Pricing of swing options in a mean-reverting model with jumps

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Agenda

Introduction

Swing option payoff

Spot price model, model calibration and derivatives pricing

Swing option pricing

Results

Conclusions and future research
Example of a swing option

Duration: One year

Rights of the holder: Every Friday, the holder decides on what days the following week he wants to buy 1 MWh of electricity at a fixed price.

Underlying: Nord Pool system price

Settlement: Financial, daily net payments.

Total amount: Exactly 100 MWh may be bought in total.
Previous work

Impulse control problems, variational inequalities and Feynman-Kac theorems

- Bensoussan and Lions [1984]
- Pham [1997]

Swing option pricing

- Thompson [1995]
- Jaillet, Ronn and Tompaidis [2004]
- Ibáñz, [2004]
- Dahlgren, [2005]

Schwartz model I and day ahead decisions selected from discrete set!
Contribution

Discontinuous price trajectories: Schwartz Model I driven by jump diffusion

- Model by Deng [2001]

Vector of amounts needs to be chosen at each decision date

- Ex: Every Friday, amounts for every day next week must be chosen

Select amount to be bought from any compact set rather than from discrete set

- Not always Bang-Bang solutions
Agenda

Introduction

Swing option payoffs

Spot price model, model calibration and derivatives pricing

Swing option pricing

Results

Conclusions and future research
Definition of a swing option

1. **Maturity date:** The contract runs over the period \([0, T]\).

2. **Swing action times:** The times when the holder is allowed to make decisions are denoted by \(\{T_n\}_{n=1}^{N}\), where \(0 \leq T_1 < T_2 < \ldots < T_N < T\).

3. **Swing action:** At each swing action date \(T_n\), the holder decides on the amount of energy \(B_{d n}^{d}\) MWh to be bought at the fixed Strike price \(K\) EUR/MWh over each of the \(D\) periods \((T_{d n}^{d}, T_{d n}^{d+1}]\), \(1 \leq d \leq D\). Here \(T_n = T_{n}^{1} < T_{n}^{2} < \ldots < T_{n}^{D} = T_{n+1}\), and \(T_{N}^{D+1} = T\).

4. **Allowed amounts per period:** We assume that \(B_{d n}^{d} \in \mathcal{O} \subseteq [0, \infty)\), where \(\mathcal{O}\) is either a closed interval \(\mathcal{O} = [B, \overline{B}]\) or a discrete set.

5. **Allowed amount in total:** The holder may buy at least \(\underline{M}\) MWh and at most \(\overline{M}\) MWh in total. To be of interest, \(NDB < \underline{M} \leq \overline{M} < ND\overline{B}\).

6. **Settlement:** All swing contracts are financially settled. To reduce the consequences of a default of the counterpart, net payments occur at times \(T_{n}^{d}\), \(1 \leq d \leq D\).
Agenda

Introduction

Swing option payoffs

Spot price model, model calibration and derivatives pricing

Swing option pricing

Results

Conclusions and future research
Nordpool facts

Common market for Norway, Denmark, Sweden and Finland

Spot market is day ahead auction market for each hour

Annual turnover: 166 TWh (40%)

Forward market is financially settled with spot price as underlying

Forwards exist with delivery of 1 MWh at constant load over: one day, one week, one month, one quarter, one season and one year

Net payments every day during delivery ⇒ Swaps

Liquidity of concentrated to contracts with longer delivery periods.

Need for a model of the daily average price!
Features of electricity prices

Non-storability of spot electricity and inelastic demand causes
- Seasonality (Seasons, work week)
- Mean reversion
- Spikes up and down
Model selection approach

Same approach as in Lucia and Schwartz [2002]
  • Set up the model under $P$ using one factor only.
  • Estimate parameters with a time series approach.
  • Choose $Q$ explicitly and estimate from traded derivatives.

Benefits
  • Nordpool has few liquid forwards and no liquid vanilla options $\Rightarrow$. Only simple Radon-Nikodym derivatives feasible.
  • Better dynamical performance.

Disadvantages
  • No spikes, only up and down jumps.
  • Other stochastic factors must be omitted.
Notation

$(\Omega, \mathcal{F}, P)$: Probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$

$Q$: A risk neutral probability measure, $P \sim Q$. At least one exists.

$\mathbb{E}$: Risk neutral expectation.

$\varphi_X(u) = \mathbb{E}[e^{iuX}]$. Characteristic function of r.v. $X$.

$S_t$: Spot price. Belongs to probability space above.

$B_t = B_0 e^{rt}$: Risk free bank account.

$F^T_t = \mathbb{E}[S_T | \mathcal{F}_t]$: Forward price.

$G^T_t = e^{-r(T-t)}(F^T_t - K)$: Price of financially settled forward contract with strike $K$. 

Pricing of swing options in a mean-reverting model with jumps – p. 12
Spot price model under $P$

$P$-dynamics with seasonality, mean reversion and jumps:

$$
\begin{align*}
S_t & = \exp(f(t) + X_t) \quad f(t) \text{ seasonal trend} \\
\quad dX_t & = -\alpha X_t dt + dL_t, \quad L_t \text{ compensated jump diffusion,}
\end{align*}
$$

with

- $L_t = \sigma W_t + U_t^J - \lambda J \mathbb{E}_P[J] t$, $U_t^J$ Compound Poisson indep. of $W_t$
- Jumps $J$ arrive with intensity $\lambda J$ and have density $f_J$
- Model by Deng [2002]. Lucia-Schwarz [2002]-model if $\lambda J = 0$.
- $\int_{\mathbb{R}} e^{2y} f_J(y) dy < \infty$ so $S_t$ has finite variance under $P$. 

Pricing of swing options in a mean-reverting model with jumps – p. 13
Spot price model under $Q$

$Q$–dynamics for $X_t$:

$$dX_t = -\alpha X_t dt + d\tilde{L}_t$$

Where

- $\tilde{L}_t = \sigma \tilde{W}_t + U_t J$, $d\tilde{W}_t = dW_t + \lambda dt$
- $\lambda \in \mathbb{R}$: Constant market price of diffusive spot price risk.
- Changes long term mean of $X_t$. Used by Lucia and Schwarz.
- Jump risk not priced, so $f_J$ invariant under change of measure.
- Analytical expression for forward prices since affine.
- Only $\lambda$ needs to be estimated from forwards.
Forward prices

Forward prices from transform analysis by Duffie, Pan and Singleton

\[ F(t, T) = \exp \left( f(T) + (\log S_t - f(t))e^{-\alpha(T-t)} \right) \]
\[ \times \exp \left( -\frac{\sigma \lambda}{\alpha} \left( 1 - e^{-\alpha(T-t)} \right) + \frac{\sigma^2}{4\alpha} \left( 1 - e^{-2\alpha(T-t)} \right) \right) \]
\[ \times \exp \left( \lambda_J \int_{0}^{T-t} \mathbb{E}[\exp(Je^{-\alpha s}) - (1 + Je^{-\alpha s})] ds \right) \]
\[ \equiv F_{season} \times F_{diffusion} \times F_{jump}, \]

Analytical expression available if double exponential jumps

\[ f_J(x|\lambda_1, \mu_1, \lambda_2, \mu_2) = \begin{cases} \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{e^{-x/\mu_1}}{\mu_1}, & x \geq 0 \\ \frac{\lambda_2}{\lambda_1 + \lambda_2} \frac{e^{x/\mu_2}}{\mu_2}, & x < 0 \end{cases} \]
\[ \lambda_1 + \lambda_2 = \lambda_J \]
Let $V \in C^{1,2}$ with a bounded first $x$–derivative and define

$$D_x V = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} + (\sigma \lambda - \alpha x) \frac{\partial V}{\partial x}$$

$$\mathcal{I}_x V = \int_{\mathbb{R}} \left( V(t, x + y) - V(x) - y \frac{\partial V}{\partial x} \right) f_J(y) \, dy,$$

then the generator $\mathcal{L}_x$ of $X_t$ is given by

$$\mathcal{L}_x V = D_x V + \lambda J \mathcal{I}_x V.$$
Feynman-Kac theorem

Consider a simple European derivative with payoff $Y = H(S_T)$ at time $T$. Then under some technical conditions on $H$

$$V_t = e^{-r(T-t)} E[H(S_T)|F_t],$$

and $V_t = V(t, x)$ is the unique solution to the parabolic PIDE

$$\begin{cases} 
\frac{\partial V}{\partial t} + \mathcal{L}_x V - rV = 0 \\
V(T, x) = H(e^{f(T)} + x). 
\end{cases}$$

Existence and uniqueness of solution to the PIDE: Change of variable to log-forward price and apply results from Bensoussan and Lions [1984] and Pham [1997].
Least squares estimation of $f(t)$

Seasonal trend: $f(t|\Theta) = A_0 + \sum_{n=1}^{N} A_n \cos(2\pi f_n t + B_n)$

- $f_1 = 1/365$, $f_2 = 4/365$, $f_3 = 12/365$, $f_4 = 52/365$, $f_5 = 104/365$
- $\Theta = (A_0, A_n, B_n)_{n=1}^{N}$ to be determined from data

Non-linear least squares

- $Y_t = \log S_t$ and explicit Euler discretization with $\Delta t = 1$ day gives

$$y_t = (1 - \alpha)y_{t-1} + f(t|\Theta) - (1 - \alpha)f(t-1|\Theta) + \epsilon_t$$

- $\epsilon_t$ white noise
- Non-linear least squares optimization gives optimal $(\Theta, \alpha)$. 
Estimation of the other parameters

Remove the trend: \( X_t = \log S_t - f(t|\Theta) \) with estimated \( \Theta \).

Estimate \( \alpha \) from moment condition
\[
\text{Cov}(X_{t+\tau}, X_t) = e^{-\alpha \tau} \text{Var}(X_t).
\]

Estimate the remaining parameters with FFT-based ML
- The r.v.'s \( Z_t \equiv X_{t+1} - X_t e^{-\alpha} \) are i.i.d.
- Likelihood function from inverse Fourier transform
\[
L = -\sum_{t=1}^{T-1} \log \left( \int_{\mathbb{R}} \varphi_Z(u) e^{-iuZ_t} \frac{du}{2\pi} \right),
\]
\[
\varphi_Z(u) = \exp \left( -iu \frac{\lambda J \mathbb{E}[J]}{\alpha} (1 - e^{-\alpha}) - \frac{\sigma^2 u^2}{4\alpha} (1 - e^{-2\alpha}) + \lambda J \int_0^1 \{ \varphi_J(ue^{-\alpha s}) - 1 \} ds \right)
\]

Market price of risk \( \lambda \) estimated from cross section of forward prices.
Agenda

Introduction

Swing option payoffs

Spot price model, model calibration and derivatives pricing

Swing option pricing

Results

Conclusions and future research
Payoff from a swing action

Choosing $B_n^d$: Equivalent to receiving $B_n^d$ forward contracts with delivery of 1 MWh during day $d$ (i.e. during $[T_n^{d-1}, T_n^d]$).

Total amount bought: $\Delta_n \equiv \sum_{d=1}^{D} B_n^d$ MWh is total amount bought through this swing action.

Payoff: Let $\Delta_n \in [(k - 1)\bar{B}, k\bar{B})$, where $1 \leq k \leq D$ and pick the most expensive contracts. This yields the payoff $g$ as

$$g(T_n, s, \Delta_n) = \sum_{d=1}^{k-1} \bar{B} \bar{G}_k(T_n, s) + [\Delta_n - (k - 1)\bar{B}] \bar{G}_k(T_n, s).$$

where $\bar{G}_k(T_n, s)$ are forward contracts sorted by price when $S_{T_n} = s$.

\[
\text{A swing action is characterised by one number } \Delta_n!\]
The set of *admissible swing action strategies* \( \mathcal{A} \) consists of all sequences \( \{\Delta_n\}_{n=1}^N \) such that

(a) \( \Delta_n \in [0, \bar{D}\bar{B}] \).

(b) \( \sum_{n=1}^N \Delta_n \in [\bar{M}, \bar{M}] \).

(c) \( \Delta_n \in \mathcal{F}_{T_n} \).
More definitions and results

\[ z = \sum_{j=1}^{n} \Delta_j, \quad T_n < t \leq T_{n+1} \]: Total amount bought up to time \( t \).

\[ V(T_n, s, z) \]: Value of a swing option immediately before the decision, given that \( z \) MWh already have been bought and \( S_t = s \)

\[ V(t, T_n, s, z) \]: Value of a swing option with first decision occurring at time \( T_n > t \), given that \( z \) MWh already have been bought and \( S_t = s \), i.e.

\[ V(t, T_n, s, z) = e^{-r(T_n-t)} \mathbb{E}[V(T_n, S_{T_n}, z) | \mathcal{F}_t] \]
Pricing theorem

Consider the swing option defined above and let
\[ T_j = \min(T_n : T_n \geq t : n \in \{1, \ldots, N\}) \]. Then its value \( V_t = V(t, s, z) \) is given by

\[
V(t, s, z) = \begin{cases} 
\sup \left\{ \Delta_n \right\}_{n=j}^{N} \in \mathcal{A} \left[ \mathbb{E} \left[ \sum_{n=j}^{N} g(T_n, S_{T_n}, \Delta_n) e^{-r(T_n-t)} | \mathcal{F}_{T_j} \right] \right], & t = T_j; \\
V(t, T_j, s, z), & T_{j-1} < t < T_j. 
\]

Moreover, there exists at least one optimal swing action plan \( \{\Delta^*_n\}_{n=1}^{N} \in \mathcal{A} \) such that the supremum is attained.
Sketch of proof and algorithm

At time $t = T_N$: Choose $\Delta_N$, receive $g(T_N, s, \Delta_N)$. Maximizing this given $z \leq M$ gives $V(T_N, s, z)$.

At time $T_{N-1} < t < T_N$: $V(t, s, z) = V(t, T_N, s, z)$.

At time $t = T_{N-1}$: Choose $\Delta_{N-1}$, receive

$$g(T_{N-1}, s, \Delta_{N-1}) + V(T_{N-1}, T_N, s, z + \Delta_{N-1})$$

Maximizing this gives $V(T_{N-1}, s, z)$.

Continue backwards recursively.

$V(T_{N-1}, T_N, s, z)$ computed by solving PIDE numerically
Finite differences

Operator splitting finite differences assuming zero gamma as \( s \to \infty \).

Crank-Nicholson finite differences for the differential operator \( D_x \).

Explicit finite differences for the integral operator \( I_x \):

- Extrapolate \( V \) consistently with zero gamma boundary condition.

Faster than Monte-Carlo. More accurate than trees!
Agenda

Introduction

Swing option payoffs

Spot price model, model calibration and derivatives pricing

Swing option pricing

Results

Conclusions and future research
### Estimation results

<table>
<thead>
<tr>
<th>$f_n$</th>
<th>$f_0 = \infty$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>3.3873</td>
<td>0.2930</td>
<td>0.0411</td>
<td>-0.0251</td>
<td>-0.0310</td>
<td>-0.0133</td>
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<tr>
<td>$B_n$</td>
<td>n.a.</td>
<td>0.4063</td>
<td>1.2807</td>
<td>0.9712</td>
<td>0.6386</td>
<td>1.7629</td>
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</tbody>
</table>

**Table 1:** Estimated parameters of the seasonal function, where $f_0 = \infty$ corresponds to the constant $A_0$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\alpha$</th>
<th>$\sigma$</th>
<th>$\lambda_1$</th>
<th>$\mu_1$</th>
<th>$\lambda_2$</th>
<th>$\mu_2$</th>
<th>$\lambda$ (MPR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lucia-Schwartz</td>
<td>0.0280</td>
<td>0.0711</td>
<td>n.a.</td>
<td>n.a.</td>
<td>n.a.</td>
<td>n.a.</td>
<td>0.0095</td>
</tr>
<tr>
<td>Deng</td>
<td>0.0280</td>
<td>0.0370</td>
<td>0.1432</td>
<td>0.0897</td>
<td>0.2355</td>
<td>0.0556</td>
<td>0.0199</td>
</tr>
</tbody>
</table>

**Table 2:** Estimated parameters of the OU-process in the LS and Deng models.
Swing option examples (I)

Figure 1: Prices of swing options with daily and weekly decisions, \( M = 0, \overline{M} = 100, \) \( K = 30 \) as a function of \( s = S_t \) in the Lucia-Schwartz and Deng models.
Swing option examples (II)

Figure 2: Prices of swing option with daily decisions, $\underline{M} = 0$, $\overline{M} = 100$, $K = 60$ as a function of $s = S_t$ in the Lucia-Schwartz and Deng models.
Swing option examples III

Biggest impact of jumps when $N$ is high and $\bar{M}$ small

<table>
<thead>
<tr>
<th>$N$</th>
<th>2</th>
<th>4</th>
<th>12</th>
<th>52</th>
<th>364</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deng</td>
<td>842</td>
<td>887</td>
<td>1024</td>
<td>1197</td>
<td>1264</td>
</tr>
<tr>
<td>LS</td>
<td>841</td>
<td>884</td>
<td>1010</td>
<td>1167</td>
<td>1228</td>
</tr>
<tr>
<td>Difference</td>
<td>0.12%</td>
<td>0.34%</td>
<td>1.39%</td>
<td>2.57%</td>
<td>2.93%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\bar{M}$</th>
<th>10</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>364</th>
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</thead>
<tbody>
<tr>
<td>Deng</td>
<td>197</td>
<td>774</td>
<td>1197</td>
<td>1518</td>
<td>1559</td>
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<tr>
<td>LS</td>
<td>185</td>
<td>746</td>
<td>1167</td>
<td>1494</td>
<td>1536</td>
</tr>
<tr>
<td>Difference</td>
<td>6.49%</td>
<td>3.75%</td>
<td>2.57%</td>
<td>1.61%</td>
<td>1.50%</td>
</tr>
</tbody>
</table>

Table 3: Dependence on $N$ and $\bar{M}$ for $\bar{M} = 0$. Current spot price: $s = 30$ EUR/MWh. Option parameters: $N = 52$, $\mathcal{O} = \{0, 1\}$ and $K = 30$ EUR/MWh.
Introduction

Swing option payoffs

Spot price model, model calibration and derivatives pricing

Swing option pricing

Results

Conclusions and future research
Conclusions

Swing options can be priced by combining dynamic programming and numerical solution of PIDEs.

Jumps are frequent and large compared to the diffusion.

Swing option prices may differ 2-35% between the models:

- Larger difference for OTM options.
- Difference increases with increasing "timing flexibility", i.e. many decision dates $N$ and small $\bar{M}$ compared to $\bar{B}$.
Future research

More complex models for the underlying

- Spikes, stochastic volatility, stochastic seasonal trend
- Natural gas or crude oil
- Forward-curve (HJM) model alternative for these commodities.

Theoretical and computational issues

- Monte Carlo methods for American contracts since PIDE-approach too time consuming with many factors.