Spread Options as Compound Exchange Options
with Applications to American Crack Spreads

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Abstract

This paper presents a new analytic approximation for pricing and hedging European and American spread options where the spread option price is represented as the price of an exchange option on two in-the-money European call options on the assets. We also extend the formula stated by Kirk [1996] to obtain approximate prices of American spread options on commodities and equity indices. However, the new exchange option approach has many advantages over other approximations that are illustrated via simulations and by comparing the model’s pricing and hedging performance using recent data on American crack spread options. Finally, the compound exchange option approximation is based on two correlated geometric Brownian motion price processes and this assumption has a straightforward extension to local volatility functions and local correlation. In this case it captures the volatility smiles in market prices of options on each underlying asset as well as the correlation "frown" that is observed in market prices of spread options.

1. Introduction

A spread option is an option whose pay-off depends on the price spread between two correlated underlying assets. Spread options are an important tool for trading and hedging correlation and they are widely traded on several exchanges and in over-the-counter markets. If the asset prices are $S_1$ and $S_2$ the payoff to a spread option of strike $K$ is $\max\{\omega(S_1 - S_2) - K, 0\}$ where $\omega = 1$ for a call and $\omega = -1$ for a put. To price such an option, Ravindran [1993], Shimko [1994], Kirk [1996] and others assume each asset price process is a geometric Brownian motion with constant volatility and that these processes have a constant non-zero correlation: we label this the '2GBM' framework.
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for short. The framework is tractable but it captures neither the implied volatility smiles that are derived from market prices of univariate options nor the implied correlation 'frown' that is evident from market prices of spread options.

Kirk [1996] states an analytic approximation to the price of a European spread option under the 2GBM assumption in the special case that spreads are valued on futures or forward contracts. Yet correlation frowns are a prominent feature in spread option markets because the pay-off to a spread option decreases with correlation. Hence if market prices of out-of-the-money call and put spread options are higher than the standard 2GBM model prices with constant correlation, the implied correlations that are 'backed-out' from the 2GBM model will have the appearance of a 'frown'.

Alexander and Scourse [2004] derive approximate analytic prices of European spread options on futures or forward contracts that display both volatility smiles and a correlation frown. They assume the asset prices have a bivariate lognormal mixture distribution and hence obtain prices as a weighted sum of four different 2GBM spread option prices, each of which may be obtained using an analytic approximation such as that of Kirk [1996]. However, most spread options are traded on assets that pay dividends or have carry costs, and spread options on equity indices and options on commodity spreads are common. And in most cases the options are American, as is the case for the crack spread options that we consider later in this paper.

Other approaches to pricing and hedging spread options that are both realistic and tractable have been rather elusive. Carr and Madan [1999] and Dempster and Hong [2000] advocate models that capture volatility skews on the two assets by introducing stochastic volatility to the price processes. And the addition of price jumps can explain the implied correlation frown, as in the spark spread option pricing model of Carmona and Durrleman [2003a]. However pricing and hedging in this framework necessitates computationally intensive numerical resolution methods such as the fast Fourier transform. Other models provide only a range for spread option prices, as in Durrleman [2001] and Carmona and Durrleman [2005], who provide upper and lower bounds that can be very narrow for certain parameter values. For a detailed survey of these models and a comparison of their performances, the reader is referred to Carmona and Durrleman [2003b].

In this paper we first extend Kirk's formula so that it applies to European and American spread options on commodities, stock indices and any underlying paying coupons or dividends. Then our main result derives a new analytic approximation in which the price of a spread option is expressed as a compound exchange option. Our approach retains the simplicity offered by the 2GBM framework, but is shown to offer a greater accuracy than the approximation of Kirk [1996]. It is easy to calibrate and hedge ratios are particularly simple to compute. The model may also be extended from constant to local volatilities and correlation, so that it is smile and frown consistent, without requiring resolution through complex numerical methods.

In the following, section 2 derives the analytic pricing and hedging formulae for European spread options.
options on assets paying dividends or with carry costs. We summarize the exchange option pricing formula of Margrabe [1978], since this is central to our model. Then we provide a full derivation of the approximation stated in Kirk [1996], since this is not available in the literature, and extend it to allow for non-zero dividends or carry costs. Our main theoretical result is presented in section 2.3, where we derive the compound exchange option approximation to prices and hedge ratios of spread options. The section concludes with a simulation exercise that highlights some substantial problems with implementing Kirk’s approximation. Section 3 extends both Kirk’s approximation and the compound exchange option approximation to accommodate early exercise. We then demonstrate the superiority of the compound exchange option approach by analysing its pricing and hedging performance for the American crack spread options traded on the New York Mercantile Exchange (NYMEX) during 2005. The final section summarises our results and concludes.

2. Pricing European Spread Options

2.1. Margrabe’s Exchange Option Pricing Formula

When the strike of the spread option is zero the option is called an exchange option, since the buyer has the option to exchange one underlying asset for the other. If \( S_{1,t} \) and \( S_{2,t} \) are the spot prices of two assets at time \( t \) then the payoff to an exchange option at the expiry date \( T \) is given by \( \max\{S_{1,T} - S_{2,T}, 0\} \). Assume that the risk-neutral price dynamics are governed by two correlated geometric Brownian motions with constant volatilities given by:

\[
\frac{dS_{i,t}}{S_{i,t}} = (r - q_i)dt + \sigma_i dW_{i,t} \quad i = 1, 2
\]

(1)

where, \( W_{1,t} \) and \( W_{2,t} \) are Wiener processes under risk neutral measure, \( r \) is the (assumed constant) risk-free interest rate and \( q_1 \) and \( q_2 \) are the (assumed constant) dividend yields of the two assets. The volatilities \( \sigma_1 \) and \( \sigma_2 \) are also assumed to be constant as is the returns correlation:

\[
\langle dW_{1,t}, dW_{2,t} \rangle = \rho dt
\]

Using risk-neutral valuation the price of an exchange option is given by

\[
P_t = E_Q \left\{ e^{-r(T-t)} \max\{S_{1,T} - S_{2,T}, 0\} \right\} = e^{-r(T-t)} E_Q \left\{ S_{2,T} \max\{x_T - 1, 0\} \right\}
\]

where \( x_t = \frac{S_{1,t}}{S_{2,t}} \) follows the process

\[
\frac{dx_t}{x_t} = (q_2 - q_1)dt + \sigma dW
\]

with

\[
dW = \rho dW_1 + \sqrt{1-\rho^2} dW_2
\]

\[
\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}
\]
Since the both the assets grow at the risk-free rate, the relative drift of $S_1$ with respect to $S_2$ due to $r$ is zero. But as the dividend yields of the assets may be different, $x_t$ drifts at the rate of $(q_2 - q_1)$.

Margrabe [1978] shows that under these assumptions the price $P_t$ of an exchange option is given by

$$
P_t = S_{1,t} e^{-q_1(T-t)} \Phi(d_1) - S_{2,T} e^{-q_2(T-t)} \Phi(d_2)
$$

where,

$$
d_1 = \frac{\ln \left( \frac{S_{1,t}}{S_{2,t}} \right) + (q_2 - q_1 + \frac{1}{2} \sigma_1^2) (T-t)}{\sigma_1 \sqrt{T-t}}
$$

$$
d_2 = d_1 - \sigma_1 \sqrt{T-t}
$$

$$
\sigma_t = \sqrt{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2}
$$

2.2. Kirk’s Approximation

Kirk [1996] presents an approximate closed form formula for pricing European spread options on futures or forwards. The method extends that of Margrables’ for non-zero but very small strike values. When $K \ll S_{2,0}$ the displaced diffusion process $S_{2,t} + K$ can be assumed to be approximately log-normal. Then, $\frac{S_{1,t}}{S_{2,t} + K e^{-r(T-t)}}$ is also approximately log-normal and can be expressed as a geometric Brownian motion process. The derivation has not been documented in the literature, and neither were dividends included in the formula presented in Kirk [1996]. Hence we outline the main steps below. Rewrite the pay-off of the European put spread option as:

$$
max \{ K - S_{1,T} + S_{2,T}, 0 \} = (K + S_{2,T}) \max \{ 1 - \frac{S_{1,T}}{K + S_{2,T}}, 0 \}
$$

$$
= (K + S_{2,T}) \max \{ 1 - Z_T, 0 \}
$$

where

$$
Z_t = \frac{S_{1,t}}{S_{2,t}/(K e^{-r(T-t)} + S_{2,t})}
$$

Then the price at time $t$ of an option with payoff $max \{ K - S_{1,T} + S_{2,T}, 0 \}$ is

$$
P_t = \mathbb{E}_Q \left\{ Y_t e^{-r(T-t)} \max \{ 1 - Z_T, 0 \} \right\}
$$

where $Y_t = S_{2,t} + K e^{-r(T-t)}$ and by Ito’s lemma:

$$
\frac{dZ_t}{Z_t} = \frac{\partial Z}{\partial S_1} dS_1 + \frac{\partial Z}{\partial Y} dY + \frac{1}{2} \frac{\partial^2 Z}{\partial S_1^2} (dS_1)^2 + \frac{1}{2} \frac{\partial^2 Z}{\partial Y^2} (dY)^2 + \frac{1}{2} \frac{\partial^2 Z}{\partial S_1 \partial Y} (dS_1 dY)
$$

$$
= \frac{S_1}{Y} \left( \frac{dS_1}{S_1} - \frac{dY}{Y} \right) + \left( \frac{dY}{Y} \right)^2 - \frac{dS_1}{S_1} \frac{dY}{Y}
$$

(3)
We have \( dY_t = dS_{2,t} + K re^{-(T-t)} dt \) and for \( K \ll S_2 \)
\[
\frac{dY_t}{Y_t} = \frac{S_{2,t}}{Y_t} \left( (r - q_2) dt + \sigma_2 dW_{2,t} \right)
= (\bar{r} - \bar{q}_2) dt + \bar{\sigma}_2 dW_{2,t}
\]
(4)

Hence (3) can be rewritten:
\[
\frac{dZ_t}{Z_t} = (r - \bar{r} - (q_1 - \bar{q}_2)) dt + (\bar{\sigma}_2^2 - \sigma_1 \bar{\sigma}_2 \rho) dt + \sigma_1 dW_{1,t} - \bar{\sigma}_2 dW_{2,t}
\]
where
\[
\bar{\sigma}_2 = \left( \frac{S_2}{Y_t} \right) \sigma_2, \quad \bar{r} = \left( \frac{S_2}{Y_t} \right) r, \quad \text{and} \quad \bar{q}_2 = \left( \frac{S_2}{Y_t} \right) q_2
\]

Let \( W_3(t) \) be a Brownian motion that is uncorrelated with \( W_2(t) \) and such that
\[
dW_{1,t} = \rho dW_{2,t} + \sqrt{1 - \rho^2} dW_{3,t}
\]

Then,
\[
\frac{dZ_t}{Z_t} = (r - \bar{r} - (q_1 - \bar{q}_2)) dt + (\bar{\sigma}_2 - \sigma_1 \bar{\sigma}_2 \rho) dt + (\rho \sigma_1 - \bar{\sigma}_2) dW_{2,t} + \sqrt{1 - \rho^2} dW_{3,t}
\]

Define \( W_{2,t}^* = W_{2,t} - \bar{\sigma}_2 dt \). Using Girsanov’s theorem, let \( \mathbb{P} \) be the new probability measure under which both \( W_{2,t}^* \) and \( dW_{3,t} \) are martingales. The Radon-Nikodym derivative with respect to the risk-neutral probability \( \mathbb{Q} \) is then given by:
\[
\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{(-\frac{1}{2} \sigma_2^2 T + \sigma_2 W_{2,t})}
\]

We now have
\[
\frac{dZ_t}{Z_t} = (r - \bar{r} - (q_1 - \bar{q}_2)) dt + (\bar{\sigma}_2 - \sigma_1 \rho) dW_{2,t}^* + \sqrt{1 - \rho^2} \bar{\sigma}_2 dW_{3,t}
= (r - \bar{r} - (q_1 - \bar{q}_2)) dt + \sigma_t dW_t
\]

and
\[
\sigma_t = \sqrt{\sigma_1^2 + \sigma_2^2 \left( \frac{S_{2,t}}{S_{2,t} + K e^{-r(T-t)}} \right)^2 - 2 \rho \sigma_1 \sigma_2 \left( \frac{S_{2,t}}{S_{2,t} + K e^{-r(T-t)}} \right)}
\]

Note that \( Z_t \) is (approximately) log-normal and is also observable in the market. Hence the spread option can be priced by treating it as a plain vanilla option defined on an observable asset whose price process is described by \( Z_t \) and with a strike \( K = 1 \). Therefore the price \( P_t \) at time \( t \) for an option on \( S_{1,t} \) and \( S_{2,t} \) with strike \( K \), maturity \( T \) and payoff \( \max \{ \omega (S_1 - S_2 - K), 0 \} \) is given by:
\[
P_t = \omega \left( S_{1,t} e^{-q_1(T-t)} \Phi (\omega d_{1}^*) - \left( K e^{-r(T-t)} + S_{2,t} \right) e^{-r(T-t) + \bar{q}_2} \Phi (\omega d_{2}^*) \right)
\]
(5)
where \( \Phi \) denotes the standard normal distribution function and \( \omega = 1 \) for a call and \( \omega = -1 \) for a put,

\[
d_1^* = \frac{\ln \left( \frac{S_{1,t}}{K e^{-r(T-t)}} + \sigma_t \sqrt{T-t} \right) + \left( r - \bar{\rho} + \bar{q}_2 - q_1 + \frac{1}{2} \sigma_t^2 \right) (T-t)}{\sigma_t \sqrt{T-t}}
\]

\[
d_2^* = d_1^* - \sigma_t \sqrt{T-t}
\]

A slightly modified representation of Kirk’s formula is

\[
P_t^* = \frac{P_t}{K e^{-rT} + S_{2,t}} = \omega \left( Z_t \Phi(d_1^*) - \Phi(d_2^*) \right)
\] (6)

This representation reduces the dimension of the pricing problem from two to one. This form of representation is useful when we extend the formula to price American spread options, as we shall see in section 3.

### 2.3. Spread Options as Compound Exchange Options

For \( t \in [0,T] \) let

\[
\mathcal{M}_{K,T} = \left\{ x \in \mathbb{R}^+ : x \leq \frac{1}{K} \max \{ \inf(S_{1,t}), \inf(S_{2,t}) + K \} \right\}
\]

It is trivial to show that the set \( \mathcal{M}_{K,T} \) is non-empty for any two assets following the 2GBM process. Since \( S_{1,t} \) and \( S_{2,t} \) are lognormal processes we have \( S_{1,t} \geq 0 \) and \( S_{2,t} \geq 0 \) for all \( t \in [0,T] \). Note that even if \( \inf(S_{1,t}) = \inf(S_{2,t}) = 0 \) which is highly improbable, \( \mathcal{M}_{K,T} \) will still contain 1.

For any \( m(K,T) \in \mathcal{M}_{K,T} \) the payoff to a put spread option of strike \( K \) at time \( T \) can be written

\[
\max \{ K - S_{1,T} + S_{2,T}, 0 \} = \max \{ (S_{2,T} - (m(K,T) - 1)K) - (S_{1,t} - m(K,T)K), 0 \}
\]

Consider a European call option on asset 1 with strike \( m(K,T)K \) and another on asset 2 with strike \( (m(K,T) - 1)K \). We choose \( m(K,T) \geq 1 \) and to be sufficiently small so that these options are deep in-the-money (ITM) at the calibration time \( t = 0 \). Note that with our choice of \( m(K,T) \), \( m(K,T)K \) and \( (m(K,T) - 1)K \) are lower bounds for \( S_{1,t} \) and \( S_{2,t} \) for all \( t \in [0,T] \) respectively. This choice of \( m(K,T) \) admits a representation of the payoffs to the two call options as

\[
U_{1,T} = \max \{ (S_{1,T} - m(K,T)K), 0 \} = (S_{1,T} - m(K,T)K)
\]

\[
U_{2,T} = \max \{ (S_{2,T} - (m(K,T) - 1)K), 0 \} = (S_{2,T} - (m(K,T) - 1)K)
\] (7)

so the payoff to the European put spread option \( P_T \) can then be written as

\[
P_T = \max \{ U_{2,T} - U_{1,T}, 0 \}
\]

Hence the spread option is a compound exchange option on two deep in-the-money call options, with prices \( U_{1,t} \) and \( U_{2,t} \). It is easy to extend this construction to call spread options, where \( U_{1,t} \) and \( U_{2,t} \) are reversed in the spread option payoff above.
It remains to describe the processes of the two deep in-the-money call options $U_{2,t}$ and $U_{1,t}$ and to obtain the value of these stochastic processes at time $T$. Applying Ito’s lemma to (1):

$$\frac{dU_{1,t}}{U_{1,t}} = rdt + \tilde{\sigma}_1 dW_1$$

(8)

and

$$\frac{dU_{2,t}}{U_{2,t}} = rdt + \tilde{\sigma}_2 dW_2$$

(9)

where for $i = 1, 2$

$$\tilde{\sigma}_i = \frac{\partial U_{i,t}}{\partial S_i,t}$$

(10)

We assume that $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are constant throughout $[0, T]$. This is reasonable since, for deep in-the-money call options $\frac{S_t}{V_{1,t}} \approx 1$ and $\frac{\partial U_{1,t}}{\partial S} \approx 1$, where $S_t$ and $U_{1,t}$ are the prices of the underlying and option respectively. Then the risk neutral price of the put spread option at time $t$ is given by

$$P_t = \mathbb{E} \left\{ e^{-r(T-t)} \max(U_{2,T} - U_{1,T}, 0) \right\}$$

(11)

so its price may be obtained using Margrabe’s formula:

$$P_t = U_{2,t} \Phi(-d_2^*) - U_{1,t} \Phi(-d_1^*)$$

(12)

where

$$d_1^* = \frac{\ln \left( \frac{U_{1,t}}{U_{2,t}} \right) + \tilde{\sigma}^2(T-t)}{\tilde{\sigma} \sqrt{T-t}}$$

$$d_2^* = d_1^* - \tilde{\sigma} \sqrt{T-t}$$

$$\tilde{\sigma} = \sqrt{\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - 2 \rho \tilde{\sigma}_1 \tilde{\sigma}_2}$$

and

$$U_{1,t} = S_{1,t} e^{-q_1(T-t)} \Phi(d_1^*) - m Ke^{-r(T-t)} \Phi(d_2^*)$$

$$U_{2,t} = S_{2,t} e^{-q_2(T-t)} \Phi(d_1^*) - (m-1) Ke^{-r(T-t)} \Phi(d_2^*)$$

(13)

where

$$d_{1,t}^{U_1} = \frac{\ln \left( \frac{S_{1,t}}{mK} \right) + (r - q_1 + \frac{1}{2} \sigma_1^2) (T-t)}{\sigma_1 \sqrt{T-t}}$$

$$d_{2,t}^{U_1} = d_{1,t}^{U_1} - \sigma_1 \sqrt{T-t}$$

$$d_{1,t}^{U_2} = \frac{\ln \left( \frac{S_{2,t}}{(m-1)K} \right) + (r - q_2 + \frac{1}{2} \sigma_2^2) (T-t)}{\sigma_2 \sqrt{T-t}}$$

$$d_{2,t}^{U_2} = d_{1,t}^{U_2} - \sigma_2 \sqrt{T-t}$$

Under the assumption of complete markets there exists at least one option price pair $\{U_{1,t}, U_{2,t}\}$ such that (12) holds. Moreover, if the markets are assumed to be arbitrage free, then such a pair must be unique. Hence the problem of pricing a European spread option reduces to identifying that unique combination of the two ITM call options.
2.4. Calibration

To avoid arbitrage the price of a spread option should be consistent with the prices of options on $S_1$ and $S_2$. The simplest case is to set $\sigma_i$ equal to the implied volatility of $S_i$ for $i = 1, 2$. Then the implied correlation is calibrated by equating the model and market prices of the spread option.

But at which strikes should the single asset implied volatilities be chosen? Although the 2GBM model assumes constant volatility, we know that the market implied volatilities are not constant with respect to strike. So the strikes $K_1$ and $K_2$ at which the implied volatilities $\sigma_1$ and $\sigma_2$ are calculated may have a significant influence on the results, and the calibrated value of the implied correlation of a spread option with strike $K$ will not be independent of this choice.

The problem with Kirk’s approach is that the implied volatility and correlation parameters are ill-defined. There are infinitely many pairs $(S_{1,0}, S_{2,0})$ for which $S_{1,0} - S_{2,0} = K$ and hence very many possible choices of $K_1$ and $K_2$. Similarly there are infinitely many combinations of market implied $\sigma_1$, $\sigma_2$, and $\rho$ that yield the same $\sigma$ in equation (5). Hence to calibrate the model some ‘convention’ needs to be applied. We have tried using the single asset’s at-the-money (ATM) forward volatility to calibrate spread options of all strikes. We have also tried several different strike conventions and found the best results were obtained on setting $K_1 = S_{1,0} - \frac{K}{2}$ and $K_2 = S_{2,0} + \frac{K}{2}$. Still, we shall see below that a major drawback of Kirk’s approximation is that for large values of $K$, the log-normality approximation does not hold and the formula is no longer valid.

By contrast the implied volatility and correlation parameters are not ill-defined in the calibration of the compound exchange option (CEO) approximation. The single asset options’ strikes are determined by $m(K,T)$, i.e. $K_1 = m(K,T)K$ and $K_2 = (m(K,T) - 1)K$. And the implied correlation between the call options, which is the same as that of the underlying assets on which these options are written because they follow the same Wiener processes, is calibrated by equating market prices of the spread options to (12). For $\alpha$ and $\beta > 0$ we set

$$\rho(K,T)^{\alpha} = \min\{S_{1,0}, S_{2,0}\} / \beta m(K,T)K$$

so that all three parameters are determined by the value of $m(K,T)$ for given values of $\alpha$ and $\beta$. Appendix A outlines our reasons for specifying this functional form.

To investigate the extent of the error in Kirk’s approximation for spread options with high strikes, we have simulated prices for spread options of different strikes and compare the performance of Kirk’s approximation with that of the new exchange option approximation. We have used prices $S_1 = 65$ and $S_2 = 50$, and spread option strikes ranging between 9.5 and 27.5 with a step size of 1.5 and maturity 30 days. We also fixed the values $\alpha = 2$ and $\beta = 1.57$. The spread option prices were simulated using quadratic local volatility and local correlation functions that are assumed to be dependent only on the price levels of the underlying assets and not on time. The at-the-money forward volatilities were both 30% and the at-the-money forward correlation was 0.80.
Figure 1 compares the implied correlations calibrated from the two approaches. They illustrate the poor performance of Kirk’s approximation for high strike values. Using Kirk’s approximation the root mean square percentage calibration error (RMSE), i.e. where each error is expressed as a percentage of the option price, was 9% using the strike convention and 9.3% using the constant ATM volatility to determine $\sigma_1$ and $\sigma_2$. By contrast the exchange option model’s pricing errors are extremely small (the RMSE was 0.53%) and the implied correlation values in figure 1 show greater stability.

**Figure 1: Implied Correlations from Kirk’s and CEO Approximations**

Kirk 1 implied volatilities are calculated using $K_1 = S_{1,0} - \frac{K}{2}$ and $K_2 = S_{2,0} + \frac{K}{2}$ and Kirk 2 uses ATM constant volatility.

This simulation exercise illustrates the main problem with Kirk’s approximation. However, when we apply a convention for fixing the strikes of the implied volatilities $\sigma_1$, $\sigma_2$ and take these from the single asset option prices, and then calibrate the implied correlation to the spread option price, we obtain unrealistic results except for options with very low strikes. For high strike options the model’s lognormality assumption is not valid. A possible ‘quick fix’ could be to try changing the strike convention so that it can be different for each spread option, but this is very ad hoc.\(^1\)

\(^1\)Attempts to use the exchange option strike convention with $K_1 = m(K, T)K$ and $K_2 = (m(K, T) - 1)K$ led to even greater pricing errors.
3. Pricing and Hedging American Spread Options

The price of American style options on single underlying assets is mainly determined by the type of the underlying asset, the prevailing discount rate, and the presence of any dividend yield. The option to exercise early suggests that these options are more expensive than their European counterparts but there are many instances when it is not optimal to exercise an option early. American calls on non-dividend paying stocks and calls or puts on forward contracts are two examples where it is never optimal to exercise the option early (see James [2003]). Since no traded options are perpetual the expiry date forces the price of American options to converge to the price of their European counterparts. Before expiry, the prices of American calls and puts are always greater than or equal to the corresponding European calls and puts.

3.1. The Early Exercise Premium of a Spread Option

In the free boundary pricing methods of McKean [1965], Kim [1990], Carr et al. [1989], Jacka [1991], and others the price of an American option with payoff \( \max\{\omega(S_t - K), 0\} \) on one underlying asset with price process (1) is given by:

\[
P(S_t, t) = P^E(S_t, T, \omega) + \omega \int_t^T qS_t e^{-q(s-t)} \Phi(\omega d_1(S_t, B_t, s-t)) \, ds - \omega \int_t^T rKe^{-r(s-t)} \Phi(\omega d_2(S_t, B_t, s-t)) \, ds
\]

(15)

where \( \omega = 1 \) for a call and -1 for a put.

In the case of multiple underlying assets the behaviour of American options is similar to that of single asset American options, with some notable exceptions. Rubinstein [1991] was the first to note that an American exchange option is equivalent to a standard option in a modified yet equivalent financial market. The problem pricing reduces to that of pricing a plain vanilla option by taking one of the assets as the numeraire instead of the money market account with the corresponding equivalent martingale measure. Then the prices of such claims can be found using the early exercise premium (EEP) representation (see Detemple [2005]).

We can express the price of an American spread option as a sum of its European counterpart and an early exercise premium. Consider the simple case of an American exchange option on two assets. Let \( S_{1,t} \) and \( S_{2,t} \) be the prices of two assets at time \( t \) given by equation (1) and \( x_t = \frac{S_{1,t}}{S_{2,t}} \), as in section 2.1. Then the payoff to an exchange option at maturity \( T \) is \( S_{2,T} \max\{x_T - 1, 0\} \).

Let \( P^E_t \) and \( P^A_t \) be the prices of a European and American exchange option respectively. The EEP
representation gives

\[ P_t^A(x_t) = P_t^E(x_t) + \int_t^T q_1 x_t e^{-q_1(s-t)} \Phi(d_1(x_t, B_s, s-t, q_1, q_2, \sigma_x)) dt \]

\[ - \int_t^T q_2 e^{-q_2(s-t)} \Phi(d_2(x_t, B_s, s-t, q_1, q_2, \sigma_x)) dt \]

where,

\[ d_1(x_t, B_t, T-t, q_1, q_2, \sigma_x) = \frac{\ln(x_t) + (q_2 - q_1 + \frac{1}{2} \sigma_x^2) (T-t)}{\sigma_x \sqrt{T-t}} \]

\[ d_2(x_t, B_t, T-t, q_1, q_2, \sigma_x) = d_1(x_t, B_t, T-t, q_1, q_2, \sigma_x) - \sigma_x \sqrt{T-t} \]

and \( \sigma_x \) is as defined in section 2.1. This shows that for the early exercise premium to be positive we require \( q_1 > 0 \).

Now consider the case when the two underlying assets are futures contracts. Since futures do not have dividends, the above equation implies \( P_t^A = P_t^E \). Hence whilst an American option on a single futures contract may be worth more than the corresponding European option this is not necessarily the case for American options on multiple assets. Broadie and Detemple [1997] provide a detailed discussion of pricing American options on two assets stating properties of the exercise region and giving a recursive integral equation which is satisfied by the early exercise boundary. At present there are no efficient methods available to calculate the early exercise boundary, even in the two asset case.

### 3.2. Extension of Kirk’s Formula

In section 2.1 the random variable \( Z \) was approximately log normal for small \( K \) values and this allowed one to express the price of a European put spread as that of an ordinary European put. By the same construction we can use \( Z \) to express the price of an American put spread as an ordinary American put on \( Z \) with strike 1. The intrinsic value of the option at time \( t \) is given by,

\[ \max\{Ke^{-r(T-t)} - S_{1,t} + S_{2,t}, 0\} = \max\{Y_t - S_{1,t}, 0\} \]

The above resembles the payoff of an exchange option written on \( Y_t \) and \( S_{1,t} \), and both processes are observable in the market. Recalling equations (1) and (4) we have,

\[ P_t^A(Z_t) = P_t^E(Z_t) - \int_t^T q_1^* Z_t e^{-q_1^*(s-t)} \Phi(-d_1(Z_t, B_s, s-t, q_1^*, q_2^*, \sigma)) dt \]

\[ + \int_t^T q_2^* e^{-q_2^*(s-t)} \Phi(-d_2(Z_t, B_s, s-t, q_1^*, q_2^*, \sigma)) dt \]

where \( q_1^* = q_1 \) and \( q_2^* = (r - \bar{r} + \bar{q}_2) \).

\[ ^2\text{If } q_1 = 0 \text{ and } q_2 > 0 \text{ then } P_t^A < P_t^E. \]
At the early exercise boundary, i.e., when \( Z_t = B_t \), the price given by equation (17) equals \( 1 - B_t \):

\[
1 - B_t = P_t^*(B_t) - \int_t^T q_1^* B_t e^{-q_1^*(s-t)} \Phi(-d_1(B_t, B_s, s-t, q_1^*, q_2^*, \sigma)) \, dt
+ \int_t^T q_2^* e^{-q_2^*(s-t)} \Phi(-d_2(B_t, B_s, s-t, q_1^*, q_2^*, \sigma)) \, dt
\]

This is the value match condition. Moreover, at \( B_t \) the slope of the price curve of equation (17) is that of \( 1 - B_t \). This is called as the high contact condition and it can be obtained by differentiating equation (18) with respect to \( B_t \), giving:

\[
\frac{\partial P_t^*(B_t, 1, T-t)}{\partial B_t} - 1 = \frac{\partial}{\partial B_t} \left( \int_t^T q_2^* e^{-q_2^*(s-t)} \Phi(-d_2(B_t, B_s, s-t, q_1^*, q_2^*, \sigma)) \, dt \right)
- \frac{\partial}{\partial B_t} \left( \int_t^T q_1^* B_t e^{-q_1^*(s-t)} \Phi(-d_1(B_t, B_s, s-t, q_1^*, q_2^*, \sigma)) \, dt \right)
\]

### 3.3. Extension of Compound Exchange Option Formula

In section 2.3 we showed how a European spread option price is equivalent to the price of an exchange option on two deep in-the-money call options. We may choose \( m(K, T) \) to be sufficiently small so that the option price processes closely imitate that of underlying assets and hence carry costs or dividends on the underlying assets can alter the prices of these in-the-money options considerably. Therefore any change in price of the underlying assets due to dividends or carry costs must be accounted for when pricing a spread option as a compound exchange option.

Consider the price process of each underlying asset and the corresponding call options. The solutions to their stochastic differential equations at time \( t \) are given by,

\[
S_{i,T} = S_{i,t} e^{(r-q_i-\frac{1}{2}\sigma_i^2)(T-t)+\sigma_i dW_i}
\]

\[
U_{i,T} = U_{i,t} e^{(r-q_i-\frac{1}{2}\tilde{\sigma}_i^2)(T-t)+\tilde{\sigma}_i dW_i}
\]

Dividing \( S_{i,T} \) by \( U_{i,T} \) and using the approximation \( \tilde{\sigma}_i \approx \sigma_i \), to remove the stochastic term

\[
q_i^* = \frac{1}{T} \left( \ln \left( \frac{S_{i,T}}{S_{i,0}} \right) - \ln \left( \frac{U_{i,T}}{U_{i,0}} \right) \right) + q_i
\]

We now rewrite equations (8) and (9) as:

\[
\frac{dU_{1,t}}{U_{1,t}} = (r - q_1^*) dt + \tilde{\sigma}_1(S_{1,t}, t) dW_1
\]

\[
\frac{dU_{2,t}}{U_{2,t}} = (r - q_2^*) dt + \tilde{\sigma}_2(S_{2,t}, t) dW_2
\]

(19)

where \( q_1^* \) and \( q_2^* \) are the equivalent dividend yields of the options.

It should be noted that even though we shall be pricing American spread options, the two call options with prices \( U_1 \) and \( U_2 \) remain European style options. Although the exchange option may...
be exercised before maturity, the call options may be exercised only at expiry. Since the compound exchange option replicates the cash flow of a spread option, when exercised they will yield the same payoff. Since most of the trades are cash settled this is adequate. Even in commodity markets where the options are exercised by the physical delivery of goods, this adjustment can be justified as the underlying future contracts’ expiry date is the same as or later than that of the spread option.

Let us now restrict our analysis to the case that there are no dividend yields or carry costs, such as when the underlying assets are future contracts. Recall that equations (8) and (9) describe the price processes of the two deep in-the-money call options. The discounted option price processes and the process followed by \( X_t = \frac{V_t}{V_t^Z} \) are martingales. This allows one to price an American spread options as an American compound exchange option using the early exercise premium representation given by equation (16). The American spread option price is thus given by

\[
P^A_t(x_t) = P^E_t(X_t) + \int_t^T q_1^* X_t e^{-q_1^*(s-t)} \Phi(d_1(X_t, B_s, s - t, q_1^*, q_2^*, \sigma_x)) dt - \int_t^T q_2^* e^{-q_2^*(s-t)} \Phi(d_2(x_t, B_s, s - t, q_1^*, q_2^*, \sigma_x)) dt
\]  

and hence American spread options on futures or non dividend paying stocks are worth the same as their European counterparts.

3.4. Empirical Results

We now test the pricing performance of the exchange option approximation using 1:1 American crack spread option data traded at NYMEX between September 2005 and May 2006. The crack spread options are on both heating oil - crude oil and gasoline - crude oil and are traded on the price differential between the futures contracts of WTI light sweet crude oil, gasoline and heating oil. Option data for American style contracts on each of these individual futures contracts were also obtained for the same time period along with the futures prices. The size of all the futures contracts is 1000 bbls.

Figures 2 and 3 depict the implied volatility skews in gasoline and crude oil on several of the days during the sample period. These pronounced negative implied volatility skews indicate that a suitable pricing model should exhibit a positive skew in implied correlation as a function of the spread option strike.

We compare the results of Kirk’s approximation with the exchange option approximation by calibrating each model to the market prices of the gasoline - crude oil crack spread over consecutive trading dates starting from 1st March 2006 to 15th March 2006, these being days of particularly high trading volumes. From figure 4 we can clearly see that Kirk’s approximation gives an error that increases drastically for high strike values, as was also the case in our simulation results. On the other hand the compound exchange option model errors were found to be close to zero for
**Figure 2: Implied Volatility of Gasoline**

![Graph showing implied volatility of gasoline across different strike prices and dates from 01 Mar 06 to 13 Mar 06.](image)

**Figure 3: Implied Volatility of Crude Oil**

![Graph showing implied volatility of crude oil across different strike prices and dates from 01 Mar 06 to 13 Mar 06.](image)

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Figure 4: Kirks and CEO Pricing Errors

Figure 5: Implied Correlation Skews of CEO Model
all strikes on all dates. Figure 5 shows that the implied correlations that are calibrated from the compound exchange option approximation exhibit a realistic, positively sloped skew on each day of the sample. However, the implied correlations computed from Kirk’s approximation were found to be equal to 0.99 for all strikes and on every day.

The price hedge ratios for Kirk’s approximation are straightforward to derive from equation (5). In the compound exchange option framework:

**Delta**

\[
\frac{\partial P}{\partial S_1} = -e^{-q_1(T-t)} \Phi (-d_1^U) \Phi (d_1^U) \\
\frac{\partial P}{\partial S_2} = e^{-q_2(T-t)} \Phi (-d_2^U) \Phi (d_2^U)
\]

**Gamma**

\[
\frac{\partial^2 P}{\partial S_1^2} = e^{-q_1(T-t)} \left( \frac{\phi (-d_1^U)}{\sqrt{T-t} U_{1,0}} \Phi (d_1^U)^2 - \frac{\phi (-d_1^U)}{\sqrt{T-t} S_{1,0}} \Phi (-d_1^U) \right) \\
\frac{\partial^2 P}{\partial S_2^2} = e^{-q_2(T-t)} \left( \frac{\phi (-d_2^U)}{\sqrt{T-t} U_{2,0}} \Phi (d_2^U)^2 - \frac{\phi (-d_2^U)}{\sqrt{T-t} S_{2,0}} \Phi (-d_2^U) \right)
\]

Hence if \( \Delta y^x \) denotes the delta of \( x \) with respect to \( y \) and similarly for the gamma:

\[
\Delta_{S_i}^P = \Delta_{U_i}^P \Delta_{S_i}^{U_i} \\
\Gamma_{S_i}^P = \Gamma_{U_i}^P \left( \Delta_{S_i}^{U_i} \right)^2 + \Gamma_{S_i}^{U_i} \Delta_{S_i}^P
\]

Figures 6 and 7 compare the two deltas and gammas of each model, calibrated on 1st March 2006 and depicted as a function of the spread option strike. The same features are evident on all other days in the sample: at every strike the exchange option delta is significantly closer to zero than the delta that is obtained through Kirk’s formula. Similar remarks apply to the gamma hedges, particularly for the gamma hedge on crude oil. Hence the use of Kirk’s approximation may lead to significant over-hedging.

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Figure 6: Delta with respect to Gasoline (left) and Crude Oil (right)

Figure 7: Gamma with respect to Gasoline (left) and Crude Oil (right)
4. Conclusion

This paper is not completed yet. We plan to extend the theoretical part to include local volatilities and correlation in the compound exchange option approximations to prices and hedge ratios. We also plan to extend the empirical work to add comparisons of (1) the pricing accuracy of the exchange option approximation with the Carmona-Durrleman bounds, and (2) a dynamic hedging study to quantify the extent of over-hedging that stems from the use of Kirk’s formula.

So far we have highlighted some significant problems with implementing Kirk’s approximation. These problems stem from the fact that the approximation is only valid for spread options with low strikes, and from the need to fix the strike convention for determining the implied volatilities in the calibration. We have thus tested several strike conventions for fixing the implied volatilities in equation (5) but in each case the market prices of the single assets options appear to be inconsistent with the spread option price, if the assumptions of Kirk’s approximation are valid. However, we should not forget that these assumptions are only valid for options with low strikes, and for these strikes (only) the pricing errors are indeed minimal. For the crack spread option data all choices of strike convention yielded almost constant correlations that were close to 1, which is unrealistic.

By contrast, the compound exchange option approximation provides accurate prices at all strikes and realistic values for implied correlation. Apart from this another advantages of the compound exchange option approximation is the ease of calibration and the simple computation of the option’s price sensitivities. We have found empirically that the compound exchange option approach specifies deltas and gammas that are significantly smaller than the deltas and gammas from Kirk’s approximation, and we thus have reason to suppose that the use of Kirk’s approximation will lead to substantial over-hedging of crack spread positions.

References


A Appendix

This appendix motivates our choice of functional form for correlation in equation 14. The first derivative of $P_t$ with respect to $m(K, T)$ is

$$\frac{\partial P_t}{\partial m(K, T)} = Ke^{-r(T-t)} \left( \Phi \left( d_2^1 \right) \Phi \left( -d_1^* \right) - \Phi \left( d_2^2 \right) \Phi \left( -d_2^* \right) \right)$$

(23)

Hence when calibrating to market prices $m(K, T)$ will have an inverse and approximately linear relationship with strike $K$. Next we note that by definition $m(K, T)K$ lies in the $K$-neighbourhood of $\min \{S_{1,t}, S_{2,t}\}$ and $m(K, T)$ also has a near linear relationship with $\min \{S_{1,t}, S_{2,t}\}$. Finally we investigated the relationship between $m(K, T)$ and implied correlation by fixing the correlation and calibrating $m(K, T)$ for different spread option strikes of the same maturity. This was achieved using a simple Newton-Raphson method as $P_t$ represents a well-behaved, monotonically increasing or decreasing function of $m(K, T)$. In most cases a suitable starting value could be chosen to be $\left( \frac{\min\{S_{1,t}, S_{2,t}\}}{K} - 2 \right)$.

The additional parameters $\alpha$ and $\beta$ allow for non-linear variation of correlation with the strike and maturity of the option. Since $m(K, T)K$ lies in the $K$-neighbourhood of $\min S_{1,t}, S_{2,t}$ we require $\beta \rho(K, T)\alpha$ to lie in the neighbourhood of one. For our data we found it perfectly adequate to fix $\alpha = 2$ and then set $\beta$ using an approximate value for the correlation. For instance, when $\rho \approx 0.7$ we found that $\beta = 2$ kept $\beta \rho(K, T)\alpha$ close to one.