MSc Economics:
Economic Theory and Applications I

General Equilibrium in a Pure Exchange Economy

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1 Exchange

• In partial equilibrium all prices other than that being studied are fixed. In general equilibrium all prices are variable.

• Thus GE theory takes account of all interactions between markets, as well as the functioning of the individual markets.

• Begin with a pure exchange economy where all agents are consumers. We will add production later.

• An economy consists of consumers $i = 1, 2, ..., I$ and commodities $n = 1, 2, ..., N$.

• A consumption bundle for a consumer is $\mathbf{x}_i = (x_{i1}, x_{i2}, ..., x_{iN}) \in \mathbb{R}_+^N$.

• An allocation $\mathbf{x}$ is a list of the consumption bundles of all consumers $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_I)$, a $N \times I$ matrix.
• Each consumer has some initial endowment \( e_i = (e_{i1}, e_{i2}, \ldots, e_{iN}) \in \mathbb{R}_+^N \). The aggregate endowment is \( e = (e_1, e_2, \ldots, e_I) \).

• In a pure exchange economy with no production, everything that is consumed must come from somebody’s initial endowment. Feasibility is thus defined by,

\[
\sum_{i=1}^{I} x_i \leq \sum_{i=1}^{I} e_i
\]
2 Walrasian Equilibrium

• Assumption 1: Agents are price-takers. Thus the market is characterised by a price vector $\mathbf{p} = (p_1, p_2, \ldots, p_N)$.

• Assumption 2: Consumers are utility maximisers. Hence,

$$\max_{\mathbf{x}_i} U_i(\mathbf{x}_i) \text{ such that } \mathbf{p} \cdot \mathbf{x}_i \leq \mathbf{p} \cdot \mathbf{e}_i$$

(GE-UMP)

• The solution is the consumer’s demand function $\mathbf{x}_i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}_i)$ given prices $\mathbf{p}$ and initial endowment $\mathbf{e}_i$.

• For an arbitrary price vector $\mathbf{p}$, aggregate demand $\sum_i \mathbf{x}_i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}_i)$ may not equal aggregate supply $\sum_i \mathbf{e}_i$. The prices then adjust to reach the equilibrium defined below,
Definition  A Walrasian equilibrium for the given pure exchange consists of a pair \((p^*, x^*)\) s.t.

a. At prices \(p^*\), the bundle \(x_i^*\) solves (GE-UMP) \(\forall i\); and

b. Feasibility in all markets:

\[
\sum_{i=1}^{I} x_i^*(p^*, p^*, e_i) \leq \sum_{i=1}^{I} e_i
\]

i.e. there is no good for which there is positive excess demand.

- The array of final consumption vectors \(x\) is the final consumption allocation.
3 Market Clearing WE

- Natural to think of an equilibrium price vector as one that clears all markets, i.e. demand equals supply in every market.

- Here we show that if all goods are desirable, then in fact all markets do clear in equilibrium.

**Definition** The aggregate excess demand function is given by,

\[ z(p) = \sum_{i=1}^{I} [x_i(p, p.e_i) - e_i] \]

- As each consumer’s demand function is continuous and h.d.0, so is \( z(p) \).

- Using this,
Theorem 1 (Walras’ Law) Assuming non-satiation of preferences, for any price vector $\mathbf{p}$ the value of the excess demand is identically zero, i.e.

$$\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) \equiv 0$$

Proof Follows from the individual’s BC and non-satiation. Multiply $\mathbf{z}(\mathbf{p})$ with $\mathbf{p}$,

$$\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = \sum_{i=1}^{I} [\mathbf{p} \cdot \mathbf{x}_i(\mathbf{p}, \mathbf{p.e}_i) - \mathbf{p.e}_i]$$

But this is zero as $\mathbf{x}_i(\mathbf{p}, \mathbf{p.e}_i)$ must satisfy the BC $\mathbf{p} \cdot \mathbf{x}_i = \mathbf{p.e}_i$ under non-satiation $\forall i = 1, ..., I$.$\blacksquare$

- Walras’ law says something quite obvious: if each individual satisfies his BC, so that the value of his excess demand is zero, then the value of the sum of the excess demands must be zero.

- Here identically zero implies that the value of excess demand is zero for all prices, i.e. not just for the equilibrium price vector.
Corollary 2 (Market Clearing) If demand equals supply in \( n - 1 \) markets and \( p_n > 0 \), then demand must equal supply in the \( n^{th} \) market.

Proof Walras’ law: \( p \cdot z = p_1 z_1 + \ldots + p_{n-1} z_{n-1} + p_n z_n = 0 \). Then if \( z_i = 0 \) \( \forall z = 1, \ldots, n - 1 \), and \( p_n > 0 \), it must be that \( z_n = 0 \). Thus if \( n - 1 \) markets clear, then so must the \( n^{th} \) market.

• But does the number (not value) of goods consumed equal the number of available supply (i.e. endowment) in a WE? To investigate apply Walras’ law to WE,

Corollary 3 (Free goods) If \( p^* \) is a WE and \( z_n(p^*) < 0 \), then \( p^*_n = 0 \), i.e. goods in excess supply at a WE must be a free good.

Proof At WE we have \( z(p^*) \leq 0 \). Then since prices are non-negative, \( p^* \cdot z(p^*) \leq 0 \). But if for \( z_n(p^*) < 0 \) we had \( p^*_n > 0 \), we would have \( p^* \cdot z(p^*) < 0 \). This violates Walras’ law.
• But we can also assume,

**Definition (Desirability)** If \( p_n = 0 \) then \( z_n(p) > 0 \ \forall n = 1, ..., N. \)

Then we have,

**Proposition 4** *If all goods are desirable, then at a WE \( z(p^*) = 0 \), i.e. demand equals supply.*

**Proof** Free goods corollary and the desirability of individual goods imply that \( p_n^* \) cannot be zero at a WE. Thus \( p_n^* > 0 \), but Walras’ law implies then that \( z(p^*) = 0. \)

• For intuition see Lecture Note.

• The big questions are now: does such an equilibrium (necessarily) exist for all economies? And if it does, is it unique for an economy?
4 A 2 × 2 Economy

4.1 Edgeworth Box Analysis

1. Edgeworth Box for a 2 × 2 Economy

- The left-hand diagram shows that the equilibrium point(s) must lie on the core, which is the segment of the contract curve between the two indifference curves.

- The right-hand diagram depicts how the price adjusts to clear the markets.
• But how do we find these points? Use offer curves, which is the locus of tangencies between the indifference curves and the budget line as the relative prices vary, i.e. the set of demanded bundles.

2. Offer Curves for (1) Consumer $A$ and (2) Consumer $B$

• The equilibria occur when the offer curves intersect.
• The slope of the budget lines at which the intersections occur are the equilibrium prices.

• The fact that at least one intersection will occur is proved in the next section.

• However there may be multiple equilibria; the substitution effect and the income effect can offset or reinforce each other in ways that make it possible for more than one set of prices to constitute an equilibrium.

3. Walrasian Equilibrium using Offer Curves: (1) Unique Equilibrium, (2) Multiple Equilibria
4.2 Example

- Consumers $A$ and $B$ and goods 1 and 2.

- Prices of the goods: $p = (p_1, p_2)$.

- Utility functions:

\[
\begin{align*}
  u_A(x_1^A, x_2^A) &= (x_1^A)^a (x_2^A)^{1-a} \\
  u_B(x_1^B, x_2^B) &= (x_1^B)^b (x_2^B)^{1-b}
\end{align*}
\]

- Endowments $e_A = (1, 0)$ and $e_B = (0, 1)$.

- We know the Marshallian demands when incomes are $m_A$ and $m_B$. E.g. for $A$,

\[
\begin{align*}
  x_1^A(p, m_A) &= \frac{am_A}{p_1} \\
  x_2^A(p, m_A) &= \frac{(1 - a)m_A}{p_2}
\end{align*}
\]
• But here $m_A = p_1$ and $m_B = p_2$. Thus,

\[
x_1^A(p) = \frac{ap_1}{p_1} = a
\]
\[
x_2^A(p) = \frac{(1 - a)p_1}{p_2}
\]
\[
x_1^B(p) = \frac{bp_2}{p_1}
\]
\[
x_2^B(p) = \frac{(1 - b)p_2}{p_2} = 1 - b
\]

• The aggregate excess demands are,

\[
z_1(p) = x_1^A(p) + x_1^B(p) - e_{A1} - e_{B1}
\]
\[
= a + \frac{bp_2}{p_1} - 1
\]
\[
z_2(p) = x_2^A(p) + x_2^B(p) - e_{A2} - e_{B2}
\]
\[
= \frac{(1 - a)p_1}{p_2} + (1 - b) - 1
\]
• Check these are h.d.0 in \( p \),

\[
\begin{align*}
  z_1(t p) &= a + \frac{btp_2}{tp_1} - 1 = z_1(p) \\
  z_2(p) &= \frac{(1 - a)tp_1}{tp_2} - b = z_2(p)
\end{align*}
\]

• Check Walras’ Law,

\[
\begin{align*}
  p \cdot z(p) &= p_1 z_1(p) + p_2 z_2(p) \\
  &= p_1 \left( a + \frac{bp_2}{p_1} - 1 \right) + p_2 \left( \frac{(1 - a)p_1}{p_2} - b \right) \\
  &= 0
\end{align*}
\]

• Can we find the WE price vector \( p^* = (p_1^*, p_2^*) \) s.t. the markets clear? For good 1 market to clear \( x_1^A(p^*) + x_1^B(p^*) \) should equal the aggregate endowment,

\[
\begin{align*}
  a + \frac{bp_2^*}{p_1^*} &= 1 \\
  \Rightarrow \frac{p_2^*}{p_1^*} &= \frac{1 - a}{b}
\end{align*}
\]
• We know by Walras’ law the market for good 2 should also clear. Check this,

\[ x_1^B(p^*) + x_2^B(p^*) = \frac{(1 - a)p_1^*}{p_2^*} + (1 - b) = 1 \]

at \( \frac{p_2^*}{p_1^*} = \frac{1-a}{b} \).

• Note only relative prices are determined in equilibrium; a normal practice is to normalise one of the prices to 1.
5 Existence of a WE

5.1 Proof of the Existence

• Use Walras’ law.

• First normalise prices to the following relative prices (remember \( z(p) \) is h.d.0) to reduce the dimension by 1,

\[
P_n = \frac{\hat{p}_n}{\sum_{m=1}^{N} \hat{p}_m}
\]

• Thus \( \sum_{n=1}^{N} \hat{p}_n = 1 \), i.e. WE solution \( p^* \) belongs to the \( N - 1 \)-dimensional unit simplex,

\[
S^{N-1} = \left\{ p \in \mathbb{R}_+^N : \sum_{n=1}^{N} p_n = 1 \right\}
\]

• Also need,
Theorem 5 (Brouwer Fixed Point Theorem)
If $f : S^{N-1} \rightarrow S^{N-1}$ is a continuous function from the unit simplex to itself, there is some $x^* \in S^{N-1}$ such that $x^* = f(x^*)$.

Proof Prove this for $N = 2$, i.e. show that for a continuous function $f : [0, 1] \rightarrow [0, 1]$, $\exists$ some $x^* \in [0, 1]$ such that $x^* = f(x^*)$, or for $g(x) = f(x) - x$, $g(x^*) = 0$. But as $0 \leq f(x) \leq 1 \forall x \in [0, 1]$, $g(0) = f(0) - 0 \geq 0$ and $g(1) = f(1) - 1 \leq 0$. Then using the intermediate value theorem, for continuous $f$ there must be some $x^* \in [0, 1]$ such that $g(x^*) = f(x^*) - x^* = 0$. •

• Then,
Proposition 6 (Existence of WE) If \( z : S^{N-1} \rightarrow \mathbb{R}^N \) is a continuous function that satisfies Walras’ law \( p.z(p) \equiv 0 \), then \( \exists \) some \( p^* \in S^{N-1} \) such that \( z(p^*) \leq 0 \).

Proof Define a map \( g : S^{N-1} \rightarrow S^{N-1} \) by

\[
g_n(p) = \frac{p_n + \max(0, z_n(p))}{1 + \sum_{m=1}^{N} \max(0, z_n(p))}
\]

for \( n = 1, ..., N \).

Intuitively \( g \) is a function that increases the price of the good if that good is in excess demand (i.e. \( z_n(p) > 0 \)). Since \( \sum_{n=1}^{N} g_n(p) = 1 \), \( g(p) \) is a point in the simplex \( S^{N-1} \). It is also a continuous function as \( z \) and the max function are continuous. Thus using Brouwer’s fixed point theorem there is a \( p^* \) s.t. \( p^* = g(p^*) \), i.e.

\[
p^*_n = \frac{p^*_n + \max(0, z_n(p^*))}{1 + \sum_{m=1}^{N} \max(0, z_n(p^*))}
\]

Is this \( p^* \) a WE? Rearranging,
\[ p_n^* \sum_{m=1}^{N} \max(0, z_n(p^*)) = \max(0, z_n(p^*)) \quad \forall n \]

Multiply each of these \( N \) equations by \( z_n(p^*) \) and sum up over \( n \),

\[
\left\{ \sum_{m=1}^{N} \max(0, z_n(p^*)) \right\} \left\{ \sum_{n=1}^{N} p_n^* z_n(p^*) \right\} = \sum_{n=1}^{N} z_n(p^*) \max(0, z_n(p^*))
\]

But the second \{ . \} is zero by Walras’ law, so,

\[
\sum_{n=1}^{N} z_n(p^*) \max(0, z_n(p^*)) = 0
\]

Now each term of this sum is \( \geq 0 \) since each term is either 0 or \((z_n(p^*))^2\). But if any term were \( > 0 \) then the equality would not hold. Hence every term must be zero, i.e.

\[ z_n(p^*) \leq 0 \quad \forall n = 1, \ldots, N \]
5.2 Convexity Issue

- Strict convexity assumption assures that the demand function is well-defined (i.e. single bundle demanded at each price) and continuous (i.e. small changes in prices lead to small changes in demand).

- Below $A$ has non-convex ICs. At $p^*$ there are two points that max $A$’s utility, but supply $\neq$ demand at either point. Here non-convexity leads to discontinuity in demand.
• Suppose however that the total supply of demand is half way between the two demands at $p^*$. 

• Replicate this once s.t. there are two agents of types $A$ and $B$. 

• If one $A$ demands $X'_A$ and the other $X''_A$, then the total demand would in fact equal total supply. Hence WE exists in the replicated economy. This can be generalised to many agents. 

• *Thus in a large economy in which the scale of non-convexities is small relative to the size of the market, Walrasian equilibria can exist.*
6 Uniqueness

- There has been much research on conditions when the equilibrium will be unique, or at least those which will limit the number of equilibria.

- E.g. need continuous differentiability of the excess demand $z(p)$, as kinks mean whole ranges of prices that are market equilibria. In this case not only are the equilibria not unique, they are not even locally unique.

- One result states that under mild assumptions the number of equilibria will be finite (Regular economy argument) and odd (Index Theory argument). Furthermore if an economy as a whole has the gross substitute property, then the equilibrium will be unique.

- We will review these briefly in reverse order.
6.1 Example of Multiple Equilibria

- Utility functions

\[ u_A(x_A^1, x_A^2) = x_A^1 - \frac{1}{8} (x_A^2)^{-8} \]
\[ u_B(x_B^1, x_B^2) = -\frac{1}{8} (x_B^1)^{-8} + x_B^2 \]

- Endowment allocations \( e_A = (2, r) \) and \( e_B = (r, 2) \), where \( r = 2^{\frac{8}{5}} - 2^{\frac{1}{5}} \).

- \( A \)'s optimisation:

\[
\max_{x_1, x_2, \lambda} \quad L = x_A^1 - \frac{1}{8} (x_A^2)^{-8} \\
\quad \quad \quad \quad -\lambda \left( p_1 x_A^1 + p_2 x_A^2 - 2p_1 - rp_2 \right)
\]
• FOCs:

\[ 1 - \lambda p_1 = 0 \Rightarrow \lambda = \frac{1}{p_1} \]

\[ (x_2^A)^{-9} - \lambda p_2 = 0 \Rightarrow x_2^A = \left( \frac{p_2}{p_1} \right)^{-\frac{1}{9}} \]

\[ p_1 x_1^A + p_2 x_2^A - 2p_1 - rp_2 = 0 \]

\[ \Rightarrow x_1^A = 2 + r \left( \frac{p_2}{p_1} \right) - \left( \frac{p_2}{p_1} \right)^{\frac{8}{9}} \]

• By symmetry,

\[ x_1^B = \left( \frac{p_1}{p_2} \right)^{-\frac{1}{9}} \]

\[ x_2^B = 2 + r \left( \frac{p_1}{p_2} \right) - \left( \frac{p_1}{p_2} \right)^{\frac{8}{9}} \]

• Thus for good 1 market clearance,

\[ 2 + r \left( \frac{p_2}{p_1} \right) - \left( \frac{p_2}{p_1} \right)^{\frac{8}{9}} + \left( \frac{p_1}{p_2} \right)^{-\frac{1}{9}} = 2 + r \]

• Three solutions: \( \frac{p_2}{p_1} = 2, 1, \frac{1}{2} \).
6.2 Gross Substitutes

**Definition** Two goods $i$ and $j$ are gross substitutes at $p$ if $\frac{\partial z_j(p)}{\partial p_i} > 0$ for $i \neq j$.

- N.B. usual definition of substitutes (net substitutes): a price increase of a good increases the Hicksian demand of the other.

**Proposition 7** If all goods are gross substitutes at all prices, then if $p^*$ is an equilibrium price vector, it is unique.

**Proof** Prove for 2 good case. Suppose that $p^*$ and $p'$ are equilibrium price vectors, i.e. $z_i(p^*) = z_i(p') = 0$. H.d.0 means that these are defined only up to a scaler multiplication. W.l.o.g. then, assume $p^*_1 = p'_1$ and $p^*_2 > p'_2$. Now lower price $p^*_2$ down to $p'_2$. As the goods are gross substitutes, then the demand for good 1 must decrease. Thus $z_1(p') < 0$, i.e. $p'$ cannot be an equilibrium price vector. ■
6.3 Index Analysis

- Consider again a 2 good economy.

- Let the price of good 2 be the numeraire and normalised to 1. Then the problem is reduced to finding $p_1$, i.e. it becomes a one-variable problem.

- Draw then the excess demand curve $z_1$ for good 1 as function of $p_1$.

- Desirability $\Rightarrow$ when $p_1$ is small, $z_1 > 0$, and when $p_1$ is large, $z_1 < 0$.

- The equilibrium is at $z_1 = 0$. 

\[ z_1 \]
\[ p_1 \]

\[ z_1 \]
\[ p_1 \]
Can state,

a. *Always odd number of equilibria.*

b. *For multiple equilibria need* \( \frac{dz_1}{dp_1} > 0 \) *for some values of* \( p_1 \). Note as \( z_1 = \sum_{i=1}^{I} (x_{i1} - e_{i1}) \), using Slutsky decomposition with price-dependent endowments,

\[
\frac{dz_1}{dp_1} = \sum_{i=1}^{I} \frac{dx_{i1}}{dp_1} = \sum_{i=1}^{I} \frac{dh_{i1}}{dp_1} + \sum_{i=1}^{I} \frac{dx_{i1}}{dm_i} (e_{i1} - x_{i1})
\]

This is positive when income effect > substitution effects. Hence cannot rule out multiple WE.

c. *If* \( \frac{dz_1}{dp_1} < 0 \) *at all equilibria, then there can only be one equilibrium.* This is an application of the **Index Theorem**. The result can be generalised to \( k \) dimensions.
6.4 Regular Economy

**Definition** An equilibrium $p_1^*$ is **regular** if
\[
\frac{dz_1(p_1^*)}{dp_1} \neq 0
\]
at the equilibrium.

- The solid line is not regular.

- Here the equilibria has no neighbourhood that has no other equilibrium in it, i.e. not **locally unique**.

- But Debreu (1970): “almost all” economies are regular (i.e. all equilibrium prices are regular), as very small perturbation will produce a regular economy. Then the equilibria are finite, and locally unique.
7 Welfare Economics

- First define efficiency,

**Definition** A feasible allocation \( \mathbf{x} \) is **Pareto efficient** if \( \not\exists \) other feasible allocation \( \mathbf{x}' \) s.t. for all agents \( \mathbf{x}' \preceq \mathbf{x} \) and for some \( \mathbf{x}' \succ \mathbf{x} \).

- i.e. one cannot make someone better off without making someone else worse off.

- Next restate the definition of the WE with desirability assumption,

**Definition** A **Walrasian equilibrium** for the given pure exchange consists of a pair \((\mathbf{p}^*, \mathbf{x}^*)\) such that
  a. If agent \( i \) prefers \( \mathbf{x}'_i \) to \( \mathbf{x}^*_i \), it must be that \( \mathbf{p}^*.x'_i > \mathbf{p}^*.e_i \); and
  b. All markets clear: \( \sum_{i=1}^{I} \mathbf{x}^*_i = \sum_{i=1}^{I} e_i \).

Then,
Theorem 8 (FTWE) \( \text{If } (x^*, p^*) \text{ is a WE, then } x^* \text{ is Pareto efficient.} \)

Proof Prove by contradiction. Suppose \((x^*, p^*)\) is a WE but \(x^*\) is not Pareto efficient, i.e. \(\exists\) some other feasible \(x'\) that Pareto dominates \(x^*\). Then,

a. Since \(x'\) is feasible,
\[
\sum_{i=1}^{I} p^* . x'_{i} \leq \sum_{i=1}^{I} p^* . e_i
\]
given non-negative prices.

b. Since \(x'\) Pareto dominates \(x^*\), for all consumers \(x'_{i} \succeq x^*_{i}\), and for some \(x'_{i} \succ x^*_{i}\).
Since \(x^*\) is an equilibrium allocation, it follows from the definition of equilibrium that \(p^* . x'_{i} \geq p^* . e_i \forall i\), and \(p^* . x'_{i} > p^* . e_i\) for some \(i\). Summing up over \(i\) we have,
\[
\sum_{i=1}^{I} p^* . x'_{i} > \sum_{i=1}^{I} p^* . e_i
\]
1. and 2. contradict. \(\blacksquare\)

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• As each WE satisfies the FOCs for utility max, the MRS between goods for each consumer must be equal to the price ratios of the goods. Since all agents face the same price ratios at a WE, it implies that all consumers must have the same MRS. Hence their ICs are tangent, and the outcome cannot be Pareto improved.

• Thus in a **decentralised** economy prices co-ordinate agents’ behaviours to attain Pareto efficiency.

• Note, however, a market equilibrium is not ‘optimal’ in any ethical sense; it may be very ‘unfair’.

• Now conversely can state that every Pareto efficient allocation is a WE,
Theorem 9 (STWE) Assume that preferences are convex, continuous, non-decreasing and locally non-satiable. Let $x^*$ be a Pareto efficient allocation which is strictly positive (i.e. $x^*_{in} > 0 \forall i, n$). Then if we redistribute endowments among all consumers suitably, $x^*$ can be obtained as a WE allocation.

• In an Edgeworth box if we pick an arbitrary Pareto efficient allocation, we know that the MRS must be equal for the two agents. Thus we can pick a price ratio equal to this common value, or equivalently draw the common tangency line separating the two ICs. If we then pick any point on this line as the initial endowment, agents who try to maximise preferences on their budget sets will end up precisely at the Pareto efficient allocation.

• To show this formally we need,
Theorem 10 (Separating Hyperplane Theorem)

If $A$ and $B$ are two non-empty, disjoint, convex sets in $\mathbb{R}^N$, then $\exists$ a linear functional $p$ s.t. $p.x \geq p.y \ \forall x \in A, y \in B$.

- So then,

**Proof** Given an allocation $x^*$, let

$$Z_i = \{z_i \in \mathbb{R}^N : z_i \succ_i x_i^*\}$$

This is the set of all consumption bundles that $i$ prefers to $x_i^*$. Note $\infty$ is in $Z_i$. Then define

$$Z^* = \sum_{i=1}^{I} Z_i = \left\{z \in \mathbb{R}^N : z = \sum_{i=1}^{I} z_i \ \text{with} \ z_i \in Z_i\right\}$$

i.e. the set of bundles for $I$ consumers that strictly Pareto dominates $x^*$. This is a convex set. Also let,

$$Z^+ = \left\{z \in \mathbb{R}^N : z \leq \sum_{i=1}^{I} x_i^*\right\}$$

i.e. a set of feasible bundles given aggregate endowments. $Z^+$ is also convex.
Now given that \( x^* \) is a Pareto efficient allocation it must be that \( Z^* \) and \( Z^+ \) do not intersect. Then by the SHT \( \exists \ p \in \mathbb{R}_+^N \) and \( m \in \mathbb{R}_+^+ \) s.t.

\[
\begin{align*}
\mathbf{p} \cdot \mathbf{z} & \leq m \quad \forall \mathbf{z} \in Z^+ \quad \text{and} \\
\mathbf{p} \cdot \mathbf{z} & \geq m \quad \forall \mathbf{z} \in Z^*
\end{align*}
\]

The claim is that \( \{ p, x^* \} \) forms a WE. In particular we must have \( \mathbf{p} \cdot \mathbf{x}^* = m \), and for any redistribution \( \mathbf{e} \) s.t. \( \mathbf{p} \cdot \mathbf{e}_i^* = \mathbf{p} \cdot \mathbf{x}_i^* \ \forall i \), \( \mathbf{(p, x^*)} \) must be a WE. In effect we have found a set of prices \( p \) which support the allocation \( x^* \) as an equilibrium. What is required is to ensure that \( p \) is a plausible price vector, i.e. it is non-negative, and if \( y_i \succ_i x_i \) then \( \mathbf{p} \cdot y_i > \mathbf{p} \cdot x_i \). These follow from the monotonicity, continuity and local non-satiation assumptions of the preferences.\[\blacksquare\]
• The convexity assumption is crucial. The diagram is an example where consumer \( A \) has a strictly concave indifference curve. Then the tangent line does not separate the ICs.

• Thus there are no points on the line that, going in one direction, both consumers are better off, i.e. there are no reallocated endowment points from which the Pareto efficient point can be attained as a WE.
In summary the FTWE states that a decentralised economy attains a Pareto efficient outcome as a WE (the “invisible hand”). However not all Pareto efficient outcomes are desirable. A government may wish to pick a more equitable point on the Parato set. The STWE states that under certain conditions this can be achieved by simply reallocating the initial endowment distribution, and letting the economy reach a WE again.

This suggests that the issues of efficiency and equity can be separated and need not involve a trade-off.

However in reality such non-distortionary lump-sum taxes do not exist, and market failures such as market power, externalities and asymmetric information mean that the STWE does not apply.