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1 Introduction

The importance of correlation as a measure of dependence has been often emphasized in the context of pricing derivatives whose payoffs depend on the joint distribution of underlying prices, or indices, or rates. There is no doubt that at present these derivatives are getting more and more popular and numerous. For example, they include a vast class of basket options, i.e., options on linear combinations of various price indexes from different markets. Another example can be found in commodity markets, where spread options, both standard and, increasingly, "multi-legged", are omnipresent.

Obviously, there are limitations to the use of the correlation, especially, in the case of complex joint distributions. However, even under these circumstances practical considerations often force one to use correlations as measure of dependence.

Once the correlation is used for pricing, an immediate question arises, namely, the question of estimating the sensitivity of derivative prices to the correlation parameters. In this paper we will discuss various ways to compute the correlation VaR.

We start with a brief recap of the correlation and its properties. We discuss various approaches to parametrization of the correlation matrix. We then introduce different methods to generate distributions of correlations matrices. Finally, we apply these methods to determine the correlation VaR of various derivative products. We conclude with a discussion.

2 Correlation

Linear correlation is the most widely used measure of dependence between random variable $X$ and $Y$ with finite variances. It is defined as

$$
\rho(X, Y) = \frac{\text{Cov}[X, Y]}{\sqrt{\sigma^2[X]\sigma^2[Y]}}
$$

where $\text{Cov}[X, Y]$ is the covariance between $X$ and $Y$, $\text{Cov}[X, Y] \equiv E[XY] - E[X]E[Y]$ and $\sigma^2[X] \equiv \text{Cov}[X, X]$, $\sigma^2[Y]$ denote the variances of $X$ and $Y$. The linear correlation is a measure of linear dependence. For the elliptical family of joint distributions, which include joint normal, lognormal and others, the linear correlation together with the variances is sufficient to describe the dependency structure. The pitfalls and limitations of the concept of a linear correlation are investigated in detail in the excellent paper by Embrechts, McNeil, Straumann [2].

If we are dealing with $n$ random variables $X_1, X_2, \ldots, X_n$ (for example returns of financial variables), then correlations $\rho_{ij}$ between different pairs $i, j$ of returns are expressed in term of matrices. Correlation matrices must satisfy the following properties ($i, j = 1, \ldots, n$):

- All entries have to be in the interval $[-1, 1]$: $-1 \leq \rho_{ij} \leq 1$
- The diagonal terms of a correlation matrix are equal to one: $\rho_{ii} = 1$
- The matrix has to be symmetric: $\rho_{ij} = \rho_{ji}$
The matrix has to be positive semidefinite, i.e., the variance of any portfolio with correlation matrix \( \rho \), is non-negative: \( \forall W, V \sigma_P^2 = (Z)^T \rho(Z) \geq 0 \), where \( W = w_1, \ldots, w_n \) is the array of weights of the portfolio, \( V = v_1, \ldots, v_n \) is the array of standard deviations of the returns, \( Z = w_1 v_1, \ldots, w_n v_n \) and \( T \) denotes transpose of a matrix.

The most common way to estimate a correlation matrix is to use historical data. Another way is to get implied correlations, i.e., to calibrate a model using market prices of correlation-dependent derivatives in the same way implied volatilities are obtained from options quotes. The advantage of the first approach is that typically more data is available for its implementation. The drawback – it is backward looking. The second approach has less data, but it is forward looking.

Finally, we remind that portfolio VaR (value-at-risk) analysis requires the ability to generate the distribution of returns of portfolio components with a given covariance matrix of returns. In order to simulate \( n \) normal correlated variables, one uses the Cholesky decomposition to factor a positive positive definite correlation matrix \( \rho \) into a unique product of lower triangular matrix and its transpose \( \rho = LL^T \). This means that for any \( j \geq i \)

\[
\rho_{ij} = \sum_{k=1}^{i} l_{ik} l_{jk}
\]

For example, for \( n = 2 \), we get

\[
L = \begin{pmatrix}
1 & 0 \\
\rho_{12} & \sqrt{1 - \rho_{12}^2}
\end{pmatrix}
\]

For \( n \) uncorrelated normal variables \( X = x_1, \ldots, x_n \), the transformed variables \( Y = LX^T \) will have a joint multivariate normal distribution with a correlation matrix \( \rho \).

3 Perturbing the correlation matrix

In order to estimate correlation risk we need an ability to generate a random sample of correlation matrices, so that we could analyze the corresponding distribution of portfolio values, i.e. calculate the correlation VaR. This task is not easy, since a correlation matrix is a fairly rigid object, especially, with respect to the requirement of positive definiteness.

In this section we will discuss four different approaches to perturbing the correlation matrix: bootstrapping, element-wise perturbation, perturbation with the help of angle parametrization of the Cholesky matrix, and perturbation of the eigen-values.

3.1 Bootstrap method

In this section we discuss the bootstrap methodology, proposed in [1]. There, the procedure is applied to three types of multi-asset equity options: a simple basket option, a maximum option and a minimum option.
In the last two cases the option delivers, respectively, the best and the worst performing asset upon exercise. In the paper the bootstrap method is used to derive the distribution of the correlation estimates. From our point of view, it is a more natural approach to generating correlation distributions than another popular method - a moving window algorithm.

The bootstrap methodology is a Monte Carlo resampling method. It was introduced by Efron [7]. Suppose that we want to estimate the density of the correlation $\rho$ between two random variables $X$ and $Y$ based on historical observations $\{X_i, Y_i\}_{i=1}^{N}$. The basic bootstrapping algorithm to create a distribution for $\rho$ is as follows.

1. Generate $N$ uniform i.i.d. random integers $n_1, n_2, \ldots, n_N$ in the interval $[1, N]$.
2. Create a sample $\hat{X} \equiv \{X_{n_i}, Y_{n_i}\}_{i=1}^{N}$ using the $n_1, \ldots, n_N$.
3. Calculate $\hat{\rho}$ using the sample $\hat{X}$.
4. Repeat the previous steps $M$ times with $M$ being a large number.

Instead of drawing single values from observations, one can use block bootstrap. This means that the data is split into blocks of length $m$ and the above procedure is applied. The authors of [1] used blocks of length $m = 3$. They quantify the error in the correlation estimates from historical data by approximating the asymptotic distribution of the correlation via block bootstrapping. The bootstrapped correlation distributions then are mapped on prices of three standard types of multi-asset options: a basket option, minimum option and maximum option. Bid-ask spreads of the prices are computed at statistical quantiles of the resulting price distributions.

### 3.2 Perturbing individual correlations

In [3] the authors propose to perturb the correlation matrix locally to a desired target matrix while ensuring that the irrelevant correlations remain the same, and the new correlation matrix remains positive semi-definite. They obtain an analytical solution for the bounds of a single correlation term of a positive semi-definite correlation matrix. The methodology is based on the elegant idea of re-ordering of the assets that define the matrix and applying the Cholesky decomposition to localize the perturbation to the last entries in the Cholesky matrix. They also present an iterative application of the single correlation stress test methodology in order to stress a number of correlations.

Suppose we want to perturb the correlation-element $\rho_{ij}$ by $\Delta \rho_{ij}$, while leaving the other correlations fixed. To this end we reorder assets $S_1, S_2, \ldots, S_n$:

\[
(S_1, \ldots, S_{i-1}, S_i, \ldots, S_{j-1}, S_{j+1}, \ldots, S_n, S_i, S_j)
\]  

(2)

Let the corresponding correlation matrix be $\tilde{\rho}$. Let $L = (l_1, \ldots, l_n)$ be the corresponding lower triangular Cholesky matrix, where the $l_i$ denote the columns of the Cholesky matrix. If we perturb the elements of the last two columns $l_{n-1}$ and $l_n$, it will have effect only on the lower rightmost $2 \times 2$ sub-matrix of the
correlation matrix $\tilde{\rho}$, which corresponds exactly to a stress test on the correlation between assets $i$ and $j$. Thus, if two last columns of $L$ are

$$l_{n-1} = (0, 0, \ldots, 0, a, b)^T, \quad l_n = (0, 0, \ldots, 0, 0, c)^T$$

(3)

then perturbing values $a,b$ and $c$ by $\delta_a, \delta_b$ and $\delta_c$, we can calculate the corresponding correlation matrix $\hat{\rho}$ from the perturbed Cholesky matrix

$$\hat{L} = (l_1, \ldots, \hat{l}_{n-1}, \hat{l}_n)$$

(4)

with

$$l_{n-1} = (0, 0, \ldots, 0, a + \delta_a, b + \delta_b), \quad l_n = (0, 0, \ldots, 0, 0, c + \delta_c)$$

(5)

Since the diagonal terms of perturbed correlation matrix must be equal to 1, there are constrains on $\delta_a, \delta_b, \delta_c$. One can also find achievable values for $\Delta \rho_{ij}$. More precisely,

$$\Delta \rho_{ij} \in \left[ \min(-ab(1 \pm \sqrt{1 + c^2/b^2})), \max(-ab(1 \pm \sqrt{1 + c^2/b^2})) \right]$$

(6)

3.3 Perturbing angles in the angle representation of the correlation matrix

The TAP parametrization of the correlation matrix through a unique set of angles was put forward in [4]. This is an extension of results proposed in [5], which were the first to apply the results of [6] in a financial context.

The essential idea is that the correlation matrix can be parameterized through a unique lower-triangular matrix, where the entries are angles taking values in $[0, \pi]$. This angle-representation maps to a Cholesky matrix from which we can compute the correlation matrix. The good thing about this approach is that we automatically satisfy the correlation constraints specified in Sec 2.

We start with the Cholesky decomposition of the correlation matrix, see Sec. 2. In this decomposition the correlation matrix $\rho$ is represented as a product of a lower-triangular matrix $L$ and its transpose:

$$\rho_{ij} = \sum_{k=1}^{N} l_{ik} l_{jk}$$

The elements of the lower-triangular matrix $L$ are then parameterized in terms of cosines and sines of $N(N - 1)/2$ angles $\theta_{ij}(j < i)$ between 0 and $\pi$.

$$l_{ij} = \begin{cases} \cos \theta_{ij} \prod_{k=1}^{j-1} \sin \theta_{ik} & j < i \\ \prod_{k=1}^{i-1} \sin \theta_{ik} & j = i \end{cases}$$

(7)

The angles are found via a robust and efficient procedure which makes the whole approach very attractive. Moreover, through these angles the space of all correlation matrices is covered.

One implication of having the angle representation of correlation matrices is that if we have an algorithm to sample the angles, then we also have an algorithm to sample the correlations with some width around the
base correlations, thus providing an alternative way to generate the correlation VaR. To illustrate the idea we consider the case of four assets.

The Cholesky matrix will have the following expression:

\[
L = \begin{pmatrix}
1 & 0 & 0 & 0 \\
c_{21} & s_{21} & 0 & 0 \\
c_{31} & s_{31}s_{32} & s_{31} & 0 \\
c_{41} & s_{41}s_{42} & s_{41}s_{42}s_{43} & s_{41}s_{42}s_{43}
\end{pmatrix}
\]

where we use the shorthands \( s_{ij} \equiv \sin(\theta_{ij}) \) and \( c_{ij} \equiv \cos(\theta_{ij}) \). Since the correlation is symmetric, we only need to show its lower triangular part:

\[
\rho = \begin{pmatrix}
1 & 0 & 0 \\
c_{21} & 1 \\
c_{31} & c_{21}c_{31} + s_{21}s_{31}c_{32} \\
c_{41} & c_{21}c_{41} + s_{21}s_{41}c_{42} & c_{31}c_{41} + s_{31}s_{41}(c_{32}c_{42} + s_{32}s_{42}c_{43})
\end{pmatrix}
\]

(8)

Clearly the correlation elements \( \rho_{ij} \) are within the intervals \([-1, 1]\).

The simplest algorithm for generating distribution of correlation matrices is to generate random angles \( \theta_{ij} \) around the base angles \( \theta_{0ij} \) with some distribution \( \pi(\theta_{ij} | \theta_{0ij}) \), which is symmetric and centered around the base-correlation \( \theta_{0ij} \) for every \( i, j \). Random angles are simulated using the historical distribution. From our analysis we find that the angles tend to be distributed around the mean, with a standard deviation in the order of \( \sigma = 5\% \). Based on this analysis we propose two perturbation methods. In the first approach we perturb all angles using one standard normal variate \( z \sim N(0, 1) \),

\[
\hat{\theta}_{ij} = \arctan(\tan(\theta_{ij} + \frac{\pi}{2})(1 + \sigma z)) + \frac{\pi}{2}
\]

In the second approach another all angles \( \theta_{ij} \) are perturbed by i.i.d. standard normal variates \( z_{ij} \sim N(0, 1) \),

\[
\hat{\theta}_{ij} = \arctan(\tan(\theta_{ij} + \frac{\pi}{2})(1 + \sigma z_{ij})) + \frac{\pi}{2}
\]

(9)

We are, of course, interested in perturbations of the correlations around their base value. In general, it is not transparent how the correlations are distributed given the angle distributions. If we look at Eq. 8 and consider the first column of correlations it is clear that the distribution of the correlation elements is given by

\[
\pi(\rho_{i1} | \theta_{0i1}) = \int \delta(\rho_{i1} - \cos(\theta_{i1})) \pi(\theta_{i1} | \theta_{0i1}) d\theta_{i1}
\]

In the example Eq. 8 it means that we have general distributions of elements \( \rho_{21}, \rho_{31}, \rho_{41} \). Using these distributions we can now generate distributions of \( \rho_{32}, \rho_{41} \), and finally of \( \rho_{43} \). Similar inductive approach can be used to construct distributions of correlation coefficients from angle distributions in general \( n \)-dimensional case. A more complex task is to solve the inverse problem: to construct a proper angle distribution some target distribution for the correlation.
3.4 Perturbing eigenvalues

In this section we consider the generation of random correlation matrices around the base correlation matrix through the perturbation of eigen-values. We can express the correlation matrix in terms of its eigen-system \((\Lambda, V)\) via

\[
\rho_{ij} = \sum_{k,l=1}^{n} V_{ik} \Lambda_{kl} V_{lj}
\]

where \(\Lambda_{kl} \equiv \lambda_k \delta_{kl}\) is the diagonal matrix with the eigen-values \(\lambda_i\) on the diagonal. We assume that the eigen-values by \(\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n \geq 0\). Furthermore the eigen-values satisfy the constraint \(\sum_{k=1}^{n} \lambda_i = n\).

We consider the following four algorithms to perturb the eigen-values. The first algorithm is as follows.

1. Generate \(n\) i.i.d random standard normal variates \(z_i \sim N(0, \sigma_i)\), \((i = 1, \ldots, n)\)

2. Compute the perturbed eigenvalues

\[
\hat{\lambda}_i = \lambda_i e^{\sigma_i z_i}
\]

The other three algorithms are variations of the following algorithm.

1. Generate a random index \(K \in [1, \ldots, n]\) according to some distribution \(p_i \geq 0\) with \(\sum_{i=1}^{n} p_i = 1\).

2. Generate an i.i.d. standard normal variable \(z \sim N(0, 1)\).

3. Compute the perturbed eigen-value \(\lambda_K\)

\[
\hat{\lambda}_K = \lambda_K e^{\sigma_K z}
\]

The three cases we will consider are

1. Perturb the largest eigen-values more: \(p_i = \frac{\lambda_i}{n}\)

2. Perturb the eigen-values uniformly: \(p_i = \frac{1}{n}\)

3. Pick one specific eigen-value: \(p_i = \delta_{iK}\)

Having perturbed eigenvalues, we define the perturbed correlation matrix via

\[
\hat{\rho}_{ij} = \sum_{k,l=1}^{n} V_{ik} \hat{\Lambda}_{kl} V_{lj}
\]

where \(\hat{\Lambda}_{kl} \equiv \hat{\lambda}_k \delta_{kl}\) is the diagonal matrix with the perturbed eigen-values \(\hat{\lambda}_i\). To ensure that \(\sum_{i=1}^{n} \hat{\rho}_{ii} = n\) we need to renormalize the random correlation matrix \(\hat{\rho}\):

\[
\tilde{\rho}_{ij} = \frac{\hat{\rho}_{ij}}{\sqrt{\hat{\rho}_{ii} \hat{\rho}_{jj}}}
\]
4 Correlation VaR

The question we want to ask is the sensitivity of the value of a trade, or portfolio to the correlation matrix. To this end we need an efficient and practical method to perturb the correlation matrix. We will use the methods discussed in the previous section to compute the sensitivity of a portfolio to correlation.

As we already mentioned before, how we perturb the correlations, i.e. what probabilities to we attach to a particular perturbation of the correlation matrix is an important and difficult choice. We will take a practical point of view here and leave the more satisfying theoretical results for a future work.

In calculating the correlation VaR, we will perturb the correlation matrix according to some perturbation scheme, which effectively means that we have some distribution $\pi(\rho | \rho^0)$ and compute the density for the portfolio value as a function of the correlation as follows.

$$\pi(v) = \int \delta(v - V(\rho)) \pi(\rho | \rho^0) d\rho \quad (10)$$

From this we can then determine the correlation VaR.

5 Some examples

In this section we compute the distribution of the basket value for two basket options as we perturb the correlation matrix. Here we define the basket payoff as

$$\left( \sum_{i=1}^{n} w_i \frac{X_i(T)}{X_i(0)} - 1 \right)^+$$

where $X_i(0), X_i(T)$ denotes the value of the $i$-th underlyer today $t = 0$ and at expiry $t = T$. For a given correlation matrix $\rho$ we compute the value of the basket option using a Monte Carlo simulation, i.e. we simulate a sample of random $X_i(T)$ and compute the expectation value. We repeat this for a sample of random correlation matrices using one of the methods described in the previous sections. Using Eq. (10) we then compute the density of the basket value. In order to get a distribution of the eigenvalues or the angles, we perform a bootstrap on historical prices. In Fig. [I] we plot the histogram for the biggest eigenvalue of the historical correlation matrix and the histogram for the angle $\theta_{21}$. Based on this analysis, we make the assumption that the eigenvalues have a lognormal distribution and generate the random angles using Eq. (9). Obviously there are a lot of things to consider here.

We consider a basket consisting of nine international industrial indices, five Goldman Sachs commodities indices and prices for nine physical commodities such as natural gas and metals. In the first example the weights are given by

$$w_i = \frac{1}{n}$$

where $n$ is the number of assets, in this particular example $n = 23$. In the second example we consider a spread option on two baskets, with the first basket being a basket on the industrial indices and the second
basket on everything else, i.e. either commodities indices or commodities prices. The positive weights are
given by
\[ w^1_i = \frac{1}{m} \]
and the negative weights are given by
\[ w^2_i = -\frac{1}{k} \]
In this example \( m = 9 \) and \( k = 14 \). The basket payoff in this case is given by
\[
\left( \sum_{i=1}^{m} w^1_i \frac{X(T)}{X(0)} + \sum_{i=m+1}^{n} w^2_{i-m} \frac{X(T)}{X(0)} \right)^+ \]
The base correlation matrix \( \rho^0_{ij} \) is calculated from historical returns. We use three years of historical data. Given the base correlation matrix we can compute the corresponding Cholesky decomposition and the lower-triangular angle matrix \( \theta_{ij} \). In order to estimate the variations of the angles, we compute the angles using a sliding window. Table 3 shows the values of a subset of the angles for different years. We calculated the correlation matrix and the corresponding angle matrix for every year using that year’s data. In general we find that the standard deviation of the angle is in the order of 5%. We will use this number to perturb the angles.

We apply the methods we discussed earlier to simulate correlation matrices and compute the density of the basket values as a function of the correlation perturbations. In tables 1 and 2 we give the mean, standard deviation, and 95% left VaR for the two basket values as a function of the different methods that we discussed in this paper. We use 1000 simulations of correlation matrices, and 10000 simulations in the underlying Monte Carlo to evaluate the basket. The corresponding histograms are in Figures 2, 3. Let us first consider Figure 2. It is clear that the distribution of the basket values depends on the method of perturbing the correlation matrix. If we choose a uniform density over the eigenvalues the smallest eigenvalues gets too much weight and the density is very narrow. As we put more weight on the larger eigenvalues the density will widen. The case where we only perturb the largest-eigenvalue can be considered as the worst-case scenario. As for the perturbation through the angles we see that, as expected, that the case with a correlated perturbation of the angles leads to a wider distribution of the value. We get similar results for the second example as shown in Figure 3.

In Figure 4 we compare the bootstrap method with the largest eigenvalue perturbation and the parallel shift of angles for the two examples we consider. It shows that the distribution of the bootstrap case is wider than the case of the other two methods.

6 Discussion and conclusions

We have described various methods to perturb the correlation matrix and compute the correlation VaR. The simplest method is the bootstrap method, which is essentially a resampling of the historical timeseries. The
second method involves the local perturbation of the correlation matrix elements. The third method uses a perturbation of the correlation matrix through perturbation of the angles in the TAP parametrization. Finally we propose a method to perturb the correlation matrix via the perturbation of the eigenvalues.

We provide some numerical results for these methods applied to two basket options. The different methods lead to different results for the correlation VaR. This is not surprising since there is clearly a lot of freedom in how to choose the perturbation and underlying densities. Given this freedom one could argue for using the most simple method, the bootstrapping of the correlation matrix. However there is a clear advantage in having a method to perturb which has a control parameter available. This allows us to do stress testing. This is essentially what both the angle and eigenvalue perturbation methods provide.

To summarize, the present work shows that the use of some parametrized method, be it angles or eigenvalues, gives us a convenient tool to stress-test complicated portfolios of instruments and to do it in a well-defined manner. A consistent application of the correlation VaR measures described in this paper improves our understanding of product sensitivity to correlation and provides us with the useful approach to product comparison.

<table>
<thead>
<tr>
<th>Method</th>
<th>Largest eigenvalue</th>
<th>Weighted eigenvalues</th>
<th>Uniform eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean (%)</td>
<td>6.463</td>
<td>6.463</td>
<td>6.455</td>
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<tr>
<td>Standard deviation (%)</td>
<td>0.113</td>
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<tr>
<td>95% CorVaR (%)</td>
<td>0.183</td>
<td>0.163</td>
<td>0.156</td>
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<table>
<thead>
<tr>
<th>Method</th>
<th>Parallel shift angles</th>
<th>Uncorrelated angles</th>
<th>Bootstrap</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean (%)</td>
<td>6.457</td>
<td>6.458</td>
<td>6.459</td>
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<tr>
<td>Standard deviation (%)</td>
<td>0.142</td>
<td>0.098</td>
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<tr>
<td>95% CorVaR (%)</td>
<td>0.226</td>
<td>0.159</td>
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Table 1: Basket 1 with value 6.469. Sample size is 1000.

<table>
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<th>Method</th>
<th>Largest eigenvalue</th>
<th>Weighted eigenvalues</th>
<th>Uniform eigenvalues</th>
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<tr>
<td>Mean (%)</td>
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<td>Standard deviation (%)</td>
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<td>95% CorVaR (%)</td>
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<table>
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<th>Method</th>
<th>Parallel shift angles</th>
<th>Uncorrelated angles</th>
<th>Bootstrap</th>
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<tr>
<td>Mean (%)</td>
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<tr>
<td>Standard deviation (%)</td>
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<td>95% CorVaR (%)</td>
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Table 2: Basket 2 with value 11.460. Sample size is 1000.
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Table 3: Sample of correlation angles for different subsets of the data.
References


Figure 1: In this figure we plot the histogram for the largest eigenvalue of the correlation matrix. The histogram is generated by...
Figure 2: In this figure we plot the histograms for the value of the first basket option with all different methods discussed in the article. The sample size is 1000.
Figure 3: In this figure we plot the histograms for the value of the second basket option with all different methods discussed in the article. The sample size is 1000.
Figure 4: Comparison of the two examples for the three different cases: bootstrap, largest eigenvalue, parallel shift angles. The left-hand graph shows the results for the first basket option. The right-hand graph shows the results for the second basket option.