ \forall i = 1, ..., n \]

\[
f(x^*) = y
\]

which immediately implies that

\[
\frac{w_i}{w_j} = \frac{\partial f(x^*)/\partial x_i}{\partial f(x^*)/\partial x_j} \ \forall i, j = 1, ..., n
\]
i.e. the technical rate of substitution equals the price ratio, or the economic rate of substitution - at what rate two factors can be substituted at a constant cost. This, together with the technological constraint then yields the optimal input levels \( x^*(w, y) \) given the required output \( y \); and hence these are called the conditional factor demand functions.

Now the optimal value function for the CMP is the cost function \( c(w, y) \), which is the minimum cost required to produce the given output \( y \) at prices \( w \). \( c(w, y) \) possesses the following properties that follow directly from its definitions,

1. Non-decreasing in \( w \). Thus if \( w' \geq w \), then \( c(w', y) \geq c(w, y) \).

2. Homogeneous of degree 1 in \( w \), i.e. \( c(tw, y) = tc(w, y) \) \( \forall t > 0 \). This is stating that the composition of the cost minimising bundle is unchanged with a scaler multiplication of factor prices. This follows from the fact that the conditional factor demands depend on relative prices.

3. Concave in \( w \), i.e. \( c(tw_1 + (1-t)w_2, y) \geq tc(w_1, y) + (1-t)c(w_2, y) \) \( \forall 0 \leq t \leq 1 \).

**Proof.** By definition of cost minimisation, where \( w_3 = tw_1 + (1-t)w_2 \),

\[
\begin{align*}
  w_1.x_1 & \leq w_1.x_3 \Rightarrow tw_1.x_1 \leq tw_1.x_3 \\
  w_2.x_2 & \leq w_2.x_3 \Rightarrow (1-t)w_2.x_2 \leq (1-t)w_2.x_3
\end{align*}
\]

Add the two,

\[
tc(w_1, y) + (1-t)c(w_2, y) \leq \{tw_1 + (1-t)w_2\}.x_3 = c(w_3, y)
\]

i.e. always can do at least as well with a possibility of pursuing the optimal input choices at the extreme prices. (See Fig 3.2)

4. Continuous in \( w \) for \( w \gg 0 \).

**Proposition 3.2 (Shephard’s Lemma)** Suppose \( c(w, y) \) is differentiable in prices \( w \) for a fixed \( y \). Let \( x^* \) denote the optimal input vector at that price. Then,

\[
x^*_i = \frac{\partial c}{\partial w_i} \quad \forall i = 1, 2, ...
\]
Proof. This time we use the Envelope Theorem for constrained optimisation. Here our Lagrangian is given by (3.1). Thus when factor price $w_i$ changes its effect on the cost function $c(w,y)$ is given by,

$$
\frac{dc(w,y)}{dw_i} = \frac{\partial L}{\partial w_i} = x^*_i
$$

Once again this relationship between $c(w,y)$ and $x$ allows us to state, from the properties of the cost function, the following properties for the cost function and the conditional factor demand functions,

1. $c(w,y)$ is non-decreasing in factor prices. This follows immediately from Shephard’s Lemma above, where $x^*_i(w,y) \geq 0$.

2. $c(w,y)$ is homogeneous of degree 1 in $w$ $\Rightarrow$ $x^*_i(w,y)$ are h.d.0.

3. $c(w,y)$ is concave in $w$ $\Rightarrow$ the matrix of price derivatives of the factor demand functions

$$
\begin{pmatrix}
\frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} \\
\frac{\partial x_2}{\partial w_1} & \frac{\partial x_2}{\partial w_2}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial^2 c}{\partial w_1^2} & \frac{\partial^2 c}{\partial w_1 \partial w_2} \\
\frac{\partial^2 c}{\partial w_2 \partial w_1} & \frac{\partial^2 c}{\partial w_2^2}
\end{pmatrix}
$$

is symmetric negative semi-definite. It follows then that the cross-price effects are symmetric and the own-price effects are non-positive, i.e.

$$
\frac{\partial x_i}{\partial w_j} = \frac{\partial x_j}{\partial w_i} \forall i,j \quad \text{and} \quad \frac{\partial x_i}{\partial w_i} \leq 0 \quad \forall i
$$
3.6.1 Example: Cobb-Douglas Technology

Consider the CMP for the Cobb-Douglas production function \( y = L^a K^b \),

\[
\min_{L,K} \quad wL + rK \quad \text{subject to} \quad L^a K^b \geq Y
\]

for a given output level \( Y \). The Lagrangian is,

\[
\min_{L,K} \quad \mathcal{L}(L, K, \lambda) = wL + rK + \lambda(Y - L^a K^b)
\]

The FOCs are,

\[
\begin{align*}
\mathcal{L}_L &= w - \lambda \frac{aY}{L^*} = 0 \\
\mathcal{L}_K &= r - \lambda \frac{bY}{K^*} = 0 \\
\mathcal{L}_\lambda &= Y - L^{*a} K^{*b} = 0
\end{align*}
\]

Solving for \( \lambda \) yields,

\[
\lambda = \frac{1}{k} w^{\frac{a}{\alpha + \beta}} r^{\frac{\beta}{alpha + \beta}} Y^{\frac{1}{alpha + \beta}} \quad \text{where} \quad k = a^{\frac{\alpha}{alpha + \beta}} b^{\frac{\beta}{alpha + \beta}}
\]

Substituting this back into the first two FOCs yields the factor demand functions given \( Y \),

\[
\begin{align*}
L^* &= \frac{a}{k} \left( \frac{r}{w} \right)^{\frac{b}{\alpha + \beta}} Y^{\frac{1}{\alpha + \beta}} \\
K^* &= \frac{b}{k} \left( \frac{w}{r} \right)^{\frac{a}{\alpha + \beta}} Y^{\frac{1}{\alpha + \beta}}
\end{align*}
\]

It is easy to see that these conditional demands are homogeneous of degree 0, and that the own-price effects are non-positive. Check the symmetry in the cross-price effects:

\[
\frac{\partial L^*}{\partial r} = \frac{ab}{(a + b) k} w^{-\frac{b}{\alpha + \beta}} r^{-\frac{a}{\alpha + \beta}} Y^{\frac{1}{\alpha + \beta}} = \frac{\partial K^*}{\partial w}
\]

The cost function is then,\(^1\)

\[
c(w, r, Y) = w L^* + r K^* = w \left( \frac{a + b}{k} \right) W^{\frac{a}{\alpha + \beta}} r^{\frac{\beta}{\alpha + \beta}} Y^{\frac{1}{\alpha + \beta}}
\]

\(^1\)In Varian Ch 4.3 example the cost function (for \( A = 1 \)) is given as \( \left[ \left( \frac{w}{a} \right)^{\frac{a}{\alpha + \beta}} + \left( \frac{w}{b} \right)^{\frac{b}{\alpha + \beta}} \right] W^{\frac{a}{\alpha + \beta}} r^{\frac{\beta}{\alpha + \beta}} Y^{\frac{1}{\alpha + \beta}} \).

One can check that this expression and ours are equivalent by substituting the expression for \( k \).
Again it is easy to see that this function is both non-decreasing, and homogeneous of degree 1, in \((w, r)\). Note that the marginal cost is then,

\[
MC(w, r, Y) = \frac{\partial c(w, r, Y)}{\partial Y} = \frac{1}{k} w^{\frac{a}{a+b}} r^{\frac{b}{a+b}} Y^{\frac{1}{a+b} - a - b}
\]

demonstrating that the Lagrange multiplier in the CMP is simply the marginal cost.\(^2\)

Check also Shephard’s Lemma,

\[
\frac{\partial c}{\partial w} = \frac{a}{k} \left( \frac{w}{r} \right)^{\frac{b}{a+b}} Y^{\frac{1}{a+b}} = L^* \\
\frac{\partial c}{\partial r} = \frac{b}{k} \left( \frac{w}{r} \right)^{\frac{a}{a+b}} Y^{\frac{1}{a+b}} = K^*
\]

Finally note that unlike with the PMP example, none of the above results require any constraints on the size of \(a + b\). Thus all of the results are valid whether the production function is DRS, CRS or IRS.

### 3.6.2 Example: Leontief Technology

The Leontief technology is given by

\[
f(x_1, x_2) = \min[ax_1, bx_2]
\]

The isoquants are then right-angles which are not differentiable at the corner. Thus one cannot use the normal first-order condition to find the solution. However we know that the firm will not waste any input with a positive price, so it must operate at the corner where \(y = ax_1 = bx_2\). Hence the conditional factor demands are

\[
(x_1^*, x_2^*) = \left( \frac{y}{a}, \frac{y}{b} \right)
\]

and the cost function is given by

\[
c(w_1, w_2, y) = w_1 \frac{y}{a} + w_2 \frac{y}{b} = y \left( \frac{w_1}{a} + \frac{w_2}{b} \right)
\]

\(^2\)This actually follows directly from the Envelope Theorem. For our constrained optimisation, in considering changes in the parameter \(y\),

\[
\frac{dc(w, y)}{dy} = \frac{\partial L}{\partial y} = \lambda
\]
3.6.3 Example: Linear Technology

Consider a linear technology

\[ f(x_1, x_2) = ax_1 + bx_2 \]

This implies that the two input goods are perfect substitutes, and the firm will use whichever is cheaper. Hence immediately we know that the conditional factor demands are

\[
(x_1^*, x_2^*) = \begin{cases} 
(\frac{y}{a}, 0) & \text{if } \frac{w_1}{a} < \frac{w_2}{b} \\
(0, \frac{y}{b}) & \text{if } \frac{w_1}{a} > \frac{w_2}{b} \\
\{(x_1, x_2) : ax_1 + bx_2 = y; \ x_1, x_2 \geq 0\} & \text{if } \frac{w_1}{a} = \frac{w_2}{b}\end{cases}
\]

The cost function is then

\[
c(w_1, w_2, y) = \min \left[ \frac{w_1}{a}, \frac{w_2}{b} \right] y
\]

Here it is very important to note that this problem cannot be solved using the equality constraint Lagrangians optimisation. The solution is a corner solution as long as \( \frac{w_1}{a} \neq \frac{w_2}{b} \) rather than an interior solution, and thus the FOC will not be satisfied. For this then the Kuhn-Tucker conditions are required (see Varian p.57-8).

3.7 Duality

We have seen that given any technology, it is straightforward to derive its cost function by simply solving the CMP. The question can be raised then as to whether this process can be reversed, i.e. given a cost function, can we “solve for” a technology that could have generated that cost function. If this is so then the cost function will contain essentially the same information that the production function contains. Then any concept defined in terms of the properties of the production function has a “dual” definition in terms of the properties of the cost function and vice versa. This is the principle of duality.

The answer to this question is “yes, provided the technology is convex and monotonic.” To see this define a special form of the input requirement set for a given \( c(w, y) \),

\[
V^*(y) = \{ x : w \cdot x \geq w \cdot x(w, y) = c(w, y) \ \forall w \geq 0\}
\]
i.e. all the possible input factor combinations that would produce at least $y$ at a cost greater than or equal to the minimised cost function $c(w, y)$ for all factor prices $w$. We can now try and relate this input requirement set, constructed from $c(w, y)$, to the usual input requirement set $V(y)$ constructed from $f(x)$. The claim is, if $V(y)$ represents a convex and monotonic technology, then the two are identical. This is because for a convex, monotonic input requirement set $V(y)$, each point on the boundary is a cost-minimising factor demand for some price vector $w \geq 0$. If the technology is non-convex or non-monotonic, $V^*(y)$ will be a convexified, monotonised version of $V(y)$. The difference $V^*(y) - V(y)$ is an area that will not be a solution to a CMP and therefore contains no economically relevant information. Thus the cost function summarises all of the economically relevant aspects of the firm’s technology.

### 3.7.1 Example: Leontief Technology

Consider a Leontief technology $f(x_1, x_2) = \min[x_1, x_2]$. We know from Section 3.6.2 that the conditional factor demands are

$$ (x^*_1, x^*_2) = (y, y) $$

and the cost function is given by

$$ c(w_1, w_2, y) = w_1x_1 + w_2x_2 = (w_1 + w_2)y $$
This is the minimised cost of producing $y$ for an input price vector $(w_1, w_2)$. Conversely if we fix the cost at $c(w_1, w_2, y)$, then one can draw the isocost lines for each values of $(w_1, w_2)$ for given $y$ (i.e. inputs $(x_1, x_2)$ that can produce $y$ at that cost when input prices are $(w_1, w_2)$),

$$w_1x_1 + w_2x_2 = c(w_1, w_2, y) \iff x_2 = \frac{c(w_1, w_2, y)}{w_2} - \frac{w_1}{w_2}x_1$$

For our Leontief technology the isocosts are then,

$$x_2 = \left(\frac{w_1 + w_2}{w_2}\right)y - \frac{w_1}{w_2}x_1 = \left(\frac{w_1}{w_2} + 1\right)y - \frac{w_1}{w_2}x_1$$

Now for a given value of $y$, these isocosts will pass through the point $(x_1, x_2) = (y, y)$ for all values of $(w_1, w_2)$ (can check by substituting $x_1 = x_2 = y$ in the equation). Also at the extremes $(w_1, w_2) = (0, w_2)$ and $(w_1, 0)$ the isocosts are horizontal $x_2 = y$ and vertical lines $x_1 = y$ respectively. Hence the boundary traced out by isocosts for different values of $(w_1, w_2)$ is a right-angle shape with the corner at $(y, y)$, which is equivalent to the production function $\min[x_1, x_2]$, as predicted.

### 3.8 Aggregation

Define the aggregate supply of competitive producers by the aggregate production possibility set,

$$Y = \sum_{f=1}^{F} Y^f = \left\{ y \in \mathbb{R}^N : y = \sum_{f=1}^{F} y^f \text{ for some } y^f \in Y^f \right\}$$

where $y^f$ are the production plans for firm $f$, and $y$ is the associated aggregated production plan. Then,

**Proposition 3.3** An aggregate production plan $y$ maximises the aggregate profit $p.y$ if and only if each firm’s production plan $y^f$ maximises its individual profit $p.y^f$.

**Proof.** First check the “only if” ($\Rightarrow$) by contradiction. Suppose that $y = \sum_{f=1}^{F} y^f$ maximises aggregate profit but some firm $k$ could have higher profits by choosing $y^k$. Then
aggregate profits could be higher by choosing plan $y^k$ for firm $k$ and the same $\{y^f\}$ for firms $f \neq k$ which contradicts the fact that $y$ is the maximising production plan.

Next check the "if" ($\Leftarrow$). Let $\{y^f\}$ for firms $f = 1, ..., F$ be a set of profit-maximising production plans for the individual firms. Suppose that $y = \sum_{f=1}^{F} y^f$ is not profit-maximising at prices $p$. This means that there is some other production plan $y' = \sum_{f=1}^{F} y'^f$ with $y'^f \in Y^f$ that has higher aggregate profits,

$$p \cdot \sum_{f=1}^{F} y'^f > p \cdot \sum_{f=1}^{F} y^f \quad \iff \quad \sum_{f=1}^{F} p \cdot y'^f > \sum_{f=1}^{F} p \cdot y^f$$

But for the right-hand side inequality to be valid there must be at least one firm for whom $p \cdot y'^f > p \cdot y^f$. This is again a contradiction. ■

This implies then that, unlike Consumer Theory, for Producer Theory the predictions of the theory carries through to the aggregate level.
4.1 Monopoly Profit Maximisation

A monopoly is a price-maker, as opposed to a competitive firm who is a price-taker. This is because a monopoly firm can now affect the output good price by varying its output level. Thus its maximisation problem is now,

$$\max_y \pi(y) = p(y)y - c(y)$$

where $p(y)$ is the inverse demand function. The FOC is then

$$p(y) + p'(y)y = c'(y)$$

i.e. the optimal output level is where the marginal revenue of an increase in output equals its marginal cost. Here unlike with a competitive firm, the marginal revenue now includes the effect of a drop in the price due to the increased output. Now this can be rewritten as,

$$p(y) \left( 1 - \frac{1}{\epsilon(y)} \right) = c'(y)$$

where $\epsilon(y) = -\frac{p}{y} \frac{dy}{dp}$ is the price elasticity of demand facing the monopolist. It follows then that at the optimal level of output, the elasticity of demand must be greater than 1, as otherwise the marginal revenue will be a negative value.
4.1.1 Example: Linear Demand

Consider a downward-sloping linear demand function for a monopoly

\[ y(p) = \frac{a}{b} - \frac{1}{b}p \quad \text{where} \quad a, b > 0 \]

The marginal revenue line is then twice as steep as the indirect demand,

\[ p(y) = a - by \]
\[ MR = a - 2by \]

The price elasticity of demand is:

\[ -\frac{p}{y} \frac{dy}{dp} = -\left( \frac{a - by}{y} \right) \left( -\frac{1}{b} \right) = \frac{a}{by} - 1 \]

This is decreasing in \( y \), and the unit elasticity occurs at \( y = \frac{a}{2b} \). Hence the optimal output level for the monopoly will be less than \( \frac{a}{2b} \). (Can you show this diagrammatically?)

4.2 Monopoly Inefficiency

We can see immediately from (4.1) that \( p(y_m) > c'(y_m) \), i.e. under monopoly price exceeds marginal cost. Thus the monopoly output is inefficient. To investigate this further, assume for simplicity that the market demand curve \( x(p) \) is generated by maximising the utility of a single representative consumer with a quasi-linear utility function,

\[ U(x, y) = u(x) + y \]

The \( y \)-good is a proxy for “everything else”, which can be most conveniently thought of as money left over for purchasing other goods after the consumer makes the optimal expenditure on the \( x \)-good. The price of \( y \)-good is normalised to 1. Solving the Lagrangian

\[ L = \{u(x) + y\} - \lambda(px + y - m) \]

\[ \Rightarrow \begin{align*}
L_x &= u'(x) - \lambda p = 0 \\
L_y &= 1 - \lambda = 0
\end{align*} \]

yields the inverse demand for good \( x \),

\[ u'(x) = p \]
Now consider the social welfare which is a sum of the consumer surplus and the monopolist’s profit,

\[ W(x) = [u(x) - p(x)x] + [p(x)x - c(x)] = u(x) - c(x) \]

The socially optimal level of output is then defined by

\[ W'(x^*) = u'(x^*) - c'(x^*) = p(x^*) - c'(x^*) = 0 \]

However the monopoly will choose an output where

\[ p(x_m) + p'(x_m)x_m = c'(x_m) \]

At this level the derivative \( W'(x_m) \) is,

\[ W'(x_m) = p(x_m) - c'(x_m) = -p'(x_m)x_m > 0 \]

This is positive as \( p'(x) < 0 \). Thus in a monopoly equilibrium, increasing output will increase utility.

The welfare loss from this quantity distortion (see Fig 4.1) is known as the monopoly deadweight loss, and is measured by

\[ \int_{x_m}^{x^*} [p(x) - c'(x)]dx > 0 \]

![Figure 4.1: Monopoly Deadweight Loss](image-url)
4.3 Price Discrimination

According to Pigou (1920), *The Economics of Welfare*, there are three classifications of price discriminations.

4.3.1 First-degree Price Discrimination

The monopoly quantity distortion is fundamentally linked to the fact that if the monopolist wants to increase the quantity it sells, it must lower its price on all its existing sales. In fact, if the monopolist were able to perfectly discriminate among its customers in the sense that it could make a distinct offer to each customer, knowing the customer’s preferences for its product, then the monopoly quantity distortion would disappear. This is known as the first-degree price discrimination.

To see this formally, let us once again assume quasilinear utility functions $u_i(x) + y_i$ for consumer $i$, which is for simplicity normalised such that $u_i(0) = 0 \forall i$. We denote the consumers’ maximum willingness-to-pay for some consumption $x_i$ by $r_i(x)$, which is the solution to the equation

$$u_i(0) + y_i = u_i(x) + y_i - r_i(x)$$

In other words the consumers are indifferent between not consuming and consuming $x$ and paying $r_i(x)$. By virtue of our normalisation then, $r_i(x) \equiv u_i(x)$.

Suppose then that the monopolist makes a take-it-or-leave-it offer to each consumer $i$ of the form $(x_i, r_i(x_i))$. By the definition of $r_i(.)$ the consumer will accept the offer, allowing the monopolist to extract a payment of exactly $u_i(x_i)$ from consumer $i$ in return for $x_i$ units of its product, leaving the consumer with a surplus of exactly zero from consumption of the good. Given this fact, the monopolist will choose the quantities it sells to the $I$ consumers $(x_1, ..., x_I)$ to solve

$$\max_{(x_1, ..., x_I) \geq 0} \sum_{i=1}^{I} u(x_i) - c \left( \sum_{i=1}^{I} x_i \right)$$

As this is equivalent to maximising the aggregate surplus in the market, the outcome is socially optimal. Thus the Pareto efficient level of output $x^*$ (i.e. the competitive output level) will be produced. In terms of wealth distribution though this may not be desirable,
with the monopolist capturing all the surplus from trade. This can however be corrected through lump-sum redistribution of wealth, at least in principle.

In practice, however, there are significant constraints that prevent the monopolist from charging fully discriminatory prices. They may include the costs of assessing separate charges for different consumers, the monopolist’s lack of information about consumer preferences, and the possibility of consumer resale. With these constraints the best the monopolist can do is to name a single per-unit price.

### 4.3.2 Second-degree Price Discrimination

The second-degree price discrimination (also known as non-linear pricing) occurs when prices differ depending on the number of units of the goods bought. This is possible when consumers are heterogeneous, leading to different consumers demanding different units of the goods at a particular price. Each consumer’s demand is, however, private information, and therefore the monopoly has to construct a pricing scheme $p(x)$ that will allow the consumers to self-select.

To see this, assume two consumers with quasi-linear utility functions $u_i(x) + y_i$, $i = 1, 2$. We assume consumer 2 to be the high-demand consumer, i.e. $u_2(x) > u_1(x)$ and $u'_2(x) > u'_1(x) \forall x$.\(^1\) Now to consume any positive amount the consumers have to be better off in consuming than not,

\[
\begin{align*}
    u_1(x_1) - p(x_1)x_1 & \geq 0 \\
    u_2(x_2) - p(x_2)x_2 & \geq 0
\end{align*}
\]

These are called the participation constraints. Each consumer also must prefer his consumption to the consumption of the other customer,

\[
\begin{align*}
    u_1(x_1) - p(x_1)x_1 & \geq u_1(x_2) - p(x_2)x_2 \\
    u_2(x_2) - p(x_2)x_2 & \geq u_2(x_1) - p(x_1)x_1
\end{align*}
\]

\(^1\)The assumption that the consumer with the larger total willingness-to-pay also has the larger marginal willingness-to-pay is sometimes known as the single crossing property, since it implies that any two indifference curves for the agents can intersect at most once.
These are the **self-selection constraints**. In general for these problems, only one of the constraints binds (i.e. the equality holds) for each consumer. So assume that (4.3) binds. Then (4.5) implies that \( p(x_1)x_1 \geq u_2(x_1) \). But we know that \( u_2(x) > u_1(x) \) \( \forall x \) and hence this implies that \( p(x_1)x_1 > u_1(x_1) \), which contradicts (4.2). Thus (4.5) must bind for consumer 2. Next for consumer 1 assume that (4.4) binds. As we now know that (4.5) must bind, by adding both self-selection constraints we get,

\[
u_2(x_2) - u_2(x_1) = u_1(x_2) - u_1(x_1)
\]

But this contradicts our assumption that \( u'_2(x) > u'_1(x) \) \( \forall x \). Hence (4.2) must bind for consumer 1. What this implies is that **the low-demand consumer will be charged his maximum willingness-to-pay**, and **the high-demand consumer will be charged the highest price that will just induce him to consume \( x_2 \) rather than \( x_1 \).**

Now consider the monopolist’s profit-maximisation. Assuming constant marginal cost of production,

\[
\max_{x_1, x_2} \pi(x_1, x_2) = p(x_1)x_1 + p(x_2)x_2 - c(x_1 + x_2)
\]  

(4.6)

Substituting our binding constraints,

\[
\pi(x_1, x_2) = u_1(x_1) + [u_2(x_2) - u_2(x_1) + u_1(x_1)] - c(x_1 + x_2)
\]

The FOCs are,

\[
\begin{align*}
\pi_{x_1} &= 2u'_1(x_1) - u'_2(x_1) - c = 0 \\
\pi_{x_2} &= u'_2(x_2) - c = 0
\end{align*}
\]

The first FOC implies that \( u'_1(x_1) = u'_2(x_1) - u'_1(x_1) + c > c \), i.e. consumer 2 consumes at a level where his marginal value for the good exceeds marginal cost. Thus **the low-demand consumes an inefficiently small amount of the good**. On the other hand the second FOC implies that **the high-demand consumer consumes the socially efficient amount**. Hence the low-demand consumer is penalised in the monopolist’s self-selecting non-linear pricing scheme that maximises its profit under second-degree price discrimination.
4.3.3 Third-degree Price Discrimination

The **third-degree price discrimination** involves different purchasers being charged different prices, although each purchaser pays a constant price for each unit of the good bought. This is probably the most common form of price discrimination: for example student discount for cinema tickets.

Unlike in the second-degree price discrimination, in this case the monopolist is able to charge different pricing schemes to different groups. Thus in comparison to (4.6), the firm’s maximisation problem is this time,

\[
\max_{x_1, x_2} \pi(x_1, x_2) = p_1(x_1)x_1 + p_2(x_2)x_2 - c(x_1 + x_2)
\]  
(4.7)

The FOCs are,

\[
\pi_{x_i} = p_i(x_i) + p'_i(x_i)x_i - c = 0, \ i = 1, 2
\]

or in price elasticity term,

\[
p_i(x_i) \left(1 - \frac{1}{\epsilon_i}\right) = c \text{ for both } i = 1, 2
\]

It follows then that \(p_1(x_1) > p_2(x_2) \Leftrightarrow \epsilon_1 < \epsilon_2\). Hence the **more price elastic market is charged the lower price**. Figure 4.2 shows this diagrammatically, where more price sensitive students are charged a lower price. This discrimination clearly relies on the monopolist’s ability to distinguish different consumer groups.

![Figure 4.2: Third-degree Price Discrimination: Student Discount](image-url)
Chapter 5

Choice under Uncertainty

Varian Ch 11
Mas-Colell, Whinston and Green Ch 6

5.1 Lotteries

Suppose lottery \( f \) yields one of two prizes: either \( x \) with probability \( p \), or \( y \) with probability \( (1 - p) \). We then write \( f \) as:

\[
p \circ x \oplus (1 - p) \circ y
\]

We make some assumptions about the consumer’s perception of the lotteries open to him:

1. \( 1 \circ x \oplus (1 - 1) \circ y \sim x \), i.e. getting a prize with probability 1 is the same as getting the prize for certain.

2. \( p \circ x \oplus (1 - p) \circ y \sim (1 - p) \circ y \oplus p \circ x \), i.e. the consumer does not care the order in which the lottery is described.

3. (Reduction of compound lotteries) \( q \circ (p \circ x \oplus (1 - p) \circ y) \oplus (1 - q) \circ y \sim (qp) \circ x \oplus (1 - qp) \circ y \), i.e. a consumer’s perception of a lottery depends only on the net probabilities of receiving the prizes.
5.2 Preferences over Lotteries

As with the Consumer Theory under certainty, consumers have binary preference relation \( \succeq \) on a set of simple lotteries \( L \). As before then we assume,

A1. Rationality Preference \( \succeq \) is reflexive, complete and transitive.

A2. Continuity The sets \( \{ p \in [0, 1] : p \circ x \oplus (1 - p) \circ y \succeq z \} \) and \\
\( \{ p \in [0, 1] : z \succeq p \circ x \oplus (1 - p) \circ y \} \) are closed.

Remember then the sets of strict preferences \( \succ \) are open, implying that a small change in \( p \) will not suddenly invert the nature of ordering between the two lotteries. When these two axioms are satisfied, then as before there exists a continuous utility function \( U : L \rightarrow \mathbb{R} \) that represents \( \succeq \) such that, for any two simple lotteries \( f, g \in L \),

\[
 f \succeq g \Leftrightarrow U(f) \geq U(g)
\]

Of course \( U(.) \) is only unique up to a positive monotonic transformation. However here we are interested in the existence of a utility function with a particular convenient property. For this we require other axioms,

A3. Independence Axiom For any three lotteries \( f, g, h \in L \) such that \( f \sim g \) and \( \alpha \in (0, 1) \),

\[
 \alpha \circ f \oplus (1 - \alpha) \circ h \sim \alpha \circ g \oplus (1 - \alpha) \circ h
\]

i.e. mixing two lotteries with a third one does not alter the preference ordering.

The independence axiom is at the heart of the theory of choice under uncertainty. With theory of consumer demand, there is no reason to believe that a consumer’s preference over various bundles should be independent of the quantities of other goods he will consume (remember compliments and substitutes). However here the decision maker’s preference over two lotteries should be independent of the outcome of the third common lottery in the compound lottery. This is because, in contrast to the consumer theory, the agent does not consume \( f \) or \( g \) together with \( h \), but rather, only instead of it. Now to prove the Expected Utility Theorem, to avoid some technical details we will make two further assumptions,
A4. \( \exists \) some best lottery \( b \) and some worst lottery \( w \), i.e. for any \( f \in L \), \( b \succeq f \succeq w \).

A5. Monotonicity \( p \circ b \oplus (1-p) \circ w \succ q \circ b \oplus (1-q) \circ w \iff p > q \), i.e. if one lottery between the best and the worst prize is preferred to another it must be because it gives higher probability of getting the best prize.

Theorem 5.1 (Expected Utility Theorem) If \( (L, \succeq) \) satisfy the above axioms, there is a utility function \( U \) defined on \( L \) that satisfies the expected utility property,

\[
U(p \circ x \oplus (1-p) \circ y) = pU(x) + (1-p)U(y)
\]  

(5.1)

Proof. (Varian p.175) We will show that the following numbering system of an arbitrary lottery \( z \) is a utility function with the expected utility property (5.1):

\[
U(z) = p_z \text{ where } p_z \text{ is defined by } z \sim p_z \circ b \oplus (1-p_z) \circ w
\]  

(5.2)

Clearly then \( U(b) = 1 \) and \( U(w) = 0 \). Indeed \( p_z \) is well defined \( \forall b \succeq z \succeq w \): by continuity assumption (A2) it exists, and by monotonicity assumption (A5) it is unique.

First verify that \( U \) is a utility function. Suppose that \( x \succ y \). Then,

\[
U(x) = p_x \text{ such that } x \sim p_x \circ b \oplus (1-p_x) \circ w
\]

\[
U(y) = p_y \text{ such that } y \sim p_y \circ b \oplus (1-p_y) \circ w
\]

Then by the monotonicity axiom (A5), we must have \( p_x > p_y \), i.e. \( U(x) > U(y) \). Now check that \( U \) has the expected utility property (5.1). To do this expand the following,

\[
p \circ x \oplus (1-p) \circ y \sim p \circ [p_x \circ b \oplus (1-p_x) \circ w] \oplus (1-p) \circ [p_y \circ b \oplus (1-p_y) \circ w]
\]

\[
\sim [pp_x + (1-p)p_y] \circ b \oplus [1 - pp_x - (1-p)p_y] \circ w
\]

\[
\sim [pU(x) + (1-p)U(y)] \circ b \oplus [1 - \{pU(x) + (1-p)U(y)\}] \circ w
\]

The first line is simply substitution using (5.2), the second is the reduction of a compound lottery, the third replaces \( p_i \) by \( U(i) \) by the definition of \( U(.) \). Now the utility value of this whole expression is, by (5.2) (i.e. letting \( z = p \circ x \oplus (1-p) \circ y \)),

\[
U(p \circ x \oplus (1-p) \circ y) = pU(x) + (1-p)U(y)
\]
as desired. ■

A utility function \( U : L \rightarrow \mathbb{R} \) with the expected utility form is called a von Neumann-Morgenstern (v.N-M) expected utility function. It can generally be written in the form \( U = \sum p_i u(x_i) \). The main characteristic is that here utility is additively separable over the outcomes, and linear in the probabilities. The continuous form is given by \( \int u(x)p(x)dx \), for a probability density function \( p(x) \) on outcomes \( x \). It follows then that,

**Proposition 5.1** A utility function \( U : L \rightarrow \mathbb{R} \) has an expected utility form if and only if it is linear, i.e.

\[
U \left( \sum_{k=1}^{K} \alpha_k f_k \right) = \sum_{k=1}^{K} \alpha_k U(f_k)
\]

for any \( K \) lotteries \( f_k \in L \), and probabilities \( (\alpha_1, \alpha_2, ..., \alpha_K) \geq 0 \), \( \sum_{k=1}^{K} \alpha_k = 1 \).

It is important to note that the expected utility property is a *cardinal* property of utility functions defined on the space of lotteries:

**Proposition 5.2 (Uniqueness)** An expected utility function is unique up to a positive affine transformation.

**Proof.** First show that if \( U(.) \) is an expected utility function then so is \( V(.) = aU(.) + c \) where \( a > 0 \),

\[
V(p \circ x \oplus (1 - p) \circ y) = aU(p \circ x \oplus (1 - p) \circ y) + c
\]

\[
= a \{ pU(x) + (1 - p)U(y) \} + c
\]

\[
= p(aU(x) + c) + (1 - p)(aU(y) + c)
\]

\[
= pV(x) + (1 - p)V(y)
\]

Second show that any positive monotonic transform of \( U \) that has the expected utility property must be a positive affine transform. Consider such a positive monotonic transform \( T : \mathbb{R} \rightarrow \mathbb{R} \), then,

\[
T[U(p \circ x \oplus (1 - p) \circ y)] = pT[U(x)] + (1 - p)T[U(y)]
\]

But the LHS is also \( T[pU(x) + (1 - p)U(y)] \). The identity of this with the RHS is simply the definition of a positive affine transformation. ■
5.3 Example: The Demand for Insurance

An agent has wealth $w$. He faces the risk of monetary loss $L$, which, if it occurs, would leave her with wealth $w - L$. The probability of loss is known, and equals $p > 0$. He can insure himself against the risk by paying a fraction $\pi$ of the amount of the insurance he buys. The agent is strictly risk-averse. The question is: how much cover will he buy?

Suppose he buys $q$ units of cover. Two possible scenarios:

1. If no loss occurs, his final wealth is $w - \pi q$;
2. If loss occurs, his final wealth is $w - \pi q - L + q$.

Hence the expected utility if he buys $q$ units of cover equals

$$EU(q; p, w, L, \pi) = pu(w - \pi q - L + q) + (1 - p)u(w - \pi q)$$

Maximising w.r.t. $q$ yields the following FOC,

$$pu'(w - \pi q - L + q)(1 - \pi) - (1 - p)u'(w - \pi q)\pi = 0$$

or

$$\frac{u'(w - \pi q - L + q)}{u'(w - \pi q)} = \frac{1 - p}{p} \frac{\pi}{1 - \pi}$$

The SOC is guaranteed by the assumed risk-aversion, which implies that $u(.)$ is concave.

Now assume that the premium charged by the insurance firm is actuarially fair, i.e. competitive condition forces the firm’s profit to zero. Then,

$$(1 - p)\pi q + p(\pi - 1)q = 0$$

This implies that $\pi = p$. Substituting this back in the FOC gives an equation that $q$ must satisfy,

$$u'(w - \pi q - L + q) = u'(w - \pi q)$$

Given the strict concavity of $u(.)$, the arguments must also equate, leading to the result that $q^* = L$, i.e. the consumer will completely insure himself against the loss $L$. Note that this result depends crucially on the assumption that the consumer cannot influence the probability of loss. If the consumer’s actions do affect the probability of loss, the insurance firms may only want to offer partial insurance. This is an asymmetric information issue.
5.4 Allais Paradox (1953)

As a descriptive theory, the expected utility theorem (and by implication, its central assumption, the independence axiom), is not without difficulties. Allais for example pointed out in 1953 that, in considering lotteries with prizes (£5M, £1M, 0), with the following probabilities,

\[
\begin{align*}
f & = (0, 1, 0) \text{ vs } f' = (0.10, 0.89, 0.01) \\
g & = (0, 0.11, 0.89) \text{ vs } g' = (0.10, 0, 0.90)
\end{align*}
\]

Empirical result is then \( f \succ f' \) and \( g' \succ g \). However, the former implies,

\[
U(1) > 0.1U(5) + 0.89U(1) + 0.01U(0) \\
\Leftrightarrow 0.11U(1) > 0.1U(5) + 0.01U(0) \\
\Leftrightarrow 0.11U(1) + 0.89U(0) > 0.1U(5) + 0.9U(0)
\]

and thus \( f \succ f' \Leftrightarrow g \succ g' \). Thus the theory is violated. There are four reactions to this paradox:

1. Marshack and Savage (1954) - Choosing under uncertainty is a reflective activity in which one should be ready to correct mistakes if they are proven inconsistent.

2. Allais paradox is of limited significance for economics as a whole because it involves payoffs that are out of the ordinary and probabilities close to 0 and 1.

3. Regret Theory - The decision maker may value not only what he receives but also what he receives compared with what he might have received by choosing differently. For example here choosing \( f' \) would lead to a potential outcome, however small in probability, of getting none, while there is no such clear-cut regret potential exists between \( g \) and \( g' \).

4. Independence axiom should be replaced by something weaker.
5.5 Risk Aversion

Consider now the case where the lottery space consists solely of gambles with money prizes. Then for any lottery we can define its expected value. If an individual prefers receiving the expected value for certain as opposed to the original gamble, then the individual is said to be risk-averse. If he on the other hand prefers a gamble then he is risk-loving, and if he is indifferent, then he is risk-neutral. Thus for wealth \( w \),

**Definition 5.1** A decision maker is a risk-averter if and only if, \(^1\)

\[
\int u(w) dF(w) \leq u \left( \int w dF(w) \right) \quad \forall F(\cdot)
\]

or more concisely, \( Eu(w) \leq u(Ew) \). This is the **Jensen’s inequality**, which is a property of a concave function. One can measure the degree of risk-aversion of a decision maker by the concavity of the utility function,

**Definition 5.2 (Arrow-Pratt Coefficient of Absolute Risk-Aversion)** The risk-aversion at \( w \) is measured by the following coefficient,

\[
r_A(w) = -\frac{u''(w)}{u'(w)}
\]

The numerator is the curvature of the utility function, with the denominator providing the normalisation. Further we have the following definitions,

**Definition 5.3 (Certainty Equivalent)** The CE of a lottery is the value of money that, if received for certain, would make one indifferent between holding the money and holding the lottery, i.e.

\[
u(CE(F,u)) = \int u(w) dF(w) = Eu(w)
\]

**Definition 5.4 (Risk Premium)** For a random income \( w \), the risk premium \( \pi(w) \) is defined as the maximum amount an individual is willing to pay to eliminate the risk,

\[
u(Ew - \pi(w)) = Eu(w)
\]

\(^1\)Here I follow the Mas-Colell, Whinston and Green practice and distinguish between the utility function \( U(\cdot) \), defined on lotteries, and the utility function \( u(\cdot) \) defined on sure amounts of money. MWG call the former the v.N-M expected utility function, and the latter the Bernoulli utility function. The terminologies are not universal.
Figure 5.1: Certainty Equivalent and Risk Premium

It follows then from Definition 5.3,

\[ \pi(w) = Ew - CE \]

Using these we can state the following propositions,

**Proposition 5.3** The following properties are equivalent,

1. Risk-averse.
2. \( u(.) \) is concave.
3. \( CE(F, u) \leq \int w dF(w) \forall F(.) \).
4. \( \pi(w) \geq 0 \forall w \).

Furthermore one can compare the degrees of risk-aversion between individuals,

**Proposition 5.4** \( u_2(.) \) is more risk-averse than \( u_1(.) \) if,

1. \( r_A(w, u_2) \geq r_A(w, u_1) \forall w \).
2. \( \exists \) an increasing strictly concave function \( G(.) \) such that \( u_2(w) = G(u_1(w)) \).
3. \( CE(F, u_2) \leq CE(F, u_1) \) for any \( F(.) \).
4. \( \pi(w, u_2) \geq \pi(w, u_1) \forall w \).
Empirically it is often seen that the coefficient of absolute risk-aversion decreases with higher wealth. In these cases one may use a different measurement of risk-aversion,

**Definition 5.5 (Arrow-Pratt Coefficient of Relative Risk-Aversion)** The risk-aversion at \( w \) can be measured by the following coefficient,

\[
r_{R}(w) = -\frac{u''(w)w}{u'(w)}
\]

### 5.6 Example

Suppose the individual has a utility function over wealth \( u(w) = \sqrt{w} \). Then,

\[
r_A = -\frac{1}{2}w^{-\frac{3}{2}} = \frac{1}{2w}
\]

\[
r_R = r_Aw = \frac{1}{2}
\]

Consider a gamble that yields wealth of 16 or 4 with equal probability. The expected wealth is then,

\[
Ew = \frac{1}{2} \times 16 + \frac{1}{2} \times 4 = 10
\]

The expected utility is,

\[
EU = \frac{1}{2} \times \sqrt{16} + \frac{1}{2} \times \sqrt{4} = 3
\]

Thus the certainty equivalent is calculated as

\[
u(CE) = \sqrt{CE} = 3
\]

\[
\Rightarrow CE = 9
\]

which implies that the risk premium is

\[
\pi = Ew - CE = 10 - 9 = 1
\]
5.7 Comparison of Return and Riskiness

In comparing payoff distributions, there are two natural ways that random outcomes can be compared: according to the level of returns and according to the dispersion of returns. This leads to two ideas: that of a distribution $F(.)$ yielding unambiguously higher returns than $G(.)$, and that of $F(.)$ being unambiguously less risky than $G(.)$.

Consider first the first idea. We want to attach meaning to the expression “$F(.)$ yields unambiguously higher returns than $G(.)$.” At least two sensible criteria suggest themselves. First, we could test whether every expected utility maximiser (i.e. not necessarily risk-averse) who values more over less prefers $F(.)$ to $G(.)$. Alternatively we could verify whether, for every amount of money $x$, the probability of getting at least $x$ is higher under $F(.)$ than under $G(.)$. Fortunately these two criteria lead to the same concept.

**Definition 5.6 (First-Order Stochastic Dominance)** The distribution $F(.)$ first-order stochastically dominates $G(.)$ if, for every non-decreasing function $u : \mathbb{R} \to \mathbb{R}$ we have,

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

This is formally stating the first criteria. Mas-Colell et al. Proposition 6.D.1 then shows that this is true if and only if $F(x) \leq G(x) \forall x$ (i.e. the second criteria). On a graph of cumulative distributions (see Fig 5.2(1)) this implies that $F(.)$ is uniformly below $G(.)$.

Note that this does not imply that every possible return of $F(.)$ is larger than $G(.)$, neither does it imply that the distribution with higher means first-order stochastically dominates, as here the entire distribution matters.

Next consider the second idea. Given two distributions $F(.)$ and $G(.)$ with the same mean (i.e. $\int xdF(x) = \int xdG(x)$), we say that $G(.)$ is riskier than $F(.)$ if every risk-avoider prefers $F(.)$ to $G(.)$.

**Definition 5.7 (Second-Order Stochastic Dominance)** For any two distributions $F(.)$ and $G(.)$ with the same mean, $F(.)$ second-order stochastically dominates (or is less risky than) $G(.)$ if, for every non-decreasing concave function $u : \mathbb{R}_+ \to \mathbb{R}$ we have,

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

Figure 5.2: Examples of (1) First-Order and (2) Second-Order Stochastic Dominance

This time \( G(.) \) does not need to be uniformly above \( F(.) \) but,

\[
\int_0^x G(t)dt \geq \int_0^x F(t)dt \quad \forall x
\]  

(5.3)

See Fig 5.2(2) for an example of this. Note that \( F(.) \) and \( G(.) \) having the same mean implies that the areas below the two distribution functions are the same over the support \([0, \pi]\), where \( F(\pi) = G(\pi) = 1 \). Hence it is required that \( \text{Area (} A + C \text{)} = \text{Area (} B + D \text{)} \). The condition (5.3) then further requires that \( \text{Area (} A \text{)} \geq \text{Area (} B \text{)} \). This is an example where \( G(.) \) is a mean-preserving spread of \( F(.) \).

5.8 Extensions

5.8.1 State Dependent Utility

This incorporates the idea that the usefulness of a good often depends on the circumstances or the state of nature. For example the value of ice cream depends on the weather and thus the expected utility of the consumption of ice cream may take the form 

\[ p u(h, x_h) + (1 - p) u(c, x_c) \]

with \( h \) and \( c \) representing hot and cold states of nature. More serious case may involve health insurance, with the utility of money depending on one’s health, \( u(h, m_h) \). These are all examples of state-dependent utility functions. It turns out that the extended expected utility representation can be derived in exactly the same
way as above, by redefining the domain over which preferences are defined (see Mas-Colell et al., Ch.6).

5.8.2 Subjective Probability Theory

In reality many uncertain outcomes cannot be summarised by means of numerical objective probabilities. It would thus be very helpful if one could assert that choices are made as if individuals held probabilistic beliefs. Even better, we would like that well-defined probabilistic beliefs be revealed by choice behaviour. This is the intent of subjective probability theory. By assuming that the value of money is the same across different states (i.e. the state preferences are state uniform), then the utility of attaining a fixed payout in state $s$ must equal the subjective probability of that state occurring $\pi_s$ multiplied by the state uniform value of the payout $u(.)$, plus a constant $\beta_s$,

$$u_s(.) = \pi_s u(.) + \beta_s$$

i.e. the utility of payouts in different states can differ only by a positive affine transformation. Normalising $\pi_s$ and $\beta_s$ such that $\sum_s \pi_s = 1$ yields the implied subjective probabilities. Then once again, with the additional assumption of state uniformness of the state preferences, expected utility representation can be derived with uniquely determined probabilities.

5.9 Ellsberg Paradox (1961)

The Ellsberg paradox concerns subjective probability theory. You are told that an urn contains 300 balls, of which 100 are red and the rest are either blue or green. Consider the following gambles:

- Gamble A: receive $1,000 if the ball is red.
- Gamble A’: receive $1,000 if the ball is blue.
- Gamble B: receive $1,000 if the ball is not red.
- Gamble B’: receive $1,000 if the ball is not blue.
Empirically, $A \succ A'$ and $B \succ B'$. However *these preferences violate standard subjective probability theory*. To see why let $R$ be the event that the ball is red, and $\neg R$ be the event that the ball is not red, and define $B$ and $\neg B$ accordingly. Then,

\[
\begin{align*}
p(R) &= 1 - p(\neg R) \\
p(B) &= 1 - p(\neg B)
\end{align*}
\]

Normalise $u(0) = 0$ for convenience. Then for $A \succ A'$, $p(R)u(1000) > p(B)u(1000)$ and hence the subjective probabilities can be ordered,

\[p(R) > p(B)\]

However this implies that

\[p(\neg R) < p(\neg B)\]

and hence $p(\neg R)u(1000) < p(\neg B)u(1000)$ or $B \prec B'$. This paradox seems to be due to the fact that people think that betting for or against red is “safer” than betting for or against blue.
Chapter 6

General Equilibrium in a Pure Exchange Economy

Varian Ch 17.1-7, 21.2-3

6.1 Exchange

In the partial equilibrium set up we assumed that all prices other than the price of the good being studied are fixed. Now in the general equilibrium model all prices are variable, and equilibrium requires that all markets clear. Thus GE theory takes account of all of the interactions between markets, as well as the functioning of the individual markets. To begin with we consider the case of pure exchange, which is the special case of the GE model where all of the economic agents are consumers. Later on we will add production in the economy. In both cases the consumers’ preferences are assumed given exogenously.

So consider an economy with a finite number of consumers and a finite number of commodities. The consumers are indexed as $i = 1, 2, ..., I$, and the commodities are indexed as $n = 1, 2, ..., N$. A consumption bundle for a consumer is given by an $N$-dimension non-negative vector $x_i = (x_{i1}, x_{i2}, ..., x_{iN}) \in \mathbb{R}^N_+$. Each consumer $i$ has well-behaved preferences $\succeq_i$ on the set of consumption bundles: each commodity is a ‘good’; each consumer is locally non-satiable. Provided $\succeq_i$ satisfy
the usual condition, these preferences can be represented by a continuous utility function 
\( U_i(x_i) : \mathbb{R}_+^N \to \mathbb{R} \), which is non-decreasing in each argument.

An allocation \( x \) is a list of the consumption bundles of all consumers \( x = (x_1, x_2, ..., x_I) \), which is a \( N \times I \) matrix. The matrix describes what each agent gets of any particular commodity. The space of all possible allocations is denoted by \( X \).

Each consumer has some initial endowment \( e_i = (e_{i1}, e_{i2}, ..., e_{iN}) \in \mathbb{R}_+^N \). As with allocation, the aggregate endowment is denoted as \( e = (e_1, e_2, ..., e_I) \).

In a pure exchange economy that has no production then, everything that is consumed must come from somebody’s initial endowment. Thus for a particular allocation to be feasible, the commodities in that allocation must be a redistribution of the aggregate endowment. Formally, given an endowment \( e \), an allocation is feasible if,

\[
\sum_{i=1}^{I} x_i \leq \sum_{i=1}^{I} e_i
\]

The simple \( 2 \times 2 \) case where there are only two consumers and two commodities can be represented by the Edgeworth Box.

### 6.2 Walrasian Equilibrium

The first assumption is that with many agents, each agent behave competitively, i.e. take prices as given, independent of his actions. Thus the market is characterised by a price vector \( p = (p_1, p_2, ..., p_N) \). The second assumption is that the consumers are utility maximisers. Unlike with the Consumer Theory earlier, here the income of consumer \( i \) is the value of his initial endowment \( p.e_i \) and thus the UMP becomes,

\[
\max_{x_i} U_i(x_i) \text{ such that } p.x_i \leq p.e_i \tag{GE-UMP}
\]

The solution to this problem is the consumer’s demand function \( x_i(p, p.e_i) \), i.e. demand for each good given prices \( p \) in the economy and his initial endowment \( e_i \). However for an arbitrary price vector \( p \) it may not be possible actually to make the desired transaction for the simple reason that the aggregate demand, \( \sum_i x_i(p, p.e_i) \), may not equal the aggregate
supply $\sum_i e_i$. The prices then adjust to reach the equilibrium, that is defined below as an outcome that is both optimal and feasible,

**Definition 6.1 (Walrasian Equilibrium)** A Walrasian equilibrium for the given pure exchange consists of a pair $(p^*, x^*)$ such that

1. At prices $p^*$, the bundle $x_i^*$ solves (GE-UMP) $\forall i$; and

2. Feasibility in all markets: $\sum_{i=1}^I x_i^*(p^*, p^*, e_i) \leq \sum_{i=1}^I e_i$, i.e. there is no good for which there is positive excess demand.

The array of final consumption vectors $x$ is known as the final consumption allocation.

### 6.3 Market Clearing Walrasian Equilibrium

It is perhaps natural to think of an equilibrium price vector as being one that clears all markets; that is, a set of prices for which demand equals supply in every market. The above definition of Walrasian equilibrium however leaves the possibilities of some goods being in the state of excess supply. This would be true if those goods are undesirable. In this section we show that if all goods are desirable, then in fact all markets do clear in equilibrium. To do this first define the following function,

**Definition 6.2 (Aggregate Excess Demand Function)** The aggregate excess demand function is given by,

$$z(p) = \sum_{i=1}^I [x_i(p, p, e_i) - e_i]$$

As each consumer’s demand function is continuous and homogeneous of degree zero, the aggregate excess demand function also have these properties. Using this we can state the following law,

**Theorem 6.1 (Walras’ Law)** Assuming non-satiation of preferences, for any price vector $p$ the value of the excess demand is identically zero, i.e.

$$p.z(p) \equiv 0$$
Proof. This follows from the individual consumer’s budget constraint and non-satiation. Multiply the aggregate excess demand with $p$,

$$p.z(p) = \sum_{i=1}^{I} [p.x_i(p,p.e_i) - p.e_i]$$

But this is zero as $x_i(p,p.e_i)$ must satisfy the budget constraint $p.x_i = p.e_i$ under non-satiation $\forall i = 1,\ldots,I$.

Walras’ law says something quite obvious: if each individual satisfies his budget constraint, so that the value of his excess demand is zero, then the value of the sum of the excess demands must be zero. Here identically zero implies that the value of excess demand is zero for all prices, i.e. not just for the equilibrium price vector.

**Corollary 6.1 (Market Clearing)** If demand equals supply in $n-1$ markets and $p_n > 0$, then demand must equal supply in the $n^{th}$ market.

**Proof.** $p.z = p_1z_1 + \ldots + p_{n-1}z_{n-1} + p_nz_n$. Walras’ law states that this must equal 0. Then if $z_i = 0 \ \forall z = 1,\ldots,n-1$, and $p_n > 0$, it must be that $z_n = 0$. Thus if $n-1$ markets clear, then so must the $n^{th}$ market.

Once again Walras’ law states that if the workers are non-satiated, then the value of aggregate demand must equal the value of aggregate supply. But does this lead to the conclusion that the number of goods consumed equals the number of available supply (i.e. endowment) in a Walrasian equilibrium? To investigate this see that in applying Walras’ law to Walrasian equilibrium,

**Corollary 6.2 (Free goods)** If $p^*$ is a Walrasian equilibrium and $z_n(p^*) < 0$, then $p^*_n = 0$, i.e. goods in excess supply at a Walrasian equilibrium must be a free good.

**Proof.** At Walrasian equilibrium we have $z(p^*) \leq 0$. Then since prices are non-negative, $p^*.z(p^*) \leq 0$. But if for $z_n(p^*) < 0$ we had $p^*_n > 0$, we would have $p^*.z(p^*) < 0$ which would violate Walras’ law.

But we can also assume the following about individual goods,
**Definition 6.3 (Desirability)** If \( p_n = 0 \) \( \text{then } z_n(p) > 0 \; \forall n = 1, ..., N. \)

Then we have,

**Proposition 6.1** If all goods are desirable, then at a Walrasian equilibrium \( z(p^*) = 0 \), i.e. demand equals supply.

**Proof.** Corollary 6.2 and the desirability of individual goods imply that \( p^*_n \) cannot be zero at a WE. Thus \( p^*_n > 0 \), but Walras’ law implies then that \( z(p^*) = 0 \). □

Let us explain this intuitively. You have some endowment goods. You also see prices for all the goods in the (yet-out-of-equilibrium) market. So you know what your budget is at current prices, and you plan your consumption. As you are a non-satiated sort of person, you plan to spend all your budget. This means that in the economy as a whole, the aggregate planned expenditure (demand) must equal the current aggregate value of wealth to spend (supply). This is Walras’ law. Of course at this stage some goods may have more demand than supply, and others vice versa. To actually trade the prices adjust to reach a feasible equilibrium, i.e. there are no goods with excess demand. If at equilibrium some of your endowment goods are ones still in excess supply, then those unsold goods are now worthless in terms of your spending power (i.e. free goods). However if we additionally assume that all goods are desirable, in the sense that a zero price implies excess demand, then actually you cannot have unsold goods. Then in equilibrium demand must equal supply in every market.

The big questions are now: does such an equilibrium (necessarily) exist for *all* economies? And if it does, is it unique for an economy? These are investigated after the following \( 2 \times 2 \) example.

### 6.4 A \( 2 \times 2 \) Economy

#### 6.4.1 Edgeworth Box Analysis

Assume an economy with two consumers \( A \) and \( B \), and two goods 1 and 2. Their endowments are \( e_A = (e_{A1}, e_{A2}) \) and \( e_B = (e_{B1}, e_{B2}) \) respectively. This economy can
be represented by an Edgeworth Box depicted in the left-hand diagram of Fig 6.1. The consumers’ indifference curves that go through the endowment point are shown. However as these indifference curves are not tangent to each other, there are Pareto improving opportunities. The locus of tangential (i.e. Pareto optimal) points is the contract curve (or the Pareto set). The section of the contract curve between the two indifference curve is called the core; clearly the equilibrium point(s) must lie on the core.

The right-hand diagram of Fig 6.1 depicts how the price adjusts to clear the markets. The lines going through the endowment point are the budget lines with slope \(-\frac{p_1}{p_2}\). The consumers individually choose the points where their indifference curves are tangent to the budget lines. At budget line 1, good 1 is too expensive in comparison to good 2 (i.e. the slope of the budget line is too steep) for each market to clear: there is excess supply of good 1 and excess demand of good 2. The prices would thus adjust until for both goods demand equals supply, which occurs at budget line 2.

But how do we find these points? One way of doing this is to use offer curves. A consumer’s offer curve is the locus of tangencies between the indifference curves and the budget line as the relative prices vary, i.e. the set of demanded bundles. Now first note in the left-hand diagram of Fig 6.1, we know from the indifference curves that to Pareto improve,
consumer A would need to sell some of his endowed good 1 to buy consumer B’s good 2. So now consider first a horizontal budget line going through the endowment point. Assume $p_1$ to be fixed, this is the case where $p_2 = \infty$. At these prices A is unable to exchange any of his good 1 for B’s good 2, and he would simply have to consume his endowment. Now decrease $p_2 < \infty$ so that the slope of the budget constraint becomes downward-sloping. This means that A can now exchange some of his good 1 for B’s good 2, which is becoming relatively cheaper (i.e. the substitution effect). Thus the offer curve heads north-west at this point. However as the price of good 2 becomes even cheaper, consumer A starts feeling richer in real term (i.e. the income effect). At some point then, 1’s income effect becomes large enough that his offer curve starts to turn east. This is depicted in Fig 6.2(1). The offer curve for consumer B is also shown (Fig 6.2(2)). The equilibria occur when the offer curves intersect (Fig 6.3(1)). The slope of the budget lines at which the intersections occur are the equilibrium prices. The fact that at least one intersection will occur is proved in the next section. However there may be multiple equilibria; the substitution effect and the income effect can offset or reinforce each other in ways that make it possible for more than one set of prices to constitute an equilibrium (Fig 6.3(2)). This point will be further elaborated in Section 6.6.
6.4.2 Example

Let consumer $A$ and $B$ have utility functions $u_A(x^A_1, x^A_2) = (x^A_1)^a (x^A_2)^{1-a}$ and $u_B(x^B_1, x^B_2) = (x^B_1)^b (x^B_2)^{1-b}$ for consumption of goods 1 and 2. Each agent has an endowment of $e_A = (1, 0)$ and $e_B = (0, 1)$. The prices of the goods are given by $p = (p_1, p_2)$. Now we already know the Marshallian demand functions for Cobb-Douglas utility functions when incomes are $m_A$ and $m_B$; for example for $A$,

\[
\begin{align*}
    x^A_1(p, m_A) &= \frac{am_A}{p_1} \\
    x^A_2(p, m_A) &= \frac{(1-a)m_A}{p_2}
\end{align*}
\]

But here $m_A = p_1 \times 1 + p_2 \times 0 = p_1$ and $m_B = p_1 \times 0 + p_2 \times 1 = p_2$. Thus,

\[
\begin{align*}
    x^A_1(p) &= \frac{ap_1}{p_1} = a \\
    x^A_2(p) &= \frac{(1-a)p_1}{p_2} \\
    x^B_1(p) &= \frac{bp_2}{p_1} \\
    x^B_2(p) &= \frac{(1-b)p_2}{p_2} = 1 - b
\end{align*}
\]
The aggregate excess demand functions are then,

\[
\begin{align*}
    z_1(p) &= x_1^A(p) + x_1^B(p) - e_{A1} - e_{B1} \\
          &= a + \frac{bp_2}{p_1} - 1 \\
    z_2(p) &= x_2^A(p) + x_2^B(p) - e_{A2} - e_{B2} \\
          &= \frac{(1-a)p_1}{p_2} + (1-b) - 1
\end{align*}
\]

The first thing to note is that these aggregate excess demand functions are homogeneous of degree 0 in \( p \) as,

\[
\begin{align*}
    z_1(tp) &= a + \frac{btp_2}{tp_1} - 1 = z_1(p) \\
    z_2(p) &= \frac{(1-a)tp_1}{tp_2} - b = z_2(p)
\end{align*}
\]

Check Walras’ Law,

\[
\begin{align*}
    p_z(p) &= p_1 z_1(p) + p_2 z_2(p) \\
         &= p_1 \left( a + \frac{bp_2}{p_1} - 1 \right) + p_2 \left( \frac{(1-a)p_1}{p_2} - b \right) \\
         &= 0
\end{align*}
\]

as desired. The question is now whether we can find the Walrasian equilibrium price vector \( p^* = (p_1^*, p_2^*) \) such that the markets clear. For good 1 market to clear \( x_1^A(p^*) + x_1^B(p^*) \) should equal the aggregate endowment 1, or

\[
\begin{align*}
    a + \frac{bp_2^*}{p_1^*} &= 1 \\
    \Rightarrow \frac{p_2^*}{p_1^*} &= \frac{1-a}{b}
\end{align*}
\]

We know by Walras’ law the market for good 2 should also clear. Check this,

\[
\begin{align*}
    x_1^B(p^*) + x_2^B(p^*) &= \frac{(1-a)p_1^*}{p_2^*} + (1-b) = 1
\end{align*}
\]

at \( \frac{p_2^*}{p_1^*} = \frac{1-a}{b} \), i.e. the aggregate demand equals the aggregate endowment, as desired. Note that only relative prices are determined in the equilibrium; a normal practice is to normalise one of the prices to 1.
6.5 Existence of a Walrasian Equilibrium

6.5.1 Proof of the Existence

We will now use Walras’ law to prove the existence of an equilibrium. First thing we do is to normalise prices to the following relative prices (remember the aggregate excess demand is homogeneous of degree 0 so one can scale the prices with no effect on the outcome) to reduce the dimension by 1,

\[ p_n = \frac{\hat{p}_n}{\sum_{m=1}^{N} \hat{p}_m} \]

Thus \( \sum_{n=1}^{N} \hat{p}_n = 1 \), which means that the solution \( \mathbf{p}^* \) of the Walrasian equilibrium belongs to the \( N-1 \)-dimensional unit simplex,

\[ S^{N-1} = \left\{ \mathbf{p} \in \mathbb{R}^N_+ : \sum_{n=1}^{N} p_n = 1 \right\} \]

We also use the following theorem,

**Theorem 6.2 (Brouwer Fixed Point Theorem)** If \( f : S^{N-1} \rightarrow S^{N-1} \) is a continuous function from the unit simplex to itself, there is some \( \mathbf{x}^* \in S^{N-1} \) such that \( \mathbf{x}^* = f(\mathbf{x}^*) \).

**Proof.** We prove this for \( N = 2 \), i.e. to show that for a continuous function \( f : [0,1] \rightarrow [0,1] \), \( \exists \) some \( x^* \in [0,1] \) such that \( x^* = f(x^*) \). Consider the function \( g(x) = f(x) - x \). Then we want to find \( x^* \) where \( g(x^*) = 0 \). But as \( 0 \leq f(x) \leq 1 \ \forall x \in [0,1] \), \( g(0) = f(0) - 0 \geq 0 \) and \( g(1) = f(1) - 1 \leq 0 \). Thus using the intermediate value theorem, for continuous \( f \) there must be some \( x^* \in [0,1] \) such that \( g(x^*) = f(x^*) - x^* = 0 \). □
Then,

**Proposition 6.2 (Existence of WE)** If \( z : S^{N-1} \to \mathbb{R}^N \) is a continuous function that satisfies Walras’ law \( \mathbf{p}, z(\mathbf{p}) \equiv 0 \), then \( \exists \) some \( \mathbf{p}^* \in S^{N-1} \) such that \( z(\mathbf{p}^*) \leq 0 \).

**Proof.** (Varian p.321) Define a map \( g : S^{N-1} \to S^{N-1} \) by

\[
g_n(\mathbf{p}) = \frac{p_n + \max(0, z_n(\mathbf{p}))}{1 + \sum_{m=1}^{N} \max(0, z_n(\mathbf{p}))}
\]

for \( n = 1, \ldots, N \).

Since \( \sum_{n=1}^{N} g_n(\mathbf{p}) = 1 \), \( g(\mathbf{p}) \) is a point in the simplex \( S^{N-1} \). It is also a continuous function as \( z \) and the max function are continuous functions. Thus this satisfies the conditions for Brouwer’s fixed point theorem and hence there is a \( \mathbf{p}^* \) such that \( \mathbf{p}^* = g(\mathbf{p}^*) \), i.e.

\[
p^*_n = \frac{p^*_n + \max(0, z_n(\mathbf{p}^*))}{1 + \sum_{m=1}^{N} \max(0, z_n(\mathbf{p}^*))}
\]

It then suffices to show that \( \mathbf{p}^* \) is a Walrasian equilibrium. Rearranging this yields,

\[
p^*_n \sum_{m=1}^{N} \max(0, z_n(\mathbf{p}^*)) = \max(0, z_n(\mathbf{p}^*)) \quad \forall n
\]

Multiplying each of these \( N \) equations by \( z_n(\mathbf{p}^*) \) and summing up over \( n \),

\[
\left\{ \sum_{m=1}^{N} \max(0, z_n(\mathbf{p}^*)) \right\} \left\{ \sum_{n=1}^{N} p^*_n z_n(\mathbf{p}^*) \right\} = \sum_{n=1}^{N} z_n(\mathbf{p}^*) \max(0, z_n(\mathbf{p}^*))
\]

But the second \{\} is zero by Walras’ law, so we have,

\[
\sum_{n=1}^{N} z_n(\mathbf{p}^*) \max(0, z_n(\mathbf{p}^*)) = 0
\]

Now each term of this sum is greater than or equal to zero since each term is either 0 or \((z_n(\mathbf{p}^*))^2\). But if any term were strictly greater than zero then the equality would not hold. Hence every term must be zero, i.e.

\[
z_n(\mathbf{p}^*) \leq 0 \quad \forall n = 1, \ldots, N
\]

\[\blacksquare\]

Note for this proof of existence, desirability is not required. The only requirements are that the excess demand function be continuous and the Walras’ law, the latter in turn requiring non-satiation.

---

1Intuitively \( g \) is a function that increases the price of the good if that good is in excess demand (i.e. \( z_n(\mathbf{p}) > 0 \)).
6.5.2 Convexity Issue

Usually in General Equilibrium, the assumption of strict convexity has been used to assure that the demand function is well-defined (i.e. there is only a single bundle demanded at each price) and continuous (i.e. small changes in prices give rise to small changes in demand). Fig 6.4 depicts a situation where consumer A has non-convex indifference curves. At price \( p^* \) there are two points that maximise A’s utility, but supply is not equal to demand at either point. Thus this is an example where non-convexity leads to discontinuity in demand.

However suppose that the total supply of demand is just half way between the two demands at \( p^* \). If we replicate this once such that there are two agents of types A and B, and if one of the type As demands \( X_A' \) and the other \( X_A'' \), then the total demand by the agents would in fact equal total supply. Hence Walrasian equilibrium exists in the replicated economy. This can be generalised to many agents. Thus in a large economy in which the scale of non-convexities is small relative to the size of the market, Walrasian equilibria can exist.

![Figure 6.4: Non-existence of an equilibrium with non-convex preferences](image-url)
6.6 Uniqueness of Equilibrium

We have already seen that there can be more than one Walrasian equilibria. There has been much research on conditions when the equilibrium will be unique, or at least those which will limit the number of equilibria. For example it is easy to see that we want to assume continuous differentiability of the excess demand $z(p)$: if indifference curves have kinks in them, there will be whole ranges of prices that are market equilibria. In this case not only are the equilibria not unique, they are not even locally unique. One result states that under mild assumptions the number of equilibria will be finite (Regular economy argument) and odd (Index Theory argument). Furthermore if an economy as a whole, as characterised by an aggregate excess demand function, has the gross substitute property then the equilibrium will be unique. We will review these briefly in reverse order, after the following example.

6.6.1 Example of Multiple Equilibria

(Mas-Colell, Whinston & Green, p.521) Let the utility functions be this time, $u_A(x_1^A, x_2^A) = x_1^A - \frac{1}{5} (x_2^A)^{-8}$ and $u_B(x_1^B, x_2^B) = -\frac{1}{5} (x_1^B)^{-8} + x_2^B$, with endowment allocation of $e_A = (2, r)$ and $e_B = (r, 2)$, where $r = 2 \frac{8}{9} - 2 \frac{1}{9}$. Then A’s UMP Lagrangian optimisation is,

$$\max_{x_1, x_2, \lambda} L = x_1^A - \frac{1}{5} (x_2^A)^{-8} - \lambda (p_1 x_1^A + p_2 x_2^A - 2p_1 - rp_2)$$

The FOCs are,

$$1 - \lambda p_1 = 0 \Rightarrow \lambda = \frac{1}{p_1}$$

$$\left(x_2^A\right)^{-9} - \lambda p_2 = 0 \Rightarrow x_2^A = \left(\frac{p_2}{p_1}\right)^{-\frac{1}{9}}$$

$$p_1 x_1^A + p_2 x_2^A - 2p_1 - rp_2 = 0 \Rightarrow x_1^A = 2 + r \left(\frac{p_2}{p_1}\right) - \left(\frac{p_2}{p_1}\right)^{\frac{8}{9}}$$

And by symmetry,

$$x_1^B = \left(\frac{p_1}{p_2}\right)^{-\frac{1}{9}}$$

$$x_2^B = 2 + r \left(\frac{p_1}{p_2}\right) - \left(\frac{p_1}{p_2}\right)^{\frac{8}{9}}$$
Thus for good 1 market clearance,

\[ 2 + r \left( \frac{p_2}{p_1} \right) - \frac{s}{\pi} \left( \frac{p_2}{p_1} \right) + \frac{1}{\pi} \left( \frac{p_1}{p_2} \right) - \frac{1}{\pi} = 2 + r \]

By substituting for \( r \), this has three solutions \( \frac{p_2}{p_1} = 2, 1, \frac{1}{7} \).

### 6.6.2 Gross Substitutes

**Definition 6.4** Two goods \( i \) and \( j \) are **gross substitutes** at a price vector \( p \) if \( \frac{\partial z_j(p)}{\partial p_i} > 0 \) for \( i \neq j \).

Distinguish this from the usual definition of substitutes (more precisely **net substitutes**) when an increase in the price of one good increases the Hicksian demand of the other good. It is possible then that the income effect more than offsets the substitution effect and the Marshallian demand of the substitutable good decreases. The definition of gross substitute rules this out.

**Proposition 6.3** If all goods are gross substitutes at all prices, then if \( p^* \) is an equilibrium price vector, it is the unique equilibrium price vector.

**Proof.** We prove this for the two-good case. Suppose that \( p^* = (p^*_1, p^*_2) \) and \( p' = (p'_1, p'_2) \) are equilibrium price vectors, i.e. \( z_1(p^*_1, p^*_2) = z_2(p^*_1, p^*_2) = 0 \) and \( z_1(p'_1, p'_2) = z_2(p'_1, p'_2) = 0 \). Remember that by homogeneity, these equilibrium price vectors are only defined up to a scaler multiplication. Without loss of generality then, assume \( p^*_1 = p'_1 \) and \( p^*_2 > p'_2 \) (by if necessary multiplying \( p^* \) by \( \frac{p'_2}{p^*_2} \)). Now consider lowering price \( p^*_2 \) down to \( p'_2 \). As the two goods are gross substitutes, this means that the demand for good 1 must decrease. Thus \( z_1(p') < 0 \), which implies that \( p' \) cannot be an equilibrium price vector. ■

### 6.6.3 Index Analysis

Consider again an economy with only two goods. If we choose the price of good 2 as the numeraire the problem is reduced to finding \( \frac{p_1}{p_2} = p_1 \) when \( p_2 = 1 \), i.e. it becomes a one-variable problem. Then one can draw the excess demand curve \( z_1 \) for good 1 as function of its price (Fig 6.5). Walras’ law implies that when \( z_1 \) equals zero (i.e. where the
Figure 6.5: Index Analysis: (1) multiple equilibria example, (2) unique equilibrium example curve crosses the $x$-axis), we have an equilibrium. What we do know is that the desirability assumption implies that, when the relative price of good 1 is small, the excess demand for good 1 is positive, and when $p_1$ is large, $z_1$ is negative. Observing Fig 6.5 then we can state the following,

1. **There are always odd number of equilibria.**

2. **For there to be multiple equilibria, we require** \( \frac{dz_1}{dp_1} > 0 \) **for some values of** $p_1$. Note as we know that $z_1 = \sum_{i=1}^{I} (x_{i1} - e_{i1})$ for $I$ workers, using Slutsky decomposition with price-dependent endowments,

   \[
   \frac{dz_1}{dp_1} = \sum_{i=1}^{I} \frac{dx_{i1}}{dp_1} = \sum_{i=1}^{I} \frac{dh_{i1}}{dp_1} + \sum_{i=1}^{I} \frac{dx_{i1}}{dm_i} (e_{i1} - x_{i1})
   \]

   This is positive when the positive income effect outweighs the substitution effects, and hence we cannot rule out non-unique Walrasian equilibrium.

3. **If** \( \frac{dz_1}{dp_1} < 0 \) **at all equilibria, then there can only be one equilibrium.** This is an application of a much more general mathematical tool called the **Index Theorem**, and the result can be generalised to $k$ dimensions.
6.6.4 Regular Economy

Once again consider a two-good economy with its $z_1$ excess demand curve.

**Definition 6.5** An equilibrium $p^*_1$ is regular if $\frac{dz_1(p^*_1)}{dp_1} \neq 0$ at the equilibrium.

Hence a regular economy rules out situations such as the solid line in Fig 6.6. Here there is a locus of equilibria with no neighbourhood that has no other equilibrium in it. This implies that the equilibria are not **locally unique**. One reassuring result, due to Debreu (1970), is that “almost all” economies are regular (i.e. all equilibrium prices are regular), as very small perturbation will produce a regular economy (the dotted line in Fig 6.6). This in turn implies that the equilibria are finite, and locally unique.

6.7 Welfare Economics

Before we talk about the normative implications of Walrasian equilibria, first we define the following concept of efficiency:\(^2\)

**Definition 6.6 (Pareto Efficiency)** A feasible allocation $x$ is said to be Pareto efficient if there exists no other feasible allocation $x'$ such that all agents weakly prefer $x'$ to $x$, and some agent strictly prefers $x'$ to $x$.

---

\(^2\)Strictly speaking this is the definition of strong Pareto efficiency. See Varian p.323 for the definition of weak Pareto efficiency, and the proof that the two are equivalent when the preferences are continuous and monotonic. Weak Pareto efficiency implies that at optimal $x$, you cannot make everyone better off.
This implies that at Pareto efficient \( x \), you cannot make one person better off without making someone else worse off. Next restate the definition of the Walrasian equilibrium including the desirability assumption,

**Definition 6.7 (Walrasian Equilibrium)** A Walrasian equilibrium for the given pure exchange consists of a pair \((p^*, x^*)\) such that

1. If agent \( i \) prefers \( x_i' \) to \( x_i^* \), it must be that \( p^*.x_i' > p^*.e_i \); and

2. All markets clear: \( \sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} e_i \).

Now we can state the following,

**Theorem 6.3 (First Theorem of Welfare Economics)** If \((x^*, p^*)\) is a Walrasian Equilibrium, then \( x^* \) is Pareto efficient.

**Proof.** Prove this by obtaining a contradiction. Suppose \((x^*, p^*)\) is a Walrasian Equilibrium but \( x^* \) is not Pareto efficient. That is, suppose there is some other feasible allocation \( x' \) which is weakly preferred to \( x^* \) by all agents, and strictly by some. Then,

1. Since \( x' \) is feasible, \( \sum_{i=1}^{I} p^*.x_i' \leq \sum_{i=1}^{I} p^*.e_i \), given non-negative prices.

2. Since \( x' \) Pareto dominates \( x^* \), all consumers must like \( x_i' \) as much as \( x_i^* \), and some consumers must strictly prefer \( x_i' \) to \( x_i^* \). Since \( x^* \) is an equilibrium allocation, it follows from the definition of equilibrium that \( p^*.x_i' \geq p^*.e_i \) for all \( i \), and \( p^*.x_i' > p^*.e_i \) for some \( i \). Summing up over \( i \) we have,

\[ \sum_{i=1}^{I} p^*.x_i' > \sum_{i=1}^{I} p^*.e_i \]

1. and 2. contradict. \( \blacksquare \)

Intuitively, as each Walrasian equilibrium satisfies the FOCs for utility maximisation, the MRS between goods for each consumer must be equal to the price ratios of the goods. Since all agents face the same price ratios at a Walrasian equilibrium, it implies that all consumers must have the same MRS. Hence their indifference curves are tangent,
and the outcome cannot be Pareto improved. In other words in a decentralised economy, prices co-ordinate agents’ behaviours to attain Pareto efficiency. Note though that a market equilibrium is not ‘optimal’ in any ethical sense, since the market equilibrium may be very ‘unfair’. Now conversely we can state that every Pareto efficient allocation is a Walrasian equilibrium,

**Theorem 6.4 (Second Theorem of Welfare Economics)** Assume that preferences are convex, continuous, non-decreasing and locally non-satiable. Let \( x^* \) be a Pareto efficient allocation which is strictly positive (i.e. \( x^*_m > 0 \) \( \forall i,n \)). Then if we redistribute endowments among all consumers suitably, \( x^* \) can be obtained as a Walrasian equilibrium allocation.

The idea here is that in an Edgeworth analysis, if we pick an arbitrary Pareto efficient allocation we know that the MRS must be equal for the two agents. Thus we can pick a price ratio equal to this common value. Graphically we simply draw the common tangency line separating the two indifference curves. If we then pick any point on this tangent line to serve as an initial endowment, the agents who try to maximise preferences on their budget sets will end up precisely at the Pareto efficient allocation. To show this more formally, we require another theorem,

**Theorem 6.5 (Separating Hyperplane Theorem)** If \( A \) and \( B \) are two non-empty, disjoint, convex sets in \( \mathbb{R}^N \), then there exists a linear functional \( p \) such that \( p.x \geq p.y \) \( \forall x \in A, y \in B \).

Using this,

**Proof.** (Varian p.326 - slightly different version) Given an allocation \( x^* \), let

\[
Z_i = \{ z_i \in \mathbb{R}^N : z_i \geq x^*_i \}
\]

This is the set of all consumption bundles that agent \( i \) prefers to \( x^*_i \). Note there is not restriction on the set; hence \( \infty \) is also in \( Z_i \). Then define

\[
Z^* = \sum_{i=1}^{I} Z_i = \left\{ z \in \mathbb{R}^N : z = \sum_{i=1}^{I} z_i \text{ with } z_i \in Z_i \right\}
\]
i.e. the set of bundles for $I$ consumers that strictly Pareto dominates $x^*$. Since each $Z_i$ is convex from the fact that preferences are convex, $Z^*$ is convex as the sum of convex sets is convex. Also let,

$$Z^+ = \left\{ z \in \mathbb{R}^N : z \leq \sum_{i=1}^{I} x_i^* \right\}$$

i.e. a set of feasible bundles given aggregate endowments. $Z^+$ is obviously convex. Now given that $x^*$ is a Pareto efficient allocation for the economy with the given endowments, it must be that $Z^*$ and $Z^+$ do not intersect. Then by the Separating Hyperplane Theorem there exists a non-zero, $N$-dimensional vector (call it $p$) and a scaler (call it $m$) such that

$$p.z \leq m \forall z \in Z^+ \text{ and } p.z \geq m \forall z \in Z^*$$

The claim is that this $p$, combined with $x^*$, forms a Walrasian equilibrium. In particular we must have $p.x^* = m$, and for any redistribution $e$ such that $p.e_i^* = p.x_i^* \forall i$, $(p, x^*)$ must be a Walrasian equilibrium. In effect we have found a set of prices $p$ which support the allocation $x^*$ as an equilibrium. What is required is to ensure that $p$ is a plausible price vector, i.e. it is non-negative, and if agent $i$ strictly prefers $y_i$ to $x_i$, then $p.y_i > p.x_i$.

These follow from the monotonicity, continuity and local non-satiation assumptions of the preferences. ■

Note that the convexity assumption is crucial. For example Fig 6.7 shows the case where consumer A has a strictly concave indifference curve. Then the tangent line does not separate the indifference curves. This means that there are no points on the line that, going in one direction, both consumers are better off. For example consider starting from $e$, going north-west towards the Pareto efficient point. In this case B will be better off, but A will be moving down to his lower indifference curves. The same argument applies to all the points on the north-west of the Pareto efficient point. Hence there are no points on the tangent line, from which the Pareto efficient point can be attained as a Walrasian equilibrium.

In summary then, the FTWE states that a decentralised economy attains a Pareto efficient outcome as a Walrasian equilibrium (the “invisible hand” argument). However
not all Pareto efficient outcomes are desirable. A government may wish to pick a more equitable point on the Pareto set. The STWE states that under certain conditions this can be achieved by simply reallocating the initial endowment distribution, and letting the economy reach a Walrasian equilibrium again. This suggests that the issues of efficiency and equity can be separated and need not involve a trade off. However in reality such non-distortionary lump-sum taxes do not exist (see works by Mirrlees), and market failures such as market power, externalities and asymmetric information mean that the Pareto efficient outcomes may not be reached as a competitive outcome.
Chapter 7

General Equilibrium with Production

Varian Ch.18.1-18.6

7.1 Walrasian Equilibrium with Production

In this chapter we consider an economy with production. This introduces three additional features into the model, more goods to distribute, the issue of labour supply, and profits. Consider an economy with,

1. a finite number of commodities, \( n = 1, 2, ..., N \),
2. a finite number of firms, \( j = 1, 2, ..., J \),
3. a finite number of consumers, \( i = 1, 2, ..., I \).

As well as the consumers’ preferences, here the production function is also assumed given exogenously. The firms are assumed to be profit-maximisers operating in competitive markets. The profits are distributed to an individual \( i \) according to his ownership \( \theta_{ij} > 0 \) of the firm \( j \). Note then,

\[
\sum_{i=1}^{I} \theta_{ij} = 1 \quad \forall j = 1, ..., J
\]
This ownership of shares, and the initial endowments of goods, are assumed exogenously given. This ownership now adds to the value of the endowment of the consumer. Thus if $y_j(p)$ denote the production plan (i.e. negative/positive elements are inputs/outputs, and thus $p \cdot y_j$ is the profit) of the $j^{th}$ firm at prices $p$, the budget constraint for consumer $i$ is now,

$$p \cdot x_i(p) = \sum_{j=1}^{J} \theta_{ij} p \cdot y_j(p) + p \cdot e_i$$

The consumers are assumed to be utility maximisers, and thus assuming strict convex preferences we can derive the Marshallian demand functions $x_i(p)$ given prices $p$. Then the aggregate excess demand function is given by,

$$z(p) = \sum_{i=1}^{I} x_i(p) - \sum_{j=1}^{J} y_j(p) - \sum_{i=1}^{I} e_i$$

We can now repeat the exercise for the pure exchange economy. First,

**Definition 7.1 (Walrasian Equilibrium)** A Walrasian equilibrium for the given economy consists of a price vector $p^*$, an array of production plans $y_j^*$, one for each of $J$ firms, and an array of consumption plans $x_i^*$, such that

1. for each individual $i$, the bundle $x_i^*(p^*)$ maximises utility at prices $p^*$ subject to the budget constraint

   $$p^* \cdot x_i^*(p^*) = \sum_{j=1}^{J} \theta_{ij} p^* \cdot y_j^*(p^*) + p^* \cdot e_i$$

2. for each firm $j$, the production plan $y_j^*(p^*)$ maximises profits $p^* \cdot y_j^*(p^*)$ at prices $p^*$; and

3. feasibility in all markets:

   $$\sum_{i=1}^{I} x_i^*(p^*) \leq \sum_{i=1}^{I} e_i + \sum_{j=1}^{J} y_j^*(p^*)$$

   i.e. this holds element by element.

Also for all prices (not necessarily equilibrium),
Theorem 7.1 (Walras’ Law) If \( z(p) \) is as defined above, then

\[ p \cdot z(p) = 0 \quad \forall p \]

Proof. Substituting for \( z(p) \),

\[ p \cdot z(p) = \sum_{i=1}^{I} p \cdot x_i(p) - \sum_{j=1}^{J} p \cdot y_j(p) - \sum_{i=1}^{I} p \cdot e_i \]

Now using the budget constraints,

\[ p \cdot z(p) = \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij} p \cdot y_j(p) + \sum_{i=1}^{I} p \cdot e_i - \sum_{j=1}^{J} p \cdot y_j(p) - \sum_{i=1}^{I} p \cdot e_i \]

\[ = \sum_{j=1}^{J} \left( \sum_{i=1}^{I} \theta_{ij} \right) p \cdot y_j(p) - \sum_{j=1}^{J} p \cdot y_j(p) \]

\[ = 0 \]

Thus Walras’ law holds for the same reason that it holds in the pure exchange economy: each consumer satisfies his budget constraint, so the economy as a whole has to satisfy an aggregate budget constraint. Thus this is only driven by the local non-satiation assumption of individual preferences. As before then, if there is excess supply such that \( z_n(p) < 0 \), then \( p_n = 0 \), i.e. it is a free good. However desirability implies that for any commodity with \( p_n = 0 \), \( z_n(p) > 0 \). Thus desirability and non-satiation implies that all markets clear exactly in a Walrasian equilibrium, i.e.

\[ \sum_{i=1}^{I} x_i^*(p^*) = \sum_{i=1}^{I} e_i + \sum_{j=1}^{J} y_j^*(p^*) \quad \text{or} \quad z^*(p) = 0 \]

7.2 Example 1

An economy with two goods (time, which can be consumed as leisure or supplied as labour, and a consumption good), one firm, and \( I \) identical individuals. All agents (firms and consumers) behave competitively. The firm has a production function \( f(l) = l^{\frac{1}{2}} \) where \( l \) is labour input. Each individual has an equal share in the firm, and endowment of \( L \)
hours of time, and utility function $u(x_1, x_2) = x_1^a x_2^{1-a}$, where $x_1$ is leisure consumed, $x_2$ is the amount of the consumption good consumed, and $0 < a < 1$. Find the Walrasian equilibrium.

The way to solve this is to first solve the optimisation problems of each agents (i.e. the firm and the consumers) assuming the prices as given, and then find the equilibrium prices that clear the markets. Thus letting $(w, p)$ be the wage rate and the price of the consumption good, first consider the firm’s optimisation:

$$\max_l \; pl^\frac{1}{2} - wl$$

The FOC is

$$\frac{1}{2} pl^{\frac{1}{2}} - w = 0$$

$$\Rightarrow l^* = p \cdot \frac{2}{w}$$

The SOC is

$$-\frac{1}{4} pl^{\frac{3}{2}} = -\frac{1}{4} p \cdot \frac{2}{w}^{-3} < 0 \; \text{as desired}$$

Thus the firm’s labour demand function, the output supply function and the profit function are, respectively,

$$l(p, w) = \left( \frac{p}{2w} \right)^2$$

$$y(p, w) = l^\frac{1}{2} = \frac{p}{2w}$$

$$\pi(p, w) = pl^\frac{1}{2} - wl = \frac{p^2}{4w}$$

Next consider the consumers’ optimisation where the consumption bundle $(x_1, x_2) = (L - li, ci)$,

$$\max_{x_1, x_2} \; x_1^a x_2^{1-a} \; \text{subject to} \; \underbrace{wx_1 + px_2 = wL + \frac{\pi(p, w)}{I}}_{I} = wL + \frac{p^2}{4wI}$$

The Marshallian demand functions are,

$$x_1(p, wL + \frac{p^2}{4wI}) = L - li = \frac{a \cdot \left( wL + \frac{p^2}{4wI} \right)}{w}$$

$$x_2(p, wL + \frac{p^2}{4wI}) = (1-a) \cdot \frac{\left( wL + \frac{p^2}{4wI} \right)}{p}$$
Thus the individual labour supply is,

\[ l_i = (1 - a)L - \frac{a}{I} \left( \frac{p}{2w} \right)^2 \]

and hence the aggregate labour supply is,

\[ l(p, w) = (1 - a)IL - a \left( \frac{p}{2w} \right)^2 \]

The aggregate demand for the consumption goods is simply

\[ c(p, w) = (1 - a) \left( \frac{wIL}{p} + \frac{p}{4w} \right) \]

Now we can find the market clearing prices. For example for labour market clearance,

\[ (1 - a)IL - a \left( \frac{p}{2w} \right)^2 = \left( \frac{p}{2w} \right)^2 \]

\[ \Rightarrow \frac{p}{w} = 2 \left( \frac{1 - a}{1 + a} \right) \frac{IL}{p} \]

By Walras’ law the consumption goods market should also clear. Check this:

\[ (1 - a) \left( \frac{wIL}{p} + \frac{p}{4w} \right) = \frac{p}{2w} \]

\[ \Rightarrow \frac{p}{w} = 2 \left( \frac{1 - a}{1 + a} \right)^{\frac{1}{2}} \]

as desired.

### 7.3 Welfare Economics

We will now modify the Welfare Theorems.

**Theorem 7.2 (First Theorem of Welfare Economics)** If \((x^*, y^*, p^*)\) is a Walrasian Equilibrium, then \(x^*\) is Pareto efficient.

**Proof.** (Varian p.345) Suppose not, i.e. let \((x^*, y^*, p^*)\) be a Walrasian Equilibrium but there exists a Pareto dominating allocation \((x', y')\). Then,

1. Since \(x'\) is feasible, \(\sum_{i=1}^{I} p^* \cdot x'_i \leq \sum_{i=1}^{I} p^* \cdot e_i + \sum_{j=1}^{J} p^* \cdot y'_j\), given non-negative prices.
2. Since consumers are utility maximisers,
\[ \mathbf{p}^* \cdot \mathbf{x}'_i > \mathbf{p}^* \cdot \mathbf{e}_i + \sum_{j=1}^{J} \theta_{ij} \mathbf{p}^* \cdot \mathbf{y}'_j \quad \forall i = 1, ..., I \]

Summing over the consumers we have
\[ \sum_{i=1}^{I} \mathbf{p}^* \cdot \mathbf{x}'_i > \sum_{i=1}^{I} \mathbf{p}^* \cdot \mathbf{e}_i + \sum_{j=1}^{J} \mathbf{p}^* \cdot \mathbf{y}'_j \]

where we have used \( \sum_{i=1}^{I} \theta_{ij} = 1 \).

1. and 2. contradict.

**Theorem 7.3 (Second Theorem of Welfare Economics)** Assume that preferences are convex, continuous, non-decreasing and locally non-satiable, and the production sets are convex. Let \((\mathbf{x}, \mathbf{y})\) be a strictly positive Pareto efficient ‘consumption allocation-production plan pair’. Then \((\mathbf{x}, \mathbf{y})\) is the ‘allocation plan’ in a Walrasian equilibrium if we first redistribute endowments and share-holdings among consumers.

**Proof.** Uses Separating Hyperplane Theorem as before.

**7.4 Example 2**

An economy consists of two non-satiated individuals (1 and 2) who supply labour \((L)\) and demand a consumption good \((X)\). The individuals have identical preferences represented by the utility function \(U_i = X_i^a (1 - L_i)^{1-a}, 0 < a < 1, i = 1, 2\). The two individuals differ in their ability to produce the consumption good from a unit of labour: for each unit of labour supplied they produce \(w_1\) and \(w_2\) respectively. The economy’s production constraint is therefore \(X_1 + X_2 = w_1 L_1 + w_2 L_2\).

Let us find the economy’s utility possibility frontier. This is derived by maximising the utility of one individual subject to the production constraint and the constraint that the other individual obtains a specified value of utility \(U\). We will treat this as an equality-constrained optimisation problem, and assume interior solutions whose SOCs are satisfied.
The Lagrangian for individual 1’s utility maximisation problem is then,

\[ \mathcal{L}(X_1, X_2, L_1, L_2, \lambda, \mu) = X_1^a(1 - L_1)^{1-a} - \lambda [U - X_2^a(1 - L_2)^{1-a}] - \mu [X_1 + X_2 - w_1 L_1 - w_2 L_2] \]

The FOCs are,

\[
\begin{align*}
X_1 &: \quad aX_1^{a-1}(1 - L_1)^{1-a} - \mu = 0 & \quad (7.1) \\
X_2 &: \quad \lambda a X_2^{a-1}(1 - L_2)^{1-a} - \mu = 0 & \quad (7.2) \\
L_1 &: \quad -(1 - a)X_1^a(1 - L_1)^{-a} + \mu w_1 = 0 & \quad (7.3) \\
L_2 &: \quad -\lambda(1 - a)X_2^a(1 - L_2)^{-a} + \mu w_2 = 0 & \quad (7.4) \\
\lambda &: \quad U - X_2^a(1 - L_2)^{1-a} = 0 & \quad (7.5) \\
\mu &: \quad w_1 L_1 + w_2 L_2 - X_1 - X_2 = 0 & \quad (7.6)
\end{align*}
\]

Substituting (7.1) into (7.3) and (7.2) into (7.4) to eliminate \( \mu \) yields,

\[
\begin{align*}
L_1 &= 1 - \frac{(1 - a)X_1}{aw_1} & \quad (7.7) \\
L_2 &= 1 - \frac{(1 - a)X_2}{aw_2} & \quad (7.8)
\end{align*}
\]

Substituting these into (7.6) then yields, after simplification,

\[ X_1 = a(w_1 + w_2) - X_2 \]

Then we can express \( U_1 \) in terms of \( X_2 \) only using this and (7.7),

\[
\begin{align*}
U_1 &= X_1^a(1 - L_1)^{1-a} \\
&= \left( \frac{1 - a}{aw_1} \right)^{1-a} X_1 \\
&= \left( \frac{1 - a}{aw_1} \right)^{1-a} \{ a(w_1 + w_2) - X_2 \} \quad (7.9)
\end{align*}
\]

But from (7.5) using (7.8),

\[ U = \left( \frac{1 - a}{aw_2} \right)^{1-a} X_2 \]

Inverting this for \( X_2 \) and substituting into (7.8) thus yields an expression for \( U_1 \) in terms of \( w_1, w_2 \) and \( U \),

\[ U_1 = \left( \frac{1 - a}{aw_1} \right)^{1-a} \left\{ a(w_1 + w_2) - \left( \frac{aw_2}{1 - a} \right)^{1-a} U \right\} \]
Thus for any values of $U_2 = \overline{U}$,

$$U_1 = \left( \frac{1-a}{aw_1} \right)^{1-a} \left\{ a(w_1 + w_2) - \left( \frac{aw_2}{1-a} \right)^{1-a} U_2 \right\}$$  \hspace{1cm} (7.10)

which is the desired utility possibility frontier. Any point on this frontier is Pareto efficient.

Now consider the competitive equilibrium of this economy where the price of the consumption good is normalised to 1 and the wages are given by $w_i$. The Lagrangian for each individual’s UMP is, for $i = 1, 2$,

$$\mathcal{L} = X_i^a (1 - L_i)^{1-a} + \alpha_i (w_i L_i - X_i)$$

It is easily seen that the Marshallian demands are,

$$X_i = aw_i$$
$$L_i = a$$

which leads to the indirect utility function

$$V_i = a^a(1 - a)^{1-a}w_i^a$$

The question is whether this competitive equilibrium lies on the utility possibility frontier. To check this substitute for $U_2$ in (7.10),

$$U_1 = \left( \frac{1-a}{aw_1} \right)^{1-a} \left\{ a(w_1 + w_2) - \left( \frac{aw_2}{1-a} \right)^{1-a} a^a(1 - a)^{1-a}w_2^a \right\}$$

$$= a^a(1 - a)^{1-a}w_1^a$$
$$= V_1$$

Thus the equilibrium is Pareto efficient, demonstrating the FTWE.

Now consider the question whether any points on the utility possibility frontier can be attained as a competitive equilibrium with suitable redistribution of income by means of this lump-sum transfer. To do this consider a lump-sum transfer $X_i = w_i L_i - T_i$ with $T_1 = -T_2$. The Lagrangians for the UMP are now,

$$\mathcal{L} = X_i^a (1 - L_i)^{1-a} + \alpha_i (w_i L_i - T_i - X_i)$$
which leads to the Marshallian demands

\[ X_i = a(w_i - T_i) \]
\[ L_i = a + \frac{T_i(1 - a)}{w_i} \]

and the indirect utility function

\[ V_i = a^a(1 - a)^{1-a}(w_i - T_i)w_i^{-(1-a)} \]

Thus using \( T_1 = -T_2 \),

\[ V_1 = a^a(1 - a)^{1-a}(w_1 - T_1)w_1^{-(1-a)} \]
\[ V_2 = a^a(1 - a)^{1-a}(w_2 + T_1)w_2^{-(1-a)} \]

Check that this satisfies (7.10) by substituting for \( V_2 \),

\[ U_1 = \left( \frac{1 - a}{aw_1} \right)^{1-a} \left\{ a(w_1 + w_2) - \left( \frac{aw_2}{1-a} \right)^{1-a} \right\} a^a(1 - a)^{1-a}(w_2 + T_1)w_2^{-(1-a)} \]
\[ = a^a(1 - a)^{1-a}(w_1 - T_1)w_1^{-(1-a)} \]
\[ = V_1 \]

This demonstrates the STWE.
Chapter 8

Problem Sets

8.1 Theory of Choice and Consumer Theory

1. Verify that the lexicographic ordering is complete, transitive, strongly monotone, and strictly convex.

2. (Class Test 2004Q1) A consumer has an income $m$ and a utility function of the form

$$u(x_1, x_2) = a \ln x_1 + (1 - a) \ln x_2$$

(a) If the prices of the two goods are given by $p_1$ and $p_2$, derive the Hicksian demand functions for a given utility level $U$.

(b) Derive the expenditure function.

(c) Using one of the Duality identities, derive the indirect utility function.

(Turn over)
3. **(Final 2004B9) Answer both parts.**

(a) Show that a consumer’s Marshallian demand functions satisfy the following restriction:

\[
\begin{align*}
(i) & \quad \sum_{i=1}^{N} p_i \frac{\partial x_i}{\partial p_j} + x_j = 0, \quad j = 1, \ldots, N \\
(ii) & \quad \sum_{i=1}^{N} p_i \frac{\partial x_i}{\partial m} = 1
\end{align*}
\]

where there are \(N\) goods, \(p_i\) is the price of good \(i\), and \(m\) is the consumer’s income.

(b) Suppose that a consumer has an endowment of various goods \(\omega = (\omega_1, \ldots, \omega_N)\) which can be sold at market prices \(p = (p_1, \ldots, p_N)\). Specify and briefly interpret the Slutsky equation for this consumer.

4. **(Class Test 2005Q2)** Mr Watts has an income \(m\) and a utility function of the form

\[u(x_1, x_2) = x_1 x_2\]

Given prices of the two goods \(p_1\) and \(p_2\), his Marshallian demand functions are given by

\[x_1^*(p_1, p_2, m) = \frac{1}{2} \frac{m}{p_1}, \quad x_2^*(p_1, p_2, m) = \frac{1}{2} \frac{m}{p_2}\]

(a) Derive Mr Watts’ indirect utility function \(v(p_1, p_2, m)\), and hence his expenditure function \(e(p_1, p_2, U)\) for a given utility level \(U\).

(b) Now in Mr Watts’s town, the prices of goods 1 and 2 are both 1. However he learns that in the next town the price of good 1 is \(\frac{1}{4}\) while the price of good 2 is still 1. If Mr Watts’s income is £100 in his town,

(i) What is the maximum amount of income he is willing to give up to move to the next town?

(ii) Alternatively, if he can still earn £100 in the next town, what is the minimum pay-rise required to convince Mr Watts to stay in his town?

(c) Can you make any statement about the size of the Marshallian consumer surplus associated with this price change?
8.2 Producer Theory and Monopoly

1. (Class Test 2003bQ3) Consider a firm who uses a single input to produce a single output; its production function is given by $y = x^a$ where $a \in (0, 1)$. Calculate the profit function for this firm and verify that it is convex in $p$ (price of the output) and $w$ (price of the input).

2. (Class Test 2004Q2) A firm’s cost minimisation problem is given by

$$\min_{x} \ w \cdot x \ \text{subject to} \ x \in V(y)$$

where $x$ is the vector of factor inputs, $w$ is the price vector of the factors, and $V(y)$ is the input requirement set for a given output $y$. The optimised value function of the problem is then the cost function $c(w, y)$.

(a) Prove that the cost function is concave in $w$, i.e. $c(tw_1 + (1 - t)w_2, y) \geq tc(w_1, y) + (1 - t)c(w_2, y), \forall 0 \leq t \leq 1$. Illustrate this graphically.

(b) State the Shephard’s Lemma. Using this lemma and the concavity of the cost function, what predictions can you make about the properties of the price derivatives matrix of the factor input demands

$$\left( \begin{array}{c} \frac{\partial x_1}{\partial w_1} \\
\frac{\partial x_1}{\partial w_2} \\
\frac{\partial x_2}{\partial w_1} \\
\frac{\partial x_2}{\partial w_2} \end{array} \right)$$

3. (Final 2003C14) Answer both parts.

(a) A Leontief technology production function is

$$f(x_1, x_2) = \min[ax_1, bx_2]$$

where $x_1$ and $x_2$ are the two inputs and $a, b > 0$ are parameters. Define the corresponding input requirement set and isoquant. Is the technology monotonic?
(b) A competitive firm produces a single output $y$ according to the production function $y = 40z - z^2$ where $z$ is the single input. The price of $y$ is denoted by $p$ and the price of $z$ is denoted by $w$. It is necessary that $z \geq 0$.

i. What is the first-order condition for profit maximisation if $z > 0$? Check the second-order condition.

ii. For what values of $w$ and $p$ will the profit maximising $z$ be zero?

iii. For what values of $w$ and $p$ will the profit maximising $z$ be 20?

iv. What are the output supply and input demand functions?

v. What is the profit function?

vi. What is the derivative of the profit function with respect to $w$?

4. A monopolist maximises its profit $\pi(x) = p(x)x - c(x)$, where $x$ is the output, and $p(x)$ and $c(x)$ are the inverse demand function and the cost function respectively. In order to capture some of the monopoly profits, the government imposes a tax on revenue of an amount $t$ so that the monopolist’s objective function becomes $p(x)x - c(x) - tp(x)x$. Initially, the government keeps the revenue from this tax.

(a) Does this tax increase or decrease the monopolist’s output?

(b) Now the government decides to award the revenue from this tax to the consumers of the monopolist’s product. Each consumer will receive a “rebate” in the amount of the tax collected from his expenditures. Thus the representative consumer who spends $px$ receives a rebate of $tpx$ from the government. Assuming that the consumers’ utility function is of the quasilinear form $U(x) = u(x) + y$, where $y$ is the income spent on all other goods, show that the consumers’ inverse demands before and after the rebate is introduced are given by,

\[
\begin{align*}
    p_b(x) & = u'(x) \\
    p_a(x) & = \frac{u'(x)}{1-t}
\end{align*}
\]

(c) How does the monopolist’s output respond to the tax-rebate programme?
8.3 Choice Under Uncertainty

1. (Class Test 2002Q2(a)) Define the Arrow-Pratt coefficient of absolute risk aversion. Show that it is invariant to positive linear transformations of the utility function.

2. (Class Test 2003bQ4) If a decision maker prefers a 10% chance of winning $5,000 to a 20% chance of winning $2,000, explain which of the following choices are consistent with her preferences satisfying the independence axiom:

(i) she prefers a 70% chance of winning $2,000 and 10% chance of winning $5,000 to a 90% chance of winning $2,000;
(ii) she prefers a 90% chance of winning $2,000 and 10% chance of winning $5,000 to a 20% chance of winning $5,000 and a 70% chance of winning $2,000;
(iii) she prefers a 10% chance of winning $2,000 and 10% chance of winning $5,000 to a 30% chance of winning $2,000.

3. (Final 2005B10) Tommy has a utility function of the form \( u(w) = \sqrt{w} \). He has initial wealth of £4. He also has a lottery ticket that will be worth £12 with probability \( \frac{1}{2} \) and zero otherwise.

(a) Derive Tommy’s coefficients of absolute and relative risk aversion.

(b) What is the lowest price at which he would be prepared to sell the lottery ticket?

(c) Suppose that he did not have the lottery ticket. What is the maximum price he would pay to obtain it? Why does this price differ from your answer to (b)?

(d) We also know that Sammy has a utility function of the form \( u(w) = w^{\frac{3}{4}} \). Can you state in general which of the two agents is the more risk-averse? Prove your result using two separate methods.

(Turn over)
4. (Final 2006B9) In the town of Stamford there exist four prize draws $A$, $B$, $C$ and $D$, that pay out winnings of $w \in \{1, 2, 3, 4, 5\}$ with following probabilities:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0.15</td>
<td>0.25</td>
<td>0.25</td>
<td>0.30</td>
<td>0.05</td>
</tr>
<tr>
<td>$B$</td>
<td>0.10</td>
<td>0.25</td>
<td>0.30</td>
<td>0.25</td>
<td>0.10</td>
</tr>
<tr>
<td>$C$</td>
<td>0.00</td>
<td>0.40</td>
<td>0.55</td>
<td>0.00</td>
<td>0.05</td>
</tr>
<tr>
<td>$D$</td>
<td>0.05</td>
<td>0.40</td>
<td>0.50</td>
<td>0.05</td>
<td>0.00</td>
</tr>
</tbody>
</table>

(a) Mary, a mother of twins, has decided to buy one of prize draw $A$ and one of prize draw $B$ and give them to her sons, Jose and Rafael, for their birthday.

(i) If Jose’s utility gain from a prize payout $w$ is given by $u_J(w) = w^2$, which of the two prize draws would he prefer?

(ii) If he also has a choice of accepting a cash present instead, what is the minimum amount of cash that he would accept instead of either of the prize draws?

(b) It turns out that Rafael, whose utility function $u_R(w)$ is unknown, strictly prefers the prize draw $A$ to $B$. Given this information what can you say about Rafael’s preference between prize draws $C$ and $D$?

(c) Explain what is meant by First-Order Stochastic Dominance of two distributions. Hence what can you say about Rafael’s preference ordering?
8.4 General Equilibrium

1. (Class Test 2003aQ2) State, and explain using the Edgeworth Box, the First Theorem of Welfare Economics.

2. (Class Test 2004Q3) In a two-individual, two-good exchange economy, individual $A$ has utility function $u^A(x^A_1, x^A_2) = (x^A_1)^a (x^A_2)^{1-a}$, $0 < a < 1$, and individual $B$ has utility function $u^B(x^B_1, x^B_2) = \min(x^B_1, x^B_2)$. The endowment vectors of the two individuals are $e^A = (0, 1), e^B = (1, 0)$.

(a) Show that the aggregate excess demand functions are homogeneous of degree zero in prices, and confirm that they satisfy Walras’ Law. (You may use the fact that when income is $m$, the Marshallian demands are given by: (i) $(x^*_1, x^*_2) = \frac{am}{a+b+p_1}, \frac{bm}{a+b+p_2}$ for a Cobb-Douglas utility function $u(x_1, x_2) = x_1^a x_2^b$; and (ii) $x^*_1 = x^*_2 = \frac{m}{p_1 + p_2}$ for a minimum utility function $u(x_1, x_2) = \min(x_1, x_2)$.)

(b) By normalising $p^*_1$ to be 1, calculate the Walrasian equilibrium prices and allocation.

3. (Final 2005C12) Answer all parts of this question.

(a) Consider a competitive economy with complete markets, finite number of goods, households and firms. Firms are profit-maximisers owned by the households. State and prove Walras’ Law.

(b) A competitive economy with a single good consists of a single capitalist and $n$ identical workers. The capitalist does not work, and has utility function $u_c(x_c) = x_c$ for his consumption $x_c$ of the good. Each worker has utility function $u_w(x_w, l) = x_w - l^2$, where $x_w$ denotes a worker’s consumption of the good and
$l$ is his labour supply. The price of the good is $p$, and the wage $w$ is normalised to equal one. There is a single competitive firm wholly owned by the capitalist, with a production function $y = l^{\frac{1}{2}}$.

i. Find the labour demand, output supply and the profit function of the firm.

ii. Find the Marshallian demand and the labour supply of the workers.

iii. Find the equilibrium price of the good $p$.

iv. How does an increase in the number of workers affect worker utility in the equilibrium?