Imagine there exist markets for yield futures contracts as well as ordinary futures contracts for price. Intuitively one would think that a combined use of yield contracts and futures price contracts ought to provide a reasonable strategy for securing revenue.

In the paper this is made precise - it is shown that revenue can be locked in by a combined, dynamic use of these two markets. The relevant strategy is characterized: It depends only on observable price information in these two separate futures markets, one for quantity and one for price, not on the specification of parameters in utility functions of the agents involved.

The ability to trade continuously can not be dispensed with. These results should be of relevance, since markets for crop yield futures and options have been established.

*Key words: Area yield options, futures, continuous time modelling, quantity and price securing, locking in a certain revenue, CBOT yield contracts*

**Introduction**

This paper develops a strategy showing how price and quantity futures markets can be used together in order to provide a reasonable strategy for securing revenue.

Area yield crop insurance, where the index is based on average yield in a given geographical area, has been offered in India, Brazil, Canada and the USA. Parametric insurance (e.g., rainfall insurance) has been proposed in Canada, India and Mexico. A livestock mortality index has been recently
designed to cover herders against livestock losses in Mongolia (according to Mahul 2005).

In 1995, the Chicago Board of Trade (CBOT) launched its Crop Yield Insurance (CYI) futures and options contracts in the USA. Related challenges exist in the production of electricity, an example being Nord Pool (The Nordic Power Exchange), where financial futures, forward and option contracts are traded. Some of these have characteristics similar to quantity futures due to the nature of electricity power trading. Price futures could alternatively be combined with certain weather derivatives for risk management purposes.

Crop yield insurance contracts are designed to provide a hedge for crop yield risk. For example, CYI futures users can secure a certain crop yield several months into the future as a temporary substitute for a later yield-based commitment, or they can alternatively try to secure the revenue of a given acreage by combining yield contracts with futures price contracts. How this can be done is the subject of the present paper.

In the following we abstract from production costs, and assume zero local price basis (i.e., local cash price equals futures price) and zero yield basis (i.e., individual farm yield equals index yield). This is to say, we only address market risk, not idiosyncratic risk. We also ignore asymmetric information. Intuitively one would think that a combined use of yield contracts and futures price contracts ought to provide a “reasonable” strategy for insuring revenue. In the paper this intuition is made precise. It is shown that revenue can be perfectly hedged by a combined, dynamic use of these two markets. Moreover, the relevant strategy is also characterized. This strategy depends only on observable futures price information in these two separate markets.

There is a large literature on non-market based risk management and insurance of crop yield, using mostly a one period, expected utility framework. Yield contracts have been analyzed from the perspective of hedging, using a mean variance approach by Vukina, Li and Holthausen 1996. Minimizing the variance of revenue was the objective in Li and Vukina 1998. In both these papers the yield contracts traded at CBOT are explained, so we need not elaborate on this market structure here. A more recent paper is Nayak and Turvey 2000, again using a simple mean-variance model. Two other papers examine the insurance problem in the expected utility framework (see e.g., Hennessy, Babcock and Hayes 1997; Mahul and Wright 2003). The focus in this paper is somewhat different from these investigations, in that our results do not depend on any specific assumptions about utility functions: Our hedging ratios can be read straight from observed market prices. However, there are some interesting connections to this literature which we return to below.

In a recent paper, Hennessy 2001 discusses the apparent lack of interest
in revenue futures, by identifying conditions under which revenue futures are perfect substitutes for price futures. These conditions hinge upon a non-

stochastic relationship between production shock and spot price, not present in our model.

Assuming no transaction costs, basis risk and correlation between yield and price, Mahul and Wright 2003 characterize the optimal indemnity payoff net of the premium for any risk averse agent. Under the same conditions, but with the possibility of continuous resettlement, using separate yield and price futures contracts, we demonstrate how to dynamically replicate this optimal indemnity payoff. This match of the optimal insurance contract shows how essential the possibility of dynamic replication is, in order to achieve an optimal revenue insurance through trade in financial derivatives. Also we can relax the assumption about zero correlation between yield and price.

The paper is organized as follows: In the first section we present the dynamic market model, and proceed directly to the main results of the paper. In Theorem 1 (and Corollary 1) we show how a farmer can combine a pure yield futures option (pure yield futures) contract with a pure futures option (pure futures) price contract to insure a revenue similar to what a farmer could ideally secure if a futures market on revenue were to exist.

Finally we briefly discuss the possibility to extend the analysis to models containing jumps of unpredictable sizes. This brings us to incomplete financial markets. It turns out that our main results are still valid. The final section concludes.

Area Yield Futures and Options

Introduction

Imagine a country, or another area, sectioned into regions which are uniform in terms of growing conditions for a certain crop, say corn. In each area there is a quantity index $q_t$, for time $t$ running from 0 to $T$, where $T$ is the time of sale and 0 is the time of sowing. As an example, for the agricultural yield contracts in the USA that were traded at the CBOT, the values of $q$ were provided by the United States Department of Agriculture (USDA). One may think of $q_t$ as a forecast at each time $t$ of quantity, measured in bushels per acre, up for sale in this specific region at the final time $T$. On this index we assume it is possible to trade futures, and futures options contracts. In order to bring in the quantum uncertainty, we assume that this index can be modeled as a stochastic process. A farmer in this region may have production uncertainty that is well represented by this index, in
which the relevant number of contracts can be determined from each farmer’s production area.

The idea is that if the producer can buy options on this quantity index or on its corresponding futures index, the farmer can secure a prespecified quantity by buying an appropriate number of such contingent claims. This strategy is of course only 100% efficient if the farmer’s yield uncertainty is perfectly represented by the index, an unlikely event, but a careful selection of homogeneous regions may make such markets useful for practical risk management purposes. Since agricultural agents are presumably concerned about revenue in the end, rather than solely about yield, or about price, one may think that the yield market may be combined in an appropriate manner with the futures market for crop price to insure a certain revenue. The conditions under which this can be done is the topic of this paper.

A private insurance market giving the farmer insurance against quantity shortfall is of course difficult to establish, partly because of the adverse incentives this would create for the farmers, as the rich literature on this topic in agricultural economics journals show. The yield futures market may, however, avoid this difficulty, at least under some presumptions: The agents do not engage in any kind of “collective moral hazard” which effects the yield index, and there is no moral hazard in the construction of the index. There is an implicit assumption that the farmer’s actions do not influence the quantity index to any significant degree. Also the individuals in USDA constructing the index should have no economic interest attached to this market.

The model

Consider two futures markets, one where yield futures options are traded, and one where standard price futures options are traded. The quantity index \( q(t) \) at time \( t \) is measured in bushels per acre, and spot price \( p(t) \) at time \( t \) is measured in \$ per bushel. As earlier explained we abstract from production costs, and assume zero local price basis and zero yield basis. In this case we can define the revenue \( R(t) = q(t)p(t) \). A yield futures option contract will specify a real function \( g \) so that the payoff from a yield futures option contract at the expiry time \( T \) is given by \( g(q(T)) \) bushels per acre, having yield futures price at time \( t < T \) given by

\[
F^g_{t} = E^{Q}_t (g(q(T))) \cdot c. \tag{1}
\]

Here we consider an option on the futures index as a futures contract. To be specific, given is a filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}, P) \), where \( \Omega \) is the set of states of nature with generic element \( \omega \), \( P \) is a probability
measure, the “objective probability”, $\mathcal{F}$ is the set of events in $\Omega$ given by a
$\sigma-$algebra, $\mathcal{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ is a filtration satisfying the usual conditions,
where $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$, $\mathcal{F}_t$ signifying the possible events that could happen
by time $t$, or “the information available by time $t$”. We assume $\mathcal{F}_0$ to be trivial, containing only events of probability zero or one, meaning roughly
that there is no information available at time zero, and $\mathcal{F}_T = \mathcal{F}$, i.e., at
time $T$ all uncertainty is resolved. Here $Q$ appearing in equation (1) is an
equivalent martingale measure, assumed to exist, $Q$ being equivalent to the
given probability measure $P$ (i.e., the measures $P$ and $Q$ coincide on the null
sets). The symbol $E$ in (1) is the conditional expectation operator given the
“information” $\mathcal{F}_t$ possibly available by time $t$.

The constant $c$ signifies a conversion factor measured in $\$$ per bushel, so
that the futures price is measured in $\$$ per acre. For example, for the Iowa
Corn Yield Insurance Futures (ticker symbol CA) the unit of trading is the
Iowa yield estimate times $100$ (e.g., a yield of 140.3 bushels per acre gives a
contract value of $14,030$). In this section we set this conversion factor equal
to 1 without loss of generality.

Similarly an ordinary futures option contract on price will specify a real
function $h$ so that the corresponding payoff from a futures option contract at
expiration is given by $h(p(T))$ $\$$ per bushel, and the associated futures price
at any time $t$ prior to $T$ is given by $F_t^{h(p)} = E^Q_t(h(p(T)))$ measured in $\$$ per
bushel.

The linear pricing rule of quantity futures implied by the expression (1)
is, of course, far from obvious. In addition to the usual frictions in ordinary
futures markets, like no short sale possibilities of the crop, an additional
difficulty arises here, since the index $q$ is not a traded asset. In Aase 2004
this is resolved by considering the quantity $s = pq$ and identifying $s$ as a spot
price process. Based on the price processes $s$ and $p$ a no-arbitrage model is
constructed as permitted by the financial theory, where $s$ is identified as the
spot price of a leasing contract of agrarian land for the crop in the particular
region of consideration. This solves, at least in theory, the pricing problem
of these contracts.\footnote{2}

Turning to the dynamics of the two processes $p$ and $q$, we assume that
the process $q$ for quantity and $p$ for price are both defined on the given
probability space as follows:

\[ dq(t) = \mu_q(t)dt + \sigma_q(t)dB(t) \] \hspace{1cm} (2)

and

\[ dp(t) = \mu_p(t)dt + \sigma_p(t)dB(t), \] \hspace{1cm} (3)

where $B(t) = (B_1(t), B_2(t))$ is a standard two dimensional Brownian motion,
\( \sigma_q(t) = (\sigma_{q,1}(t), \sigma_{q,2}(t)) \) and \( \sigma_p(t) = (\sigma_{p,1}(t), \sigma_{p,2}(t)) \) are adapted volatility processes satisfying standard \( L^2 \)-type integrability and regularity conditions. Similarly the drift terms \( \mu_q(t) \) and \( \mu_p(t) \) are adapted stochastic processes satisfying standard \( L^1 \)-type integrability conditions.

The main result

Consider the product contracts of the form \( R^{gh}(t) = g(q(t)) h(p(t)) \). Our revenue process \( R(t) = p(t) q(t) \) would then follow as a special case, when both \( g \) and \( h \) are the identity function. We want to investigate whether we can lock in a prespecified “revenue” \( R^{gh}(t) \) at any time \( t \) prior to the expiration time \( T \) by dynamically trading in the two separate futures options markets described above. To this end imagine first that a separate market for this type of “revenue” were available. The futures price of this contract we denote by \( F^{g(q)h(p)}_t \), and it must be given as follows under our assumptions:

\[
F^{g(q)h(p)}_t = E^Q_t \{ g(q(T)) \cdot h(p(T)) \}, \quad 0 \leq t \leq T. \tag{4}
\]

Notice that this can be written

\[
E^Q_t \{ g(q(T)) \cdot h(p(T)) - F^{g(q)h(p)}_t \} = 0, \quad 0 \leq t \leq T, \tag{5}
\]

the usual starting point for analyzing futures contracts. Equation (5) implies that if the futures price \( F^{g(q)h(p)}_t \) is agreed upon at time \( t \), then no money changes hands when the futures position is initiated.

In order to better understand what follows, let us recall the main features of a simple futures contract on, say, price. For the holder of one long contract, the payoff at expiration is

\[
\int_t^T 1 \cdot dF_s = F_T - F_t = p_T - F_t \tag{6}
\]

by the principle of convergence in the futures market, where \( F_t \) is the futures price of one contract at time \( t \). If an agent holds \( \theta_s \) futures contracts at time \( s \) in the time interval \( (t, T] \), the resettlement gain at time \( T \) from this strategy would similarly be

\[
\int_t^T \theta_s dF_s. \tag{7}
\]

Consider a strategy that holds \( F^{g(q)}_s \) pure futures options on \( h(p_T) \), and \( F^{h(p)}_s \) pure futures options on \( g(q_T) \) at each time \( s \) between \( t \) and \( T \). The resettlement gain from this strategy is given by

\[
\int_t^T F^{g(q)}_s dF^{h(p)}_s + \int_t^T F^{h(p)}_s dF^{g(q)}_s. \tag{8}
\]
Using stochastic integration by parts, this can be written
\[
= g(q_T)h(p_T) - (F_t^{g(q)} F_t^{h(p)} + \int_t^T dF_s^{g(q)} dF_s^{h(p)}).
\] (9)

Returning to the basic equation (5), the starting point for analyzing futures contracts, consider the equality
\[
E_Q t (g(q_T)h(p_T) - (F_t^{g(q)} F_t^{h(p)} + \int_t^T dF_s^{g(q)} dF_s^{h(p)})) = 0.
\] (10)

If this is true, it would mean that the dynamic strategy given in (8) is equivalent to a futures contract on the product \( h(p_T)g(q_T) \) with the associated futures price equal to the projection of the expression \( (F_t^{g(q)} F_t^{h(p)} + \int_t^T dF_s^{g(q)} dF_s^{h(p)}) \) on the information filtration \( \mathcal{F}_t \), i.e.,
\[
F_t^{g(q)h(p)} = E_Q t (F_t^{g(q)} F_t^{h(p)} + \int_t^T dF_s^{g(q)} dF_s^{h(p)}).
\] (11)

We now demonstrate that the equality (10) indeed holds true under mild conditions. To this end, we will need some technical conditions, which we relegate to Appendix 1. Assuming these, we use the following notation: The futures price processes \( F_t^{h(p)} \) and \( F_t^{g(q)} \) can both be written as smooth functions \( a(p, t) \) and \( b(q, t) \) respectively. Denote by \( a_p(p, t) \) the partial derivative of the function \( a(p, t) \) with respect to its first argument, and similarly for \( b_q(q, t) \). We have the following result:

**Theorem 1** Consider the resettlement gain from the strategy given in (8). This strategy is equivalent to a futures price directly on revenue \( h(p_T)g(q_T) \) with associated futures price given in (11), which can also be written
\[
F_t^{g(q)h(p)} = F_t^{g(q)} F_t^{h(p)} + E_Q t \left( \int_t^T dF_s^{g(q)} dF_s^{h(p)} \right)
\] (12)

\[
= F_t^{g(q)} F_t^{h(p)} + E_Q t \left( \int_t^T a_p(p_s, s)(\sigma_p(s) \cdot \sigma_q(s)) b_q(q_s, s) ds \right).
\]

In the special case of zero correlation rate between yield and price, i.e., \( \sigma_p(s) \cdot \sigma_q(s) = 0 \) for all \( s \in (t, T] \), this strategy is equivalent to a futures contract on the product \( h(p_T)g(q_T) \) having futures price at each time \( t \) given by \( F_t^{g(q)h(p)} = F_t^{g(q)} F_t^{h(p)} \).
Proof: In order to prove this, according to (5) we have to show that equation (10) holds, i.e.,

\[ E^Q_t \left( g(q_T) h(p_T) - \left( F^g_t q_t F^h_t p_t + \int_t^T dF^g_s dF^h_s \right) \right) = 0, \quad t \leq T. \quad (13) \]

This would follow if

\[ E^Q_t \left( \int_t^T F^g_s dF^h_s + \int_t^T F^h_s dF^g_s \right) = 0 \quad \text{for any} \quad t \leq T \quad (14) \]

by equations (8 and 9). Consider the standard conditions (22) - (25) of Appendix 1; under these it is known, essentially by Hölder’s inequality, that the stochastic integrals in (8) both have zero conditional expectations under \( Q \), since \( F^g_t q_t \) and \( F^h_t p_t \) are both \( Q \)-martingales. Thus we get the conclusion of the theorem from the expression for the futures price in equation (4), the fact that \( F^g_t q_t \) and \( F^h_t p_t \) are both \( \mathcal{F}_t \)-measurable, and from the representations (20) and (21) of Appendix 1 for the stochastic processes \( a(p, t) \) and \( b(q, t) \).

The conclusion of the last part follows from the expression (12), since the integral

\[ \int_t^T dF^g_s dF^h_s = 0 \]

in this case. \( \square \)

Before we comment on this theorem, we briefly describe the situation with pure futures contracts only. Consider a strategy that holds \( F^q_s \) futures contracts on price and \( F^p_s \) future contracts on quantity at any time \( s \), where \( 0 \leq t \leq s \leq T \), \( t \) signifying the present. The resettlement gain from this strategy is given by

\[ \int_t^T F^q_s dF^p_s + \int_t^T F^p_s dF^q_s. \quad (15) \]

We then have the following corollary:

**Corollary 1** Consider the resettlement gain from the strategy that, for any time \( s \) between the present time \( t \) and the expiration time \( T \), holds \( F^q_s \) futures contracts on price and \( F^p_s \) contracts on quantity, given in (15). This strategy is equivalent to a futures price directly on revenue \( R_T = p_T q_T \) with associated futures price given by

\[ F^R_t = F^q_t F^p_t + E^Q_t \left( \int_t^T a(p,s)(\sigma_p(s) \cdot \sigma_q(s))b_q(q,s) ds \right). \quad (16) \]
If $\sigma_q(s) \cdot \sigma_p(s) = 0$, for all $s \in (t, T]$ i.e., a zero correlation rate between yield and price, then this strategy is equivalent to a futures contract on revenue $RT$ having futures price at each time $t$ given by $F_t^R := E_t^Q \{q(T)p(T)\} = F_t^Q F_t^P$.

Proof: Set $g(x) = x$ and $h(x) = x$ for all real $x$ in Theorem 1. □

The above results show that there is no need for a specialized futures market of, say, revenue $R = pq$ for someone who has access to the two separate markets for price and yield contracts. One can then, at least in principle, achieve exactly the same results in terms of risk management by simultaneous, dynamic trade in these two markets. Since a dynamic strategy is then needed, needless to say, we here abstract from transactions costs.

In the situation where the correlation $\sigma_{p,q}(s) := \sigma_p(s) \cdot \sigma_q(s) = 0$ for all $s \in (t, T]$, the corresponding futures price becomes particularly simple, namely the product of the corresponding futures prices $F_t^q$ and $F_t^p$. Suppose, on the other hand, that this correlation is positive. The last term in (16) is accordingly positive, which raises the futures price. This seems reasonable due to the increased risk this situation represents compared to the one with a zero covariance function: If the harvest is poor, the price is also low on the average, both contributing to a smaller revenue. If the corresponding correlation is negative, which is rather natural of this quantity, a farmer would typically not be willing to pay quite as much for this "insurance coverage" as in the two other cases, confirmed by the equations (12) and (16).

Examples and Discussion

The results of Theorem 1 and Corollary 1 do not depend on any specific assumptions about utility functions of the agents (except from some obvious axioms, like agents prefer more to less). The result is that a futures market for revenue can approximately be obtained through the combination of the two markets for yield and price futures. There exists a dynamic replication strategy in quantity futures and price futures which, under certain conditions, is equivalent to a futures contract on revenue. Moreover, this strategy can be obtained directly from futures price information in these two separate markets. There are no parameters to estimate, no assumptions about the relative risk aversion, or the subjective interest rate, or anything like that. Thus this result could be of practical interest.

The results can perhaps best be illustrated by an example.

Example 1. The strategy $(-F_t^q, -F_t^p)$ duplicates exactly the payoff $(F_t^R - qrpr)$ from one short "revenue" contract: At the initiation time $t$ the futures prices $F_t^q$ and $F_t^p$ are both set such that no money changes hands.
Instead of the resettlement strategy in (15) let us consider the very simple strategy that sells \( F^p_t \) quantity contracts, priced at \( F^q_t \) at time \( t \leq T \), and holds this position until maturity, and sells \( F^q_t \) price contracts, priced at \( F^p_t \) at time \( t \leq T \), and holds this position till maturity as well.

The payoff at expiration for the hypothetical contract on revenue would be \( (F^R_t - q_Tp_T) \), for an agent selling one such contract. On the other hand, the combined contracts described above would yield the following payoff:

\[
(F^p_t - p_T)F^q_t + (F^q_t - q_T)F^p_t,
\]

where the first term is the payoff of \( F^q_t \) short futures contracts on price \( p \), and the second term is the corresponding payoff of \( F^p_t \) short contracts on quantity \( q \).

This latter sum can be seen to be equal to

\[
(F^R_t - q_Tp_T) + (F^q_t - q_T)(F^p_t - p_T),
\]

in the situation where \( F^R_t = F^q_tF^p_t \), i.e., when the cross-correlation rate is zero, so let us for simplicity consider the case where \( \sigma_p(t) \cdot \sigma_q(t) = 0 \) for all \( t \).

Since \( F^q_t \) (\( F^p_t \)) can be considered as an “economic forecast” of \( q_T \) (\( p_T \)) at time \( t \), the remainder term in (17) should be “small of second order” (it goes to zero faster than the first term in (17) as \( t \) approaches \( T \)), in which case this strategy may function reasonably close to a hypothetical futures market for revenue. Of course, this latter ideal market does not exist, so this simple arrangement of combining existing markets for quantity and price separately may be a reasonable substitute. The strategy described in equation (15) constitutes, on the other hand, a perfect substitute in this situation, as well as in the situation when the associated cross correlation is different from zero.

Assuming no transaction costs, basis risk and correlation between yield and price, Mahul and Wright 2003 characterize the Pareto optimal indemnity payoff net of the premium for any risk averse agent, and risk neutral insurer. It is shown to be \( (F^R_t - q_Tp_T) \). This argument requires risk neutral pricing, which in our model amounts to equating the risk adjusted probability measure \( Q \) and the given one \( P \). As a consequence of this, market prices are determined as \( F^R_t = E_t(p_T)E_t(q_T) \), and the optimal revenue insurance has payoff \( (E_t(p_T)E_t(q_T) - q_Tp_T) \). From the relation in (17) it is seen that this payoff results, but in addition there is the remainder term caused by merely using the *sell and hold* strategy. If the *dynamic* resettlement strategy (15) is used instead, the correction term vanishes, and the optimal payoff is exactly achieved. It is noticeable that this result is true regardless the value of the
covariance between price and quantity, which is also consistent with standard (Pareto) optimal risk sharing theory, stating that full insurance is optimal in the situation described above.

This demonstrates an interesting connection to optimal insurance coverage, showing that the Pareto optimal net indemnity payoff can be dynamically replicated by using separate yield and price futures contracts. In the zero cross correlation case of the above example, this also gives us the rare opportunity of finding an expression for the hedging error, the last term in the expression (17), the departure from the Pareto optimal contract, resulting from merely using the sell and hold strategy when the dynamic replication strategy in (15) is indeed optimal. The rewards from being able to trade continuously are here brought forward in an explicit way.

**Jump/diffusion uncertainty model**

In an earlier paper we gave an example of a model for the price of the crop, \(p_t\), and the quantity index \(q_t\), as well as a valuation model in which market prices can be found for a large class of relevant financial contracts. In other words, we give an example how to construct an equivalent martingale measure \(Q\) for yield contracts. Since we have chosen an Itô process framework, this is done by judiciously transforming to two price processes \(q\) is not a price process), and then it is natural to choose a complete model, in which case we have to solve a linear system of equations, and use Girsanov's theorem to establish a unique market-price-of-risk process (see Aase 2002).

Suppose instead that agricultural yields are exposed to natural disasters in which case it would be natural to include changes more dramatic than continuous ones in the process dynamics for \(p\) and \(q\). Consider the following dynamics

\[
dq(t) = \mu_q(t)dt + \sigma_q(t)dB(t) + \int_{R^2} \gamma_q(t, z)\tilde{N}(dt, dz)
\]

and

\[
dp(t) = \mu_p(t)dt + \sigma_p(t)dB(t) + \int_{R^2} \gamma_p(t, z)\tilde{N}(dt, dz).
\]

The term \(\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt\) signifies a compensated Poisson random measures of an underlying two dimensional Levy process, independent of the two dimensional Brownian motion \(B\), and \(\nu(dz)\) is the associated Levy measure. The idea is that jumps of random sizes \(\gamma_i(t, z)\) occur at unpredictable time points of a Levy process, \(i = p, q\). If a jump happens to take place at time \(t\), and the underlying jump size of the Levy-process is
$z = (z_p, z_q)$, then the jump size in the quantity index $q$ is $\gamma_q(t, z)$, and in the price process $p$ the corresponding jump size is $\gamma_p(t, z)$. Without going into further technical details, we note the following: This class of models is obviously very general, and can be made to fit well most observed time series of data one can imagine. There is an integration by parts formula also for the type of processes given in the equations (18) and (19) above. Since this is an essential part of the proof of Theorem 1, our results in Theorem 1 and Corollary 1 can still be shown to be valid. In the uncorrelated cases (i) in both results we now have in addition that

$$\int_{R^2} \gamma_p(s, z)\gamma_q(s, z)\nu(dz) = 0$$

for all $s \in (t, T]$.

In the above model we are not able to construct a unique market-price-of-risk process as in the article referred to earlier, so there will exist many equivalent martingale measures $Q$ (indeed, uncountable many) that will do for pricing purposes, even if no arbitrage prevails. All these measures will coincide on the marketed subspace $M \subset L^2$ containing all the random payoffs of the type that can be generated by portfolio formation of two different, correlated assets with pricing processes like the one in (18). However, for contingent claims with components in the orthogonal complement $M^\perp$ of $M$ (here $L^2 = M \oplus M^\perp$), these components can not be hedged by the existing financial instruments. As a consequence we do not have a good pricing theory for this part, and the measures $Q$ will normally not coincide on $M^\perp$. Thus our resulting model is incomplete.

But this is of no concern to the present results. The agent can still observe the prices in the two separate futures markets, construct the dynamic strategy as time goes, and replicate the payoff of a futures (or a futures option) contract on revenue to the degree that we have explained above. Thus our results are robust to the modelling of uncertainty - more interesting models than Itô-processes can be used, in principle there are no restrictions (other than technical ones). Hence market completeness is not required for the main results to hold.

**Conclusions**

We have presented a dynamic model for the analysis of futures contracts on quantity and futures contracts on price in separate markets for such contracts, in order to construct futures contracts on revenue. Only market risk is considered.
Specifically, we have demonstrated how an agent can lock in a certain revenue by a combined trade in futures price and futures yield contracts, abstracting from production costs. This can be done perfectly if a certain dynamic strategy is used, identified in the paper. This strategy depends only on futures prices observed in the two different markets for price and yield futures, and not on the particular choice of model for the random dynamics. In consequence the result is independent of a complete market structure, and thus fairly robust. The identified dynamic strategy is, under certain conditions, equivalent to a Pareto optimal revenue insurance.

Our results do not depend upon any specific assumptions about utility functions, relative risk aversions, subjective discount rates, or other model parameters. Provided one takes into account transactions costs in a manner that is customary when hedging derivatives, it is possible to implement the main result in practice.

References


Appendix 1

Here we present the technical conditions needed for Theorem 1:

The futures price processes $F_t^{h(q)}$ and $F_t^{g(y)}$ can be both be written as some $C^{2,1}([0,T])$-functions $a(q_t, t)$ and $b(y_t, t)$, say. Since they are both $Q$-martingales, by Itô’s lemma

$$da(q_t, t) = a_q(q_t, t)\sigma_q(t)d\tilde{B}(t)$$

$$db(y_t, t) = b_y(y_t, t)\sigma_y(t)d\tilde{B}(t),$$

where $a_q(q, t)$ means the partial derivative of the function $a(q, t)$ with respect to its first argument, and similarly for $b_y(y_t, t)$, and where $\tilde{B}$ is a standard two dimensional Brownian motion with respect to the measure $Q$. We will now need the following technical conditions: We suppose the processes $a(q_t, t)$ and $b(y_t, t)$ satisfy the following:

$$\int_0^T (b(y_t, t)a_q(q_t, t))^2 (\sigma_{q,1}(t)^2 + \sigma_{q,2}(t)^2)dt < \infty \quad a.s. \quad (22)$$

$$E\left(\int_0^T (b(y_t, t)a_q(q_t, t))^2 (\sigma_{q,1}(t)^2 + \sigma_{q,2}(t)^2)dt\right) < \infty \quad (23)$$

and

$$\int_0^T (a(q_t, t)b_y(y_t, t))^2 (\sigma_{y,1}(t)^2 + \sigma_{y,2}(t)^2)dt < \infty \quad a.s. \quad (24)$$

$$E\left(\int_0^T (a(q_t, t)b_y(y_t, t))^2 (\sigma_{y,1}(t)^2 + \sigma_{y,2}(t)^2)dt\right) < \infty. \quad (25)$$
Notes

1 Alternatively it could be an option contract requiring an initial cash payment, or a hybrid.

2 In practice the hedging resulting from this use of the different markets may not be entirely accurate, but then one should perhaps have in mind that the only “perfect hedge” is found in a Japanese garden.