Credibility of Optimal Monetary Delegation: 
A Comment*

John Driffill
Birkbeck College, University of London, and CEPR, 
and

Zeno Rotondi
Capitalia and University of Ferrara

Revised: September 2005

Abstract

In his recent paper in the American Economic Review, Jensen (1997) argues that delegation of monetary policy to an independent central bank, which acts as an agent for the government, does not mitigate the problem of time-inconsistency, but merely relocates it.

*We acknowledge with thanks support for this work provided through ESRC research grant L138251003 “Imperfect Financial Markets, Business Cycles, and Growth”, which forms part of the programme on Understanding the Evolving Macroeconomy (UEM). We are grateful to participants at the Money, Macro and Finance (MMF), and UEM Conference 2002 for their helpful comments. We thank also the editors and referees of this journal for their advice and suggestions. An Appendix containing details of algebraic derivations in Section 4 of this paper can be found on the AER web site and on the authors’ site at www.econ.bbk.ac.uk/faculty/drifill
He examines a government that delegates monetary policy to an independent central bank, and that faces costs if it interferes in the policy decisions of the bank by appointing a new central banker to obtain a preferred result. He shows that delegation makes it more difficult to sustain the credibility of optimal monetary policy. We show here that this result emerges because Jensen examines a restricted range of policy actions for the government. When this restriction is lifted, the result is reversed. By means of suitable announcements of contracts for the central bank, combined with appropriate actually implemented contracts, delegated policy enables zero inflation to prevail in economies in which it could not do so without delegated policy. These economies are ones that have relatively low discount factors.

JEL Classification numbers: E31; E58; E61

Keywords: credibility, delegation, time-inconsistency, independent central banks, monetary policy

1 Introduction

In a paper published not so long ago in this journal, Henrik Jensen (1997) argued that delegation of monetary policy to an independent central bank does not improve the credibility of optimal monetary policy. He writes, "in fact, the chances of attaining credibility decrease" (his italics). He obtains this striking result from an analysis in which he assumes that the independence of the central bank is protected by re-appointment costs, which the government bears if it intervenes to alter the contract under which the central bank
operates and thereby manipulates the central bank’s policy actions at short notice. This result runs counter to many people’s intuition about the effects of re-appointment costs. Most would imagine that such costs would protect a central bank from interference from elected politicians, and thereby enhance credibility. Jensen’s result seems to substantiate Bennett McCallum’s (1995, 1997) criticism that delegation does not solve the time-inconsistency problem of monetary policy, but merely relocates it.

In this comment we show that Jensen’s result derives from an implicit limit on the range of policies that are considered. His analysis restricts the set of announced incentive schemes (or contracts) that the government might present to the central bank to maintain zero inflation. When this restriction is removed, the opposite result is restored; namely that costs of reappointing the central banker do in fact enable zero inflation to emerge as a credible policy in circumstances (i.e., when the government is sufficiently impatient) in which it would not do so in the absence of such costs.

2 The model

In this comment, we refer back extensively to Henrik Jensen’s original article, and follow his notation. But for convenience, we repeat essential features of the model here. Jensen (1997) takes the widely used framework in which the economy’s aggregate supply function is given by the standard expectations-augmented Phillips curve
where the log of the natural level of output is normalised to zero; $\pi_t, \pi^e_t$ are the actual and expected inflation rate respectively.

The government’s loss function combines squared deviations of inflation and output from their target levels. A term is added, when monetary policy is delegated to the central bank, to reflect the costs of re-appointing (or replacing) the central banker. The cost is proportional to the squared difference between the announced contract ($f^a_t$) and the realised contract ($f_t$).

We follow Jensen’s notation in denoting the loss function, when it is augmented by the term in re-appointment costs, by $\tilde{L}_t$ rather than $L_t$. Thus the government’s loss function is

$$y_t = \alpha (\pi_t - \pi^e_t)$$

$$\tilde{L}_t = [\pi_t^2 + \lambda (y_t - y^*)^2 + \varphi (f_t - f^a_t)^2].$$

The central banker, to whom monetary policy is delegated, has the same (social) preferences over output and inflation as the government, but is subject to the contract which penalises the bank for creating inflation. The bank’s loss function is

$$L^b_t = [\pi_t^2 + \lambda (y_t - y^*)^2 + 2f_t \pi_t].$$

It shows that a penalty $2f_t$ is imposed on the banker for each unit increase in the inflation rate.
In each period, four actions take place in sequence. At stage zero, the government delegates monetary policy to a central banker and announces a contract $f^a_t$. At stage one, the private sector forms expectations about inflation and sets wages. At stage two, the government sets actual conditions $f_t$ for monetary policy. Finally, at stage three, the central bank sets actual inflation.

3 Optimal Delegation

Jensen defines optimally delegated monetary policy as the policy in which the government imposes a contract $f_t = f^{opt} = \lambda_\alpha y^*$ on the central bank. Provided this is anticipated by the private sector, it induces the central bank to deliver zero inflation with no loss of output. But this policy is not credible in the one-shot game. If the government announces $f^a_t = f^{opt}$ it has an incentive to deviate from carrying it out, instead replacing it with a weaker contract.

Jensen considers whether the credibility of optimal policy can be established in the infinitely repeated game, if the private sector reverts for one period to the discretionary solution following a deviation from optimal policy. He shows that, in the absence of delegation (that is, when $\varphi = 0$ and $f_t = 0$, and the bank and the government are one and the same agent), if the discount factor, $\beta$, exceeds a critical value ($\bar{\beta} \equiv 1/\Lambda$, with $\Lambda \equiv 1 + \lambda_\alpha^2$) optimal monetary policy is a perfect Nash equilibrium and therefore is credible.

Does delegation allow optimal inflation to be achieved at lower discount
factors? Jensen considers the following strategy combinations. If inflation turned out as expected in the previous play of the game, then the government announces and also eventually implements the optimal contract \( f_t = f_{opt} = \lambda \alpha y^* \), but if inflation did not turn out as expected then the government imposes the discretionary contract \( f_t^{a,NC} = \lambda \alpha y^* \frac{\phi \Lambda}{\phi \Lambda^2 + 1} \); \( f_t^{a,NC} = f_t^{a,NC} \frac{\phi \Lambda}{\phi \Lambda^2 + 1} = \lambda \alpha y^* \frac{\phi \Lambda^2}{\phi \Lambda^2 + 1} \). For the private sector, if last period’s actual inflation was as expected, they expect zero inflation this period. If last period’s inflation was not as expected, they expect the discretionary inflation rate this period. Here the superscript \( NCD \) denotes values that emerge when there is no commitment \( (NC) \) from the government, and monetary policy is delegated \( (D) \) to the central bank. Jensen (1997) derives the values taken by these quantities.

If the government were to deviate, having announced optimal delegation, it would choose \( f_t \) so as to minimise its loss function, subject to \( \pi_t = 0 \) and \( f_t^{a,DD} = \lambda \alpha y^* \). Superscript \( DD \) denotes deviation with delegation. The government will not deviate from the proposed equilibrium if the advantage to doing so, the difference between the loss under the pre-committed solution \( \tilde{L}_{t+1}^{PR} \) and the loss under deviation \( \tilde{L}_{t+1}^{DD} \), is no greater than the discounted value of the punishment next period, the difference between the loss under discretion \( \tilde{L}^{NCD}_{t+1} \) and the loss under precommitment. The condition is

\[
\tilde{L}_{t+1}^{PR} - \tilde{L}_{t+1}^{DD} \leq \beta \left( \tilde{L}_{t+1}^{NCD} - \tilde{L}_{t+1}^{PR} \right),
\]

and this leads to a critical value for \( \beta, \hat{\beta}^D (\phi) \), which, Jensen shows, is:
\[ \beta \geq \beta^D (\varphi) \equiv \frac{1 + \varphi \Lambda^2}{\Lambda (1 + \varphi \Lambda)} \]  
(5)

It is clear, as Jensen proves, that the critical discount factor is greater when policy is delegated than when the government carries out monetary policy directly. The higher the re-appointment costs (\( \varphi \)), the higher the critical discount factor \( \beta^D (\varphi) \). The reason is that, although both the punishment subsequent to deviation and the gain from deviating become weaker the higher the reappointment costs (\( \varphi \)), the gain from deviating decreases by less than does the punishment.

4 Wider set of policy choices

An implicit restriction in Jensen’s analysis is that in the case of optimal delegation, both the announced and the actual central bank contract are set at the optimal level \( \lambda_0 y^* \). However, there is no reason why they should be the same. Any credible policy regime in which the actual contract is \( \lambda_0 y^* \) will deliver zero inflation. If the government were to announce a tougher contract, the incentive to deviate from the optimal contract would be reduced, and this fact may enable the policy to be credible with a lower discount factor. In this section we consider this possibility.

Consider the following strategies. In each period when it plays the reputational strategy (that is, when it is following its commitment policy), the government announces a "tough" contract for the central bank \( f_i^n = \omega \), with \( \omega \geq \lambda_0 y^* \). People expect inflation of zero (\( \pi_i^e = 0 \)). The government then
actually implements the optimal contract with $f_t = \lambda \alpha y^*$. With this contract in place, the central bank duly delivers inflation of the expected rate $\pi_t = 0$.

When the government cheats (deviates from the commitment policy), it announces the same contract as in the commitment policy, that is $f_t^a = \omega$, and people respond by expecting zero inflation ($\pi_t^e = 0$), but then the government implements a different contract. In fact it behaves in a discretionary way, and implements the contract that minimizes its expected loss for this period, given the announced contract and the public’s expectations of inflation. The government therefore implements $f_t = \omega \frac{\phi \Lambda}{1 + \phi \Lambda}$ (following Jensen’s equation 7) and the central bank delivers inflation

$$
\pi_t = \frac{\lambda \alpha y^*}{\Lambda} - \frac{\varphi \omega}{1 + \varphi \Lambda}
$$

following Jensen’s equation 6.

Following a period in which the government has deviated, it is punished. In this play of the game, the government follows the discretionary policy, as set out in Jensen (1997), equation 10, and reproduced in Section 3 above.

The payoffs to the government of two of the scenarios set out above – commitment and deviation – depend on the value of the announced contract parameter $\omega$. The payoffs under the three different scenarios are as follows:

In the proposed equilibrium, with the government following the pre-committed strategy, its loss is

$$
\tilde{L}_t^{PR}(\omega) = \lambda y^{*2} + \varphi(\lambda \alpha y^* - \omega)^2.
$$

8
The government’s loss on deviating from the pre-commitment strategy is

$$L_{t}^{DD}(\omega) = \frac{\lambda y^2}{\Lambda} + \frac{\omega^2 \phi}{1 + \phi \Lambda}. \quad (7)$$

And when the private sector and government follow the discretionary strategy, the government’s loss is

$$L_{t+1}^{NCD} = \frac{y^2 \lambda (\phi \Lambda + 1) \Lambda}{\varphi \Lambda^2 + 1}. \quad (8)$$

In order for the proposed solution to be sustainable with one-period punishment by reversion to discretion, we need to have

$$(\tilde{L}_{t}^{PR}(\omega) - \tilde{L}_{t}^{DD}(\omega)) \leq \beta(\tilde{L}_{t+1}^{NCD} - \tilde{L}_{t+1}^{PR}(\omega)). \quad (9)$$

How do these payoffs change as $\omega$ is varied? Figure 1 plots them as functions of $\omega$. At $\omega = \lambda \alpha y^*$, the loss under commitment ($\tilde{L}_{t}^{PR}(\omega)$ or $\tilde{L}_{t+1}^{PR}(\omega)$, equation (6)) is at a minimum. The loss for deviation ($\tilde{L}_{t}^{DD}(\omega)$) is increasing in $\omega$, but it is less than the loss under commitment when $\omega = \lambda \alpha y^*$. Thus as $\omega$ is increased above $\lambda \alpha y^*$ the temptation to cheat, $\left(\tilde{L}_{t}^{PR}(\omega) - \tilde{L}_{t}^{DD}(\omega)\right)$, diminishes. The loss under discretion, which is the loss in the punishment phase of the game, $\tilde{L}_{t+1}^{NCD}$, is independent of $\omega$. Therefore the punishment for cheating, $\left(\tilde{L}_{t+1}^{NCD} - \tilde{L}_{t+1}^{PR}(\omega)\right)$, also diminishes as $\omega$ is increased above $\lambda \alpha y^*$, but it does so slowly at first, since $\tilde{L}_{t+1}^{PR}(\omega)$ is locally constant. So a small increase in the value of $\omega$ above $\lambda \alpha y^*$ is likely to make the commitment outcome sustainable with a lower discount factor.

It is possible to find the value of the announcement $\omega$ for which the critical value of $\beta$ needed to sustain the reputational solution is at a minimum. From
(9) it is clear that the critical value of $\beta$ satisfies

$$
\beta_{\text{critical}} = \frac{L^PR_t(\omega) - L^DD_t(\omega)}{L^NCD_t - L^PR_{t+1}(\omega)}.
$$

With some algebraic manipulation, this critical value can be expressed as

$$
\beta_{\text{critical}} = \frac{\varphi\Lambda}{(1 + \varphi\Lambda)} \frac{[A - \omega']^2}{[B - \omega'] [B + \omega']},
$$

(10)
in which $\omega' \equiv \omega - \lambda\alpha y^*$, $A \equiv \frac{\lambda\alpha y^*}{\varphi\Lambda}$, $B \equiv \frac{\lambda\alpha y^*}{\sqrt{\varphi(1 + \varphi\Lambda^2)}}$.

We want to choose $\omega$ (equivalently $\omega'$) to minimize the critical value. This implies a value of $\omega'$ that lies in the range $(0, B)$. The value of $\omega'$ that minimizes $\beta_{\text{critical}}$ is

$$
\omega' = B^2 / A,
$$

and the value of $\beta_{\text{critical}}$ at its minimum point is

$$
\hat{\beta}^{D*} = \frac{1}{A(1 + \varphi\Lambda)}.
$$

Since this value for $\hat{\beta}^{D*}$ is less than $1 / \Lambda$, this result proves that the critical value of $\beta$ under delegation, when any announcement is allowed for, is less than under simple discretion.

The result is illustrated in Figure 2, in which the function (10) is plotted against $\omega'$. It shows that at $\omega' = 0$ (that is at $\omega = \lambda\alpha y^*$), the critical value of the discount factor equals $\hat{\beta}^D$ ($> 1 / \Lambda$), the value obtained by Jensen and given in equation (5). As $\omega'$ increases, the critical value of $\beta$ falls until it reaches a minimum, at $\omega' = B^2 / A$, when the critical value is $\hat{\beta}^{D*}$ as defined above. For values of the incentive $\omega'$ greater than $B^2 / A$ the value of the critical discount factor rises again, and approaches infinity as $\omega'$ approaches
B. A government with some given value of the discount factor would wish to employ the lowest value of the announced incentive scheme \( \omega' \) consistent with sustaining zero inflation in the reputational equilibrium. The left hand branch of the relationship between \( \omega' (\in (0, B)) \) and \( \beta_{\text{critical}} \) is therefore the relevant one. Delegation with an announced incentive scheme and reappointment costs therefore permits some types of government, those types whose discount factor \( \beta \) lies between \( 1/\Lambda \) and \( \beta^{D*} = \frac{1}{\Lambda(1+\varphi \Lambda)} < 1/\Lambda \), who would not have been able to sustain zero inflation in the absence of delegation with reappointment costs, to sustain zero inflation in the reputational equilibrium.

To give some indication of the quantitative significance of the results, Table 1 offers a numerical example. Four parameters determine the size of the effects: \( \lambda, \alpha, y^*, \) and \( \varphi \). We set the weight on output stabilization \( \lambda \) equal to 1, the same as the weight on inflation stabilization in the objective functions (2) and (3); for the slope of the aggregate supply function (1) we use \( \alpha = 0.4 \) so a 10\% increase in surprise inflation induces a 4\% increase in aggregate output; and for the target level of aggregate income we use \( y^* = 0.05 \), indicating that the target is 5\% above the natural level. The size of the reappointment cost parameter \( \varphi \) is harder to tie down, so we report results for a range of values. The table shows that optimal delegation as derived in this paper yields a significant reduction in the lowest discount

---

\(^1\)This minimum announcement consistent with maintaining the reputational equilibrium can be used, in order to simplify the analysis, to rule out the government’s making any other announcement. We may assume that the private sector would revert to the discretionary solution if the announcement were to differ from this one.
needed to sustain zero inflation relative to that required when there is no
delegation. When the re-appointment cost parameter is set at $\varphi = 0.5$,
Jensen’s analysis finds that a discount factor of 0.913 is needed; with no
delegation the lowest discount factor is 0.862; and when the optimal contracts
are used, without restriction on the announced contract, the lowest discount
factor, our $\beta^{D*}$, equals 0.772. This shows that there is a significant effect of
loosening Jensen’s restriction on announced incentive schemes.

5 Conclusion

This comment has shown that if the government is allowed to make any
announcement of the central bank’s inflation contract $\omega$ then it is possible
to have zero inflation using delegation with a lower discount factor than is
needed to sustain zero inflation with no delegation of policy.

It may be argued that this result is merely a curiosity, because it involves
the government in making an announcement about $\omega$ that is not honoured.
The actual value of the inflation penalty in the central bank’s contract is
always less than the announced value. This behaviour is expected, and the
private sector expects and gets zero inflation – so they are happy. Arguably
this scenario does not correspond with the behaviour of any government and
central bank in practice.

A counter-argument to such a position is that the scenario presented
above is analogous to Svensson’s (1997) suggestion that optimal inflation
and output stabilization might be achieved by giving the central bank a
target for inflation that is lower than society’s target – knowing all the while that discretionary behaviour by the central bank will lead to its generating society’s target inflation rate on average. Alternatively, as Svensson (1997) explains, the inflation bias may be more simply removed by delegating to the central bank an output target equal to the economy’s natural output level, instead of introducing either an inflation contract or a lower inflation target.

Like Jensen (1997), the analysis in our paper relies on the assumption that delegation is imperfect. Having delegated policy, the government can intervene and, at some cost, change the inflation contract after expectations have been formed but before inflation is realized. In the case that changing the contract \textit{ex post} takes longer than the lag between expectations formation and inflation realization, or that the cost (\(\phi\)) of such a change is infinitely high, then delegation is completely successful.

The key point that we would stress here is the logical one that Jensen’s result is based on an implicit assumption that may be too strong. When that assumption is relaxed, it is possible, by the government’s appropriate choice of contracts for the central bank, to sustain zero inflation with delegated monetary policy when the discount factor is too low to sustain it in the absence of delegation.
REFERENCES


Table 1

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\alpha$</th>
<th>$y^*$</th>
<th>$\varphi$</th>
<th>$\tilde{\beta}$ (Jensen)</th>
<th>$\frac{1}{\Lambda}$ (no del.)</th>
<th>$\tilde{\beta}^{D_0}$ (opt. del.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4</td>
<td>0.05</td>
<td>1</td>
<td>0.936</td>
<td>0.862</td>
<td>0.399</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
<td>0.05</td>
<td>0.5</td>
<td>0.913</td>
<td>0.862</td>
<td>0.546</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
<td>0.05</td>
<td>0.1</td>
<td>0.876</td>
<td>0.862</td>
<td>0.772</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
<td>0.05</td>
<td>0.01</td>
<td>0.864</td>
<td>0.862</td>
<td>0.852</td>
</tr>
</tbody>
</table>

The column labelled $\tilde{\beta}$ gives the lowest discount factor for which zero inflation can be sustained using Jensen’s proposed scheme of delegation; column $1/\Lambda$ gives the lowest discount factor when policy is not delegated (or $\varphi = 0$); and column $\tilde{\beta}^{D_0}$ gives the lowest discount factor when the optimal announced and implemented contracts are used.
Figure 1: Losses associated with the pre-commitment, deviation and punishment (discretion) outcomes as a function of $\omega$ the announcement of the government.
Figure 2: The critical value of the discount factor as a function of the announced penalty scheme for the central bank ($\omega$).
Appendix to ‘Credibility of Optimal Monetary Delegation: A Comment’
by John Driffill and Zeno Rotondi

This appendix provides some additional details of the derivation of the critical discount factor in Section 4 of the main text.

When the government plays the reputational strategy (that is, when it is following its commitment policy), the government announces a contract \( f_t^a = \omega \), with \( \omega \geq \lambda y^* \). People expect inflation of zero \( (\pi_t^e = 0) \). The government then actually implements the contract \( f_t = \lambda y^* \), i.e., the actual penalty on the central bank for creating inflation is less than the pre-announced one. It is in fact equal to the penalty that induces the central bank to deliver zero inflation. With this scheme in place, the central bank duly delivers inflation of the expected rate \( \pi_t = 0 \). The government’s loss is therefore

\[
\tilde{L}^{PR}_t(\omega) = \lambda y^{*2} + \varphi (\lambda y^* - \omega)^2. \tag{A.1}
\]

When the government cheats (deviates from the commitment policy), it announces the same contract as in the commitment policy, that is \( f_t^c = \omega \), and people respond by expecting zero inflation \( (\pi_t^c = 0) \), but then the government implements a different contract than it announced. It implements the contract that minimizes its expected loss for this period, given the announced contract and the public’s expectations of inflation. The best scheme to implement satisfies Jensen’s equation 7. The government therefore implements
\( f_t = \omega \frac{\varphi \Lambda}{1 + \varphi \Lambda} \) and the central bank delivers inflation
\[
\pi_t = \frac{\lambda \alpha y^*}{\Lambda} - \frac{\varphi \omega}{1 + \varphi \Lambda}.
\]
The government’s loss when it cheats is therefore
\[
\tilde{L}_{DD}(\omega) = \pi_t^2 + \lambda (\alpha \pi_t - y^*)^2 + \varphi (\omega \frac{\varphi \Lambda}{1 + \varphi \Lambda} - \omega)^2
\]
\[
= \frac{\lambda y^*}{\Lambda} + \frac{\omega^2 \varphi}{1 + \varphi \Lambda}
\]
In the punishment phase of the game, the government plays the discretionary policy. The government announces an incentive scheme \( f_t^{a,NCD} = \lambda \alpha y^* \frac{\Lambda}{\varphi \Lambda + 1} \), people expect inflation \( \pi^e_t = \lambda \alpha y^* - f_t^{a,NCD} \frac{\varphi \Lambda}{\varphi \Lambda + 1} \), the actual incentive scheme is \( f_t = f_t^{a,NCD} \frac{\varphi \Lambda}{\varphi \Lambda + 1} \), and actual inflation turns out as expected. The government’s losses are then
\[
\tilde{L}_{NCD}^{t+1} = \frac{y^* \lambda (\varphi \Lambda + 1) \Lambda}{\varphi \Lambda^2 + 1}.
\]
The critical value of the discount factor \( \beta \) satisfies
\[
\beta_{\text{critical}} = \frac{\tilde{L}_{PR}(\omega) - \tilde{L}_{DD}(\omega)}{\tilde{L}_{NCD} - \tilde{L}_{PR}(\omega)}.
\]
The numerator of this expression can be written as
\[
\tilde{L}_{PR}(\omega) - \tilde{L}_{DD}(\omega) = \lambda y^* \varphi + \varphi (\lambda \alpha y^* - \omega)^2 - \frac{\lambda y^*}{\Lambda} - \frac{\omega^2 \varphi}{1 + \varphi \Lambda};
\]
which with a little manipulation becomes
\[
\tilde{L}_{PR}^{t+1} - \tilde{L}_{DD}^{t+1} = \frac{1}{\Lambda (1 + \varphi \Lambda)} [\lambda \alpha y^* (1 + \varphi \Lambda) - \omega \varphi \Lambda] \]
\[
= \frac{\varphi^2 \Lambda}{(1 + \varphi \Lambda)} \left[ \lambda \alpha y^* \left( \frac{1 + \varphi \Lambda}{\varphi \Lambda} \right) - \omega \right]^2
\]
\[
= \frac{\varphi^2 \Lambda}{(1 + \varphi \Lambda)} \left[ \lambda \alpha y^* + \lambda \alpha y^* - \omega \right]^2 ;
\]
while the denominator gives

\[ \tilde{L}_{t+1}^{NCD} - \tilde{L}_{t+1}^{PR} = y^* \lambda \left( \frac{\Lambda + \varphi \Lambda^2}{1 + \varphi \Lambda^2} \right) - \lambda y^* - \varphi(\lambda \alpha y^* - \omega)^2; \]

which with some manipulation becomes

\[ \tilde{L}_{t+1}^{NCD} - \tilde{L}_{t+1}^{PR} = \frac{y^* \lambda^2 \alpha^2}{1 + \varphi \Lambda^2} - \varphi(\lambda \alpha y^* - \omega)^2 \]

\[ = \varphi \left[ \frac{y^* \lambda^2 \alpha^2}{\varphi(1 + \varphi \Lambda^2)} - (\lambda \alpha y^* - \omega)^2 \right] \]

\[ = \varphi \left[ \frac{y^* \lambda \alpha}{\sqrt{\varphi(1 + \varphi \Lambda^2)}} + (\lambda \alpha y^* - \omega) \right] \left[ \frac{y^* \lambda \alpha}{\sqrt{\varphi(1 + \varphi \Lambda^2)}} - (\lambda \alpha y^* - \omega) \right]. \]

Now all this can be put back together. The expression for the critical \( \beta \) can be written as

\[ \beta_{\text{critical}} = \frac{\tilde{L}_{t+1}^{PR} - \tilde{L}_{t+1}^{DD}}{\tilde{L}_{t+1}^{NCD} - \tilde{L}_{t+1}^{PR}} \]

\[ = \frac{\varphi \Lambda}{(1 + \varphi \Lambda) \left[ A - \omega' \right]^2} \left[ B - \omega' \right] \left[ B + \omega' \right], \]

in which \( \omega' \equiv \omega - \lambda \alpha y^* \), \( A \equiv \frac{\lambda \alpha y^*}{\varphi \Lambda} \), \( B \equiv \frac{\lambda \alpha y^*}{\sqrt{\varphi(1 + \varphi \Lambda^2)}} \).

We want to choose \( \omega \) to minimize the critical value. We are looking at values of \( \omega' \) that lie in the range \((0, B)\). That is equivalent to looking at values of \( \omega \) that are at least as great as in the Jensen solution and which go up to the value at which the punishment for cheating becomes zero, i.e., where the loss due to discretion equals the loss under reputation. At the minimum critical value,

\[ - \frac{2}{A - \omega'} + \frac{1}{B - \omega'} - \frac{1}{B + \omega'} = 0. \]
Multiplying through by \((A - \omega')(B - \omega')(B + \omega')\) and tidying up gives
\[
\omega' = B^2/A,
\]
and the value of the function at the minimum point is
\[
\beta^{D*} = \min_{\omega' \in (0, B)} \left( \frac{\hat{L}_t^{PR} - \hat{L}_t^{DD}}{\hat{L}_{t+1}^{NCD} - \hat{L}_{t+1}^{PR}} \right)^2
\]
\[
= \frac{\varphi \Lambda}{(1 + \varphi \Lambda)} \left[ A - \frac{B^2}{A} \right] \left[ B - \frac{B^2}{A} \right] \left[ B - \frac{B^2}{A} \right]
\]
\[
= \frac{\varphi \Lambda}{(1 + \varphi \Lambda)} \frac{A^2 - B^2}{B^2}
\]
\[
= \frac{\varphi \Lambda}{(1 + \varphi \Lambda)} \left( \frac{\lambda^2 \alpha^2 \gamma^*^2}{\varphi^2 \Lambda^2} - \frac{\lambda^2 \alpha^2 \gamma^*^2}{\varphi (1 + \varphi \Lambda^2)} \right) \frac{\varphi (1 + \varphi \Lambda^2)}{\lambda^2 \alpha^2 \gamma^*^2}
\]
\[
= \frac{1}{\Lambda (1 + \varphi \Lambda)}.
\]
Since this value is less than \(1/\Lambda\) this proves that the critical \(\beta\) under delegation with any announcement is less than under simple discretion.