Coxeter Groups and Length Functions—Review Report

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Summary During the time of my fellowship, I have been able to carry out an extensive investigation into the notion of length of subsets of Coxeter groups. Certain subsets, known as parabolic subgroups, play a vital role in Coxeter groups, and for many of these sets, a detailed description of their lengths has been found. Another type of subset that has been considered is cosets. These have been found to yield interesting partially ordered sets which can be viewed as a generalisation of the important Bruhat ordering on Coxeter groups. A further question was that of the action of conjugacy classes on the root system of Coxeter groups. An intriguing pattern had been observed in crystallographic Coxeter groups, relating to the number of negative roots in the image of a given positive root under the action of a given conjugacy class. This pattern was explained and was proved to hold for all finite crystallographic Coxeter groups. The project has provided a solid base for further investigation into lengths of subsets of Coxeter groups.

1 Introduction

In this section we give some definitions and well known results about Coxeter groups. Section 2 gives the original objectives, and Section 3 summarizes the results obtained. There follow sections on the research plan, outcomes, dissemination and public awareness activities.

A Coxeter system \((W, S)\) consists of a group \(W\) (the Coxeter group) along with a distinguished generating set \(R\). The only relations are of the form \((rs)^{m_{rs}} = 1\) for \(r, s \in R\), with the condition that \(m_{ss} = 1\) for each \(r \in R\). The elements of \(R\) are known as the fundamental reflections.

The length function has a defining part in the theory of Coxeter groups. Let \((W, R)\) be a Coxeter system. The length \(l(w)\) of a non-trivial element \(w\) in \(W\) is defined to be

\[
l(w) = \min\{l \in \mathbb{N} | w = r_1r_2\cdots r_l, \text{ some } r_i \in R\},
\]

and \(l(1) = 0\). The length function is of particular importance in the study of Coxeter groups because of its alternative description in terms of the action of \(W\) on its so-called ‘root system’. Let \(V\) be an \(\mathbb{R}\)-vector space with basis \(\Pi\) where \(\Pi = \{\alpha_r | r \in R\}\) is in one-to-one correspondence with \(R\). We may define an inner product on \(V\) which in turn may be used to define an action of \(W\) on \(V\). It is well known that this action is faithful. The root system \(\Phi\) is the set \(\{w \cdot \alpha_r : w \in W, r \in R\}\) be the set of reflections of \(W\). Then the set of positive roots is in one-to-one correspondence with \(\text{Ref}(W)\). Define \(\Phi^+ = \{\sum_{r \in R} \lambda_r \alpha_r \in \Phi | \lambda_r \geq 0 \text{ for all } r \in R\}\) and \(\Phi^- = -\Phi^+\). The sets \(\Phi^+\) and \(\Phi^-\) are called, respectively, the positive and negative roots of \(\Phi\), and it can be shown that \(\Phi = \Phi^+ \cup \Phi^-\). It can be shown that length of \(w\), an element of \(W\), is given by the cardinality of the set \(N(w)\) where

\[
N(w) = \{\alpha \in \Phi^+ | w \cdot \alpha \in \Phi^-\}.
\]

This definition of length may be extended to give the notion of the length of a subset \(X\) of \(W\), (sometimes called the Coxeter length). We simply define the length of \(X \subseteq W\), written \(l(X)\) to be the cardinality of the set \(N(X)\), where

\[
N(X) = \{\alpha \in \Phi^+ | w \cdot \alpha \in \Phi^- \text{ for some } w \in X\}.
\]
Since $N(X) = \bigcup_{x \in X} N(x)$, when $X = \{w\}$, the length of $X$ is just $l(w)$.

A main objective of the proposed research was to find out more about this new, generalised, length function. In particular, information was sought about the lengths of important subsets such as parabolic subgroups, reflection subgroups and cosets. (The standard parabolic subgroup $W_I$ of $W$ is defined to be the subgroup generated by a given subset $I \subseteq R$.) The results obtained are summarized in Section 3.

Another objective of the research concerned the following intriguing phenomenon. Let $C$ be a conjugacy class of $W$. Then for $\alpha \in \Phi^+$, let $C^- (\alpha) = \{w \cdot \alpha | w \in C\} \cap \Phi^-$. Note that if $C^- (\alpha) = \emptyset$ for any $\alpha$, then $l(C) < |\Phi^+|$. This turns out to be rare (see [3]). Experiments seemed to suggest that, for a fixed conjugacy class of a finite crystallographic Coxeter group, $|C^- (\alpha)|$ was dependent only on the height of $\alpha$ and the $W$-orbit in which $\alpha$ lay. It was hoped that a proof of this would be found. This was done; Section 3 gives more detail.

## 2 Objectives

The original objectives were:

1. To investigate the newly-defined notion of length for subsets of Coxeter groups. In particular, to consider the distribution of lengths among important subsets such as parabolic subgroups, arbitrary reflection subgroups and cosets.
2. To investigate the action of conjugacy classes on the root systems of Coxeter groups, in particular, to look at the image of roots under a given conjugacy class, and determine the number of negative roots in this image (these numbers appeared to form a pattern).

## 3 Results

We first discuss the work on length of subsets of Coxeter groups. We highlight several (but by no means all) of the results obtained. Let $W$ be a Coxeter group and $X \subseteq W$. We begin by dealing with the case when $X$ is infinite.

**Lemma 3.1** [Lemma 2.1; [3]] Let $X \subseteq W$. Then $X$ has infinite Coxeter length if and only if $X$ is infinite.

There are two main types of subsets that have been considered in more detail. These are subgroups (especially parabolic subgroups) and cosets.

### Subgroups

For $w \in W$ and a fundamental reflection $r$, it is well known that $l(rwr)$ is either $l(w) + 2$, $l(w) - 2$ or $l(w)$. If $X$ is a finite subgroup of $W$, then $l(X)$ and $l(rXr)$ are not related quite as closely. Instead we have the following:

**Lemma 3.2** [Lemma 3.1 of [6]] Suppose $X$ is a finite subgroup of $W$, $r \in R$ and let $O_r$ be the $X$-orbit of $\alpha_r$. Then

- $l(X^-) < l(X)$ if $O_r \subseteq \{\alpha_r\} \cup (\Phi^- \backslash \{-\alpha_r\})$.
- $l(X^-) > l(X)$ if $O_r \subseteq \Phi^+$ and $O_r \neq \{\alpha_r\}$.
- $l(X^-) = l(X)$ if either $O_r \subseteq \{\pm \alpha_r\}$ or both $O_r \cap \Phi^- \neq \emptyset$ and $O_r \cap \Phi^+ \neq \{\alpha_r\}$.
If we restrict ourselves to the important case of parabolic subgroups, we are in a much better position.

**Theorem 3.3** [Proposition 1.1; [3]] Let $X$ be a finite standard parabolic subgroup of $W$, and $Y$ be conjugate to $X$. Then $l(X) \leq l(Y)$, with equality if and only if $Y$ is also a standard parabolic subgroup of $W$.

Much more specific information about parabolic subgroups has been obtained. For $X$ any subgroup of $W$ we use $\mathbb{X}$ to denote the conjugacy class of $X$ in $W$ and let $\mathbb{X}_{\min}$ be the set consisting of all $Y$ of minimal length in $\mathbb{X}$, and, when it is defined, $\mathbb{X}_{\max}$ denotes those conjugates of $X$ of maximal length in $\mathbb{X}$. In addition, let $W$ be a collection of subsets of $W$ and $n_r$ be the number of subsets in $W$ whose length is $r$. Then (mimicking a Poincaré series) set

$$\Lambda_W(t) = \sum_{r=0}^{\infty} a_r t^r.$$

**Theorem 3.4** [Theorem 1.2; [7]]

(i) If $W \cong A_n$, and $X \cong A_i$, then

$$\Lambda_X(t) = t^i t^{(i+1)/2} \sum_{r=0}^{n-i} (n + 1 - i - r) \left( \binom{r + i - 1}{i - 1} t^{i+1} \right);$$

(ii) If $W \cong B_n$, and $X \cong B_i$, then

$$\Lambda_X(t) = t^{\sigma} \sum_{r=0}^{n-i} \left( \binom{r + i - 1}{i - 1} \right) t^{2ir};$$

(iii) If $W \cong D_n$, and $X \cong D_i$, then

$$\Lambda_X(t) = t^{i(i-1)} \sum_{r=0}^{n-i} \left( \binom{r + i - 1}{i - 1} \right) t^{2ir}.$$

Theorem 3.4 does not cover the connected parabolic subgroups of type $A_n$ when $W$ is of type either $B_n$ or $D_n$. For such subgroups see Lemma 3.2 and Theorems 3.10 and 4.5 of [7]. When $X$ is not connected, the situation appears to be very complicated – more information, along with tables giving the lengths of connected parabolic subgroups for the exceptional groups, is provided in [7]. An examination of maximal length parabolic subgroups appears in [8].

Lengths of parabolic subgroups are particularly important in the light of the next proposition. In order to state it we require the following definition.

**Definition 3.5** Let $X \leq W$. Define $\langle X \rangle_p$ to be the intersection of all parabolic subgroups containing $X$. Similarly, define $\langle X \rangle_{sp}$ to be the intersection of all standard parabolic subgroups containing $X$.

**Proposition 3.6** [Proposition 3.5 of [6]] Let $X \leq W$ be a finite subgroup of $W$ with $P = \langle X \rangle_p$. Then $N(X) = N(P)$.

The following was also proved in [6].

**Proposition 3.7** [Proposition 3.6 of [6]] Let $X$ be a finite subgroup of $W$ and assume that $X \in \mathbb{X}_{\min}$. Then there is a standard parabolic subgroup $W_I$ of $W$ such that $X \leq W_I$ with $N(X) = N(W_I)$.

If $X \leq W$, it is possible that for each $Y \in X$, $l(Y) = l(X)$. If this holds, we say that $X$ is a flat subgroup of $W$. The following result gives a condition which is necessary for this to occur.

**Proposition 3.8** [Proposition 3.10 of [6]] Let $X \neq 1$ be a finite subgroup of $W$, and let $W_I$ denote the standard parabolic subgroup $\langle \mathbb{X} \rangle_{sp}$ of $W$. If $X$ is flat, then $N(X) = \Phi_{W_I}^+$. 
Cosets  The following result is a generalisation of a familiar fact about the usual length function on elements of $W$.

**Proposition 3.9** [Proposition 1.6 of [5]] If $X$ is a finite subset of $W$ and $r \in R$, then

$$l(Xr) = \begin{cases} 
  l(X) + 1 & \text{if } \alpha_r \notin N(X); \\
  l(X) - 1 & \text{if } \alpha_r \in N(x) \text{ for all } x \in X; \\
  l(X) & \text{otherwise.}
\end{cases}$$

If we demand that $X$ be a subgroup of $W$ then we have

**Proposition 3.10** [Proposition 1.7 of [5]] Let $g = r_1 \cdots r_k \in W$ be a reduced expression for $g$ and let $X$ be a finite subgroup of $W$. Then

$$l(X) \leq l(Xr_1) \leq \cdots \leq l(Xr_1 \cdots r_k) = l(Xg).$$

In particular, for all $g \in G$, $l(Xg) \geq l(X)$.

(Notes that the ‘left handed’ versions of Propositions 3.9 and 3.10, comparing the length of $gX$ with the length of $X$, do not hold.) Further, we have

**Proposition 3.11** [Proposition 3.3 of [5]] Let $X$ be a finite subset of $W$ and let $s \in \text{Ref}(W)$. If

$$l(Xs) \leq l(X),$$

then there exists $x \in X$ such that $l(xs) < l(x)$.

Proposition 3.11 implies that for the given $x \in X$, $xs < x$, where $<$ is the Bruhat order on $W$. In [2], Deodhar extended the usual Bruhat order on Coxeter groups to a partial order on the (right) cosets of a standard parabolic subgroup $X$ of $W$. This construction relied on the using the unique element of minimal length in each coset. In [5] and [9] the Bruhat order was generalized further to give an order on cosets of arbitrary subgroups in a Coxeter group. A different generalization of the Bruhat order, called generalized quotients, was given in [1] by Björner and Wachs. The posets defined here have certain properties in common with generalized quotients, but they are not the same. See [6] for a discussion of this.

**Definition 3.12** Suppose that $X \leq W$.

(i) For right cosets $Xg$ and $Xh$ of $X$ we write $Xg \sim Xh$ whenever $Xgt = Xh$ for some $t \in \text{Ref}(W)$ and $l(Xg) = l(Xh)$. Let $\sim$ be the equivalence relation generated by $\sim$ on the set of right cosets of $X$ in $W$ and let $\mathcal{X}$ be the set of $\sim$ equivalence classes.

(ii) Let $x, x' \in \mathcal{X}$. We write $x \sim x'$ if there is a right coset $Xg$ in $x$ and a right coset $Xh$ in $x'$ such that $Xgt = Xh$ for some $t \in \text{Ref}(W)$ and $l(Xg) \leq l(Xh)$. The partial order $\preceq$ on $\mathcal{X}$ is defined by $x \preceq x'$ if and only if there exist $x_1, \ldots , x_m \in \mathcal{X}$ such that $x \sim x_1 \sim \ldots \sim x_m \sim x'$. We shall call $\mathcal{X}$ the $X$-poset (of $W$).

(iii) If, in (i) and (ii), we use the set of fundamental reflections instead of $\text{Ref}(W)$ we may define, analogously, the weak $X$-poset (of $W$) denoted by $\mathcal{X}_w$ with ordering $\preceq_w$.

For the coset $Xg$ we use $[Xg]$, respectively $[Xg]_w$, to denote the $\sim$ equivalence class, respectively the $\preceq_w$ equivalence class containing $Xg$. Several results about $X$-posets are shown in [5]. As an example we give the following.

**Theorem 3.13** [Theorems 1.2 and Theorem 1.3(iii); [5]] Suppose $X \leq W$ and let $\mathcal{X}$, respectively $\mathcal{X}_w$, denote the $X$-poset, respectively weak $X$-poset. Then both $\mathcal{X}$ and $\mathcal{X}_w$ have a unique minimal element, namely the $\sim$ (respectively $\preceq_w$) equivalence class containing $X$. In addition, both the weak $X$-poset and the $X$-poset are symmetric. That is if $[Xg] \preceq [Xh]$, respectively $[Xg]_w \preceq_w [Xh]_w$, then $[Xgw_0] \preceq [Xgw_0]$, respectively $[Xgw_0]_w \preceq_w [Xgw_0]_w$. 

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The second objective of the research was the study of the sets $C^-(\alpha)$ described in Section 1, where $C$ is conjugacy class of $W$ and $\alpha$ is a positive root. We call $C^-(\alpha)$ the negative orbit of $\alpha$ under $C$. (The notation is somewhat unusual as we are considering orbits under sets which are not, in general, groups.) Set $n_X^-(\alpha) = |C^-(\alpha)|$. We have the following result.

**Theorem 3.14** [Theorem 1.1 of [4]] Suppose $W$ is a finite crystallographic Coxeter group. Let $\alpha = \sum_{r \in \Pi} \lambda_r \alpha_r$ and $\beta = \sum_{r \in \Pi} \mu_r \alpha_r$ be positive roots in the same orbit $\Phi(\alpha) := W \cdot \alpha$ of $\Phi$. Then for a conjugacy class $X$ of $W$, there exists a constant $f(X) \in \{0, \pm 1\}$ dependent only on $X$ and $\Phi(\alpha)$ such that

$$n_X^-(\alpha) - n_X^-(\beta) = f(X) \left( \sum_{\alpha_r \in \Pi \cap \Phi(\alpha)} \lambda_r - \sum_{\alpha_r \in \Pi \cap \Phi(\alpha)} \mu_r \right).$$

To paraphrase: for $W$ a finite, simply laced crystallographic Coxeter group, if we fix a conjugacy class $X$, and order the roots according to height then the sequence of integers $\{n_X^-(\alpha)\}_{\alpha \in \Phi^+}$ is either constant or monotonic increasing or monotonic decreasing. This is remarkably uniform behaviour. In the non-crystallographic finite Coxeter groups $H_3$ and $H_4$ the situation is completely different, with no obvious patterns. The values of $n_X^-(\alpha)$ are given in [4] for each conjugacy class of $H_3$ and $H_4$.

4 Research Plan Review

The research plan was largely unchanged. However, I was offered a temporary lectureship at UMIST for one year from September 2003, so I put my fellowship on hold for that time. Then in July 2004, I was offered a permanent lectureship at Birkbeck College, and in today’s job market one does not turn down permanent positions. Thus I was forced to terminate the fellowship. This meant I had only completed 18 months of the research, and so obviously not as much was completed as originally hoped. I feel I ought also to mention that the death of my mother in November 2002 probably affected progress for a while.

The expenditure during the award was in line with the original spending plans. I was able to attend the ICM in Beijing (2002), the B(A)MC in Warwick (2002), as well as a one-day conference on Coxeter groups in Birmingham and the EPSRC Research Fellows Seminar in Nottingham (2003). I was also able to visit the University of Sydney for two months, and to purchase some books and a laptop computer, which was very useful when travelling. The total spent from the support fund was £6896.91. The expenditure was less than originally planned because of the fellowship being terminated early.

5 Outcomes

The transition from PhD to permanent post is usually the hardest time in the career of an academic. There is little doubt that without a research position after my PhD I would have left academia. The award of an EPSRC Fellowship enabled me to continue in my chosen career. I had the chance to focus on research and write several papers. With the generous allowance for consumables, I was able to attend conferences, both in the UK and abroad, and to spend two months working in Sydney with Professor RB Howlett, one of the top practitioners in my research area. I also attended the ICM in Beijing in 2002. I gave a talk there and discussed my work with several mathematicians. In addition, I have given seminars in the UK, at Oxford, Cambridge and Manchester. The contacts I have made during the time of my fellowship will be invaluable in
establishing future collaborations. This is probably the most important outcome of the award.

I have recently taken up a permanent lectureship in Mathematics at Birkbeck College, London. My research record was a major factor in getting this position, and the fellowship played a large part in that. It bridged the gap between PhD and first permanent job. This is one of the most important roles of such awards.

6 Dissemination

Several articles were written during or as a result of this fellowship (see references [3] – [10] below). I have spoken about my research on several occasions. I gave a seminar during my visit to the University of Sydney (2003). I also gave talks at the Algebra Seminar of the Mathematics Institute, Oxford (2002) and at the ICM, Beijing (2002). My co-author on [3] spoke at the London Algebra Colloquium about that research in 2002.

7 Public Awareness Activities

Increasing the public awareness of mathematics, and the work that mathematicians do, is of great importance. During the time of my fellowship, and since, I have given several talks to schoolchildren about mathematics and the work I do. In 2003 I was a finalist in the ‘Perspectives’ competition, run by the British Association. The aim of the competition was to encourage young scientists to explain their work to a lay audience, illustrating its relevance to society. As a finalist, I was funded to attend the British Association Festival of Science and give a poster presentation of my work. I also led a seminar at the annual conference of the Mathematical Association. I have been (and still am) involved in the Mathematics Masterclasses, organised by the Royal Institution of Great Britain. I gave a two hour masterclass at the Royal Institution in June 2002 attended by over 300 pupils.

References