The Real Option Approach to Plant Valuation: from Spread Options to Optimal Switching

René Carmona\(^1\) and Michael Ludkovski\(^2\)

\(^1\)Bendheim Center for Finance
Department of Operations Research & Financial Engineering
Princeton University

\(^2\)Department of Mathematics
University of Michigan

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Motivation: Valuing a Tolling Agreement

Stylized Version

- **Leasing an Energy Asset**
  - Fossil Fuel Power Plant
  - Oil Refinery
  - Pipeline

- **Owner of the Agreement**
  - Decides *when* and *how* to use the asset (e.g. run the power plant)
  - Has someone else do the leg work
Classical Power Plant Valuation

Real Option Approach
- Lifetime of the plant \([T_1, T_2]\)
- \(C\) capacity of the plant (in MWh)
- \(H\) heat rate of the plant (in MMBtu/MWh)
- \(P_t\) price of power on day \(t\)
- \(G_t\) price of fuel (gas) on day \(t\)
- \(K\) fixed Operating Costs
- Value of the Plant (ORACLE)

\[
C \sum_{t=T_1}^{T_2} e^{-rt} \mathbb{E}\{(P_t - HG_t - K)^+\}
\]

String of Spark Spread Options
(Flash Back)

The Calpine - Morgan Stanley Deal

- Calpine needs to refinance USD 8 MM by November 2004
- **Jan. 2004:** Deutsche Bank: no traction on the offering
- **Feb. 2004:** *The Street* thinks Calpine is ”heading South”
- **March 2004:** Morgan Stanley offers a (complex) structured deal
  - A strip of spark spread options on 14 Calpine plants
  - A similar bond offering

*How were the options priced?*

- By Morgan Stanley ?
- By Calpine ?
Enormous literature

- R.C. - V. Durrleman
  - Journal of Computational Finance
  - SIAM Review
- C. Alexander
- S. Borovkova
- M. Dempster
- . . . . . .
Plant Operation Model: the Finite Mode Case

- Markov process (state of the world) \( X_t = (X_t^{(1)}, X_t^{(2)}, \cdots) \)
  (e.g. \( X_t^{(1)} = P_t, \quad X_t^{(2)} = G_t, \quad X_t^{(3)} = O_t \) for a dual plant)

- Plant characteristics
  - \( \mathbb{Z}_M \overset{\Delta}{=} \{0, \cdots, M - 1\} \) modes of operation of the plant
  - \( H_0, H_1, \cdots, H_{M-1} \) heat rates
  - \( \{C(i, j)\}_{(i, j) \in \mathbb{Z}_M} \) regime switching costs \( (C(i, j) = C(i, \ell) + C(\ell, j)) \)
  - \( \psi_i(t, x) \) reward at time \( t \) when world in state \( x \), plant in mode \( i \)

- Operation of the plant (control) \( u = (\xi, T) \) where
  - \( \xi_k \in \mathbb{Z}_M \overset{\Delta}{=} \{0, \cdots, M - 1\} \) successive modes
  - \( 0 \leq \tau_{k-1} \leq \tau_k \leq T \) switching times
Plant Operation Model: the Finite Mode Case

- $T$ (horizon) length of the tolling agreement
- Total reward

$$H(x, i, [0, T]; u)(\omega) \triangleq \int_0^T \psi_{us}(s, X_s) \, ds - \sum_{\tau_k < T} C(u_{\tau_k-}, u_{\tau_k})$$
\( \mathcal{U}(t) \) acceptable controls on \([t, T]\)
(adapted càdlàg \( \mathbb{Z}_M \)-valued processes \( u \) of a.s. finite variation on \([t, T]\))

**Optimal Switching Problem**

\[
J(t, x, i) = \sup_{u \in \mathcal{U}(t)} J(t, x, i; u),
\]

where

\[
J(t, x, i; u) = \mathbb{E}\left[H(x, i, [t, T]; u) \mid X_t = x, u_t = i\right]
\]

\[
= \mathbb{E}\left[\int_0^T \psi_{us}(s, X_s) \, ds - \sum_{\tau_k < T} C(u_{\tau_k-}, u_{\tau_k}) \mid X_t = x, u_t = i\right]
\]
Consider problem with at most $k$ mode switches

$$\mathcal{U}^k(t) \triangleq \{ (\xi, T) \in \mathcal{U}(t) : \tau_\ell = T \text{ for } \ell \geq k + 1 \}$$

Admissible strategies on $[t, T]$ with at most $k$ switches

$$J^k(t, x, i) \triangleq \operatorname{esssup}_{u \in \mathcal{U}^k(t)} \mathbb{E} \left[ \int_t^T \psi_{u_s}(s, X_s) \, ds - \sum_{t \leq \tau_k < T} C(u_{\tau_k-}, u_{\tau_k}) \mid X_t = x, u_t = i \right].$$
Alternative Recursive Construction

\[ J^0(t, x, i) \triangleq \mathbb{E}\left[ \int_t^T \psi_i(s, X_s) \, ds \mid X_t = x \right], \]

\[ J^k(t, x, i) \triangleq \sup_{\tau \in \mathcal{S}_t} \mathbb{E}\left[ \int_t^\tau \psi_i(s, X_s) \, ds + \mathcal{M}^{k,i}(\tau, X_\tau) \bigg| X_t = x \right]. \]

**Intervention operator** \( \mathcal{M} \)

\[ \mathcal{M}^{k,i}(t, x) \triangleq \max_{j \neq i} \left\{ -C_{i,j} + J^{k-1}(t, x, j) \right\}. \]

**Hamadène - Jeanblanc** \((M=2)\)
Variational Formulation

Notation

- $\mathcal{L}_X X$ space-time generator of Markov process $X_t$ in $\mathbb{R}^d$
- $\mathcal{M}\phi(t, x, i) = \max_{j \neq i} \{-C_{i,j} + \phi(t, x, j)\}$ intervention operator

Assume

- $\phi(t, x, i)$ in $C^{1,2}([0, T] \times \mathbb{R}^d) \setminus D) \cap C^{1,1}(D)$
- $D = \bigcup_i \{(t, x) : \phi(t, x, i) = \mathcal{M}\phi(t, x, i)\}$
- (QVI) for all $i \in \mathbb{Z}_M$:
  1. $\phi \geq \mathcal{M}\phi$,
  2. $\mathbb{E}_x \left[\int_0^T \mathbf{1}_{\phi \leq \mathcal{M}\phi} \, dt\right] = 0$,
  3. $\mathcal{L}_X\phi(t, x, i) + \psi_i(t, x) \leq 0$, $\phi(T, x, i) = 0$,
  4. $\left(\mathcal{L}_X\phi(t, x, i) + \psi_i(t, x)\right)\left(\phi(t, x, i) - \mathcal{M}\phi(t, x, i)\right) = 0$.

Conclusion

$\phi$ is the optimal value function for the switching problem
Assume

- $X_0 = x$ & $\exists (Y^x, Z^x, A)$ adapted to $(\mathcal{F}_t^X)$

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^x|^2 + \int_0^T \|Z_t^x\|^2 \, dt + |A_T|^2 \right] < \infty$$

and

$$Y_t^x = \int_t^T \psi_i(s, X_s^x) \, ds + A_T - A_t - \int_t^T Z_s \cdot dW_s,$$

$$Y_t^x \geq \mathcal{M}^{k,i}(t, X_t^x),$$

$$\int_0^T (Y_t^x - \mathcal{M}^{k,i}(t, X_t^x)) \, dA_t = 0, \quad A_0 = 0.$$  

Conclusion: if $Y_0^x = J^k(0, x, i)$ then

$$Y_t^x = J^k(t, X_t^x, i)$$
System of Reflected Backward SDE’s

QVI for optimal switching: *coupled system* of reflected BSDE’s for \((Y^i)_{i \in \mathbb{Z}_M}\),

\[
Y_t^i = \int_t^T \psi_i(s, X_s) \, ds + A_T^i - A_t^i - \int_t^T Z_s^i \cdot dW_s,
\]

\[
Y_t^i \geq \max_{j \neq i} \{-C_{i,j} + Y_t^j\}.
\]

Existence and uniqueness Directly for \(M > 2\)?

\(M = 2\), *Hamadène - Jeanblanc* use difference process \(Y^1 - Y^2\).
Discrete Time Dynamic Programming

- Time Step $\Delta t = T/M^\#$
- Time grid $S^{\Delta} = \{m\Delta t, m = 0, 1, \ldots, M^\#\}$
- Switches are allowed in $S^{\Delta}$

DPP

For $t_1 = m\Delta t$, $t_2 = (m + 1)\Delta t$ consecutive times

$$J^k(t_1, X_{t_1}, i) = \max \left( \mathbb{E} \left[ \int_{t_1}^{t_2} \psi_i(s, X_s) \, ds + J^k(t_2, X_{t_2}, i) | \mathcal{F}_{t_1} \right], \mathcal{M}^{k,i}(t_1, X_{t_1}) \right)$$

$$\simeq \left( \psi_i(t_1, X_{t_1}) \Delta t + \mathbb{E} \left[ J^k(t_2, X_{t_2}, i) | \mathcal{F}_{t_1} \right] \right) \lor \left( \max_{j \neq i} \left\{ -C_{i,j} + J^{k-1}(t_1, X_{t_1}, j) \right\} \right).$$

(1)

Tsitsiklis - van Roy
Recall

\[
J^k(m \Delta t, x, i) = \mathbb{E} \left[ \sum_{j=m}^{\tau^k} \psi_i(j \Delta t, X_{j \Delta t}) \Delta t + \mathcal{M}^{k,i}(\tau^k \Delta t, X_{\tau^k \Delta t}) \big| X_{m \Delta t} = x \right].
\]

Analogue for \(\tau^k\):

\[
\tau^k(m \Delta t, x_{m \Delta t}, i) = \begin{cases} 
\tau^k((m + 1) \Delta t, x_{(m+1) \Delta t}, i), & \text{no switch;} \\
 m, & \text{switch,}
\end{cases}
\]

and the set of paths on which we switch is given by \(\{ \ell : \hat{J}^\ell(m \Delta t; i) \neq i \}\) with

\[
\hat{J}^\ell(t_1; i) = \arg \max_j \left( -C_{i,j} + J^{k-1}(t_1, x_{t_1}^\ell, j), \psi_i(t_1, x_{t_1}^\ell) \Delta t + \hat{E}_{t_1} [J^k(t_2, \cdot, i)](x_{t_1}^\ell) \right).
\]

The full recursive \textit{pathwise} construction for \(J^k\) is

\[
J^k(m \Delta t, x_{m \Delta t}, i) = \begin{cases} 
\psi_i(m \Delta t, x_{m \Delta t}) \Delta t + J^k((m + 1) \Delta t, x_{(m+1) \Delta t}, i), & \text{no switch;} \\
 -C_{i,j} + J^{k-1}(m \Delta t, x_{m \Delta t}, j), & \text{switch to } j.
\end{cases}
\]
Remarks

- Regression used solely to update the optimal stopping times $\tau^k$
- Regressed values never stored
- Helps to eliminate potential biases from the regression step.
Algorithm

1. Select a set of basis functions \((B_j)\) and algorithm parameters \(\Delta t, M^\#, N^p, \bar{K}, \delta\).

2. Generate \(N^p\) paths of the driving process: \(\{x^\ell_{m\Delta t}, m = 0, 1, \ldots, M^\#, \ell = 1, 2, \ldots, N^p\}\) with fixed initial condition \(x^\ell_0 = x_0\).

3. Initialize the value functions and switching times \(J^k(T, x^\ell_T, i) = 0, \tau^k(T, x^\ell_T, i) = M^\# \forall i, k\).

4. Moving backward in time with \(t = m\Delta t, m = M^\#, \ldots, 0\) repeat the Loop:
   - Compute inductively the layers \(k = 0, 1, \ldots, \bar{K}\) (evaluate \(E[J^k(m\Delta t + \Delta t, \cdot, i)|\mathcal{F}_{m\Delta t}]\) by linear regression of \(\{J^k(m\Delta t + \Delta t, x^\ell_{m\Delta t+\Delta t}, i)\}\) against \(\{B_j(x^\ell_{m\Delta t})\}_{j=1}^{N_B}\), then add the reward \(\psi_i(m\Delta t, x^\ell_{m\Delta t}) \cdot \Delta t\)
   - Update the switching times and value functions

5. end Loop.

6. Check whether \(\bar{K}\) switches are enough by comparing \(J^{\bar{K}}\) and \(J^{\bar{K}-1}\) (they should be equal).

Observe that during the main loop we only need to store the buffer \(J(t, \cdot), \ldots, J(t + \delta, \cdot);\) and \(\tau(t, \cdot), \ldots, \tau(t + \delta, \cdot)\).
Convergence

- Bouchard - Touzi
- Gobet - Lemor - Warin
Example 1

\[ dX_t = 2(10 - X_t) \, dt + 2 \, dW_t, \quad X_0 = 10, \]

- Horizon \( T = 2, \)
- Switch separation \( \delta = 0.02. \)
- Two regimes
- Reward rates \( \psi_0(X_t) = 0 \) and \( \psi_1(X_t) = 10(X_t - 10) \)
- Switching cost \( C = 0.3. \)
Value Functions

$J^k(t, x, 0)$ as a function of $t$
Exercise Boundaries

\[ k = 2 \ (\text{left}) \]
\[ k = 7 \ (\text{right}) \]

NB: Decreasing boundary around \( t = 0 \) is an artifact of the Monte Carlo.
Spread Options
Optimal Switching
Extensions

One Sample

State process and boundaries

Cumulative wealth

Time Units

Carmona
Spread Options and Optimal Switching
Example 2: Comparisons

Spark spread \( X_t = (P_t, G_t) \)

\[
\begin{align*}
\log(P_t) &\sim OU(\kappa = 2, \theta = \log(10), \sigma = 0.8) \\
\log(G_t) &\sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4)
\end{align*}
\]

- \( P_0 = 10, \ G_0 = 10, \ \rho = 0.7 \)
- Agreement Duration \([0, 0.5]\)
- Reward functions
  \[
  \begin{align*}
  \psi_0(X_t) &= 0 \\
  \psi_1(X_t) &= 10(P_t - G_t) \\
  \psi_2(X_t) &= 20(P_t - 1.1 G_t)
  \end{align*}
  \]
- Switching costs
  \[
  C_{i,j} = 0.25 |i - j|
  \]
## Numerical Comparison

<table>
<thead>
<tr>
<th>Method</th>
<th>Mean</th>
<th>Std. Dev</th>
<th>Time (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explicit FD</td>
<td>5.931</td>
<td>–</td>
<td>25</td>
</tr>
<tr>
<td>LS Regression</td>
<td>5.903</td>
<td>0.165</td>
<td>1.46</td>
</tr>
<tr>
<td>TvrR Regression</td>
<td>5.276</td>
<td>0.096</td>
<td>1.45</td>
</tr>
<tr>
<td>Kernel</td>
<td>5.916</td>
<td>0.074</td>
<td>3.8</td>
</tr>
<tr>
<td>Quantization</td>
<td>5.658</td>
<td>0.013</td>
<td>400*</td>
</tr>
</tbody>
</table>

**Table:** Benchmark results for Example 2.
Example 3: Dual Plant & Delay

\[
\begin{aligned}
\log(P_t) &\sim OU(\kappa = 2, \theta = \log(10), \sigma = 0.8), \\
\log(G_t) &\sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4), \\
\log(O_t) &\sim OU(\kappa = 1, \theta = \log(10), \sigma = 0.4),
\end{aligned}
\]

- \( P_0 = G_0 = O_0 = 10, \rho_{pg} = 0.5, \rho_{po} = 0.3, \rho_{go} = 0 \)
- Agreement Duration \( T = 1 \)
- Reward functions
  \[
  \begin{align*}
  \psi_0(X_t) &= 0 \\
  \psi_1(X_t) &= 5 \cdot (P_t - G_t) \\
  \psi_2(X_t) &= 5 \cdot (P_t - O_t) \\
  \psi_3(X_t) &= 5 \cdot (3P_t - 4G_t) \\
  \psi_4(X_t) &= 5 \cdot (3P_t - 4O_t).
  \end{align*}
  \]
- Switching costs \( C_{i,j} \equiv 0.5 \)
- Delay \( \delta = 0, 0.01, 0.03 \) (up to ten days)
Numerical Results

<table>
<thead>
<tr>
<th>Setting</th>
<th>No Delay</th>
<th>$\delta = 0.01$</th>
<th>$\delta = 0.03$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base Case</td>
<td>13.22</td>
<td>12.03</td>
<td>10.87</td>
</tr>
<tr>
<td>Jumps in $P_t$</td>
<td>23.33</td>
<td>22.00</td>
<td>20.06</td>
</tr>
<tr>
<td>Regimes 0-3 only</td>
<td>11.04</td>
<td>10.63</td>
<td>10.42</td>
</tr>
<tr>
<td>Regimes 0-2 only</td>
<td>9.21</td>
<td>9.16</td>
<td>9.14</td>
</tr>
<tr>
<td>Gas only: 0, 1, 3</td>
<td>9.53</td>
<td>7.83</td>
<td>7.24</td>
</tr>
</tbody>
</table>

Table: LS scheme with 400 steps and 16000 paths.

Remarks

- High $\delta$ lowers profitability by over 20%.
Example 4: Exhaustible Resources

Include $l_t$ current level of resources left ($l_t$ non-increasing process).

\[ J(t, x, c, i) = \sup_{\tau,j} \mathbb{E} \left[ \int_t^\tau \psi_i(s, X_s) \, ds + J(\tau, X_\tau, l_\tau, j) - C_{i,j} \mid X_t = x, l_t = c \right]. \]  

(5)

- Resource depletion (boundary condition) $J(t, x, 0, i) \equiv 0$.
- Not really a control problem $l_t$ can be computed on the fly.

Mining example of Brennan and Schwartz varying the initial copper price $X_0$

<table>
<thead>
<tr>
<th>Method/ $X_0$</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS '85</td>
<td>1.45</td>
<td>4.35</td>
<td>8.11</td>
<td>12.49</td>
<td>17.38</td>
<td>22.68</td>
</tr>
<tr>
<td>PDE FD</td>
<td>1.42</td>
<td>4.21</td>
<td>8.04</td>
<td>12.43</td>
<td>17.21</td>
<td>22.62</td>
</tr>
<tr>
<td>RMC</td>
<td>1.33</td>
<td>4.41</td>
<td>8.15</td>
<td>12.44</td>
<td>17.52</td>
<td>22.41</td>
</tr>
</tbody>
</table>
Can accommodate **outages**
Can include switch separation as a form of **delay**
Can be extended ([R.C. - M. Ludkovski](#)) to treat
- Gas Storage
- Hydro Plants
What Remains to be Done

- Need to improve delays
- Need **convergence analysis**
- Need better analysis of **exercise boundaries**
- Need to implement duality upper bounds
  - we have approximate value functions
  - we have approximate exercise boundaries
  - so we have lower bounds
  - need to extend Meinshausen-Hambly to optimal switching set-up
Extending the Analysis Adding Access to a Financial Market

**Porchet-Touzi**

- Same (Markov) factor process $X_t = (X_t^{(1)}, X_t^{(2)}, \cdots)$ as before
- Same plant characteristics as before
- Same operation control $u = (\xi, T)$ as before
- Same maturity $T$ (end of tolling agreement) as before
- **Reward** for operating the plant

\[
H(x, i, T; u)(\omega) \triangleq \int_0^T \psi_{us}(s, X_s) \, ds - \sum_{\tau_k < T} C(u_{\tau_k-}, u_{\tau_k})
\]
Access to a financial market (possibly incomplete)

- $y$ initial wealth
- $\pi_t$ investment portfolio
- $Y_T^{y,\pi}$ corresponding terminal wealth from investment

**Utility function** $U(y) = -e^{-\gamma y}$

**Maximum expected utility**

$$v(y) = \sup_{\pi} \mathbb{E}\{U(Y_T^{y,\pi})\}$$
Indifference Pricing

- With the power plant (tolling contract)

\[ V(x, i, y) = \sup_{u, \pi} \mathbb{E}\{U(Y_T^Y, \pi) + H(x, i, T; u)\} \]

**INDIFFERENCE PRICING**

\[ \bar{p} = p(x, i, y) = \sup\{p \geq 0; V(x, i, y) \geq v(y)\} \]

Analysis of
- BSDE formulation
- PDE formulation