Two and Three factor models for Spread Options Pricing

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Outline

- Spot Price Dynamics
- Forward Prices
- Spreads on Forwards
- Model Calibration
- Conclusions
Spot Price Dynamics
Spot Price Dynamics

- Energy and energy commodities markets are unique
- Well known peculiarities with energy commodities:
  - Markets are **illiquid**
  - **Storage costs** (or impossibility of storage) translate into peculiar price behavior
  - Structural issues lead to **high volatility** levels
  - Prices exhibit strong **mean-reversion** tendencies
  - Electricity in particular contains **fast mean-reverting jumps**
Spot Price Dynamics

- Geometric Brownian motion fails to capture price fluctuations
- Mean-reversion is an essential feature

\[ d \ln S(t) = \kappa (\theta - \ln S(t)) \, dt + \sigma \, dX(t) \]

where \( X(t) \) is a \( \mathbb{P} \) Wiener process

- One-factor models:
  - Fail to capture the term structure of forward rates
  - Fail to treat the long-run mean reversion as dynamic
Spot Price Dynamics

- **Pilipovic** (1997) first proposed the two-factor mean-reverting model

\[
\begin{align*}
    dS_t &= \beta (\theta_t - S_t) \, dt + \sigma_S S_t \, dW_t^{(1)} \\
    d\theta_t &= \alpha \theta_t \, dt + \sigma_\theta \theta_t \, dW_t^{(2)} \\
    d[W^{(1)}, W^{(2)}]_t &= \rho \, dt
\end{align*}
\]

- Long-run mean $\theta$ blows up
- No-invariant distribution in long-run
- Does not lead to closed form option prices
Spot Price Dynamics

- A **stationary two-factor mean-reverting** model:

\[ S_t = \exp\{g_t + X_t\} , \]

\[ dX_t = \beta (Y_t - X_t) dt + \sigma_X dW_t \]

\[ dY_t = \alpha (\phi - Y_t) dt + \sigma_Y dZ_t \]

\[ d[W, Z]_t = \rho dt \]

- Seasonality is modeled through \( g_t \)
- \( X_t \) mean-reverts to \( Y_t \)
- \( Y_t \) mean-reverts to \( \phi \)
Spot Price Dynamics

- Since both $X$ and $Y$ are Gaussian Ornstein-Uhlenbeck processes one finds

$$Y_t = \phi + (Y_s - \phi) e^{-\alpha(t-s)} + \sigma_Y \int_s^t e^{-\alpha(t-u)} dZ_u$$

$$X_t = G_{s,t} + e^{-\beta(t-s)} X_s + M_{s,t} Y_s + \sigma_X \int_s^t e^{-\beta(t-u)} dW_u + \sigma_Y \int_s^t M_{u,t} dZ_u$$

- Here, $G_{s,t}$ and $M_{s,t}$ are deterministic functions of the model parameters and time
Stochastic Volatility Spot Models

- A **three-factor model**: stochastic long-run mean and stochastic volatility

\[
S_t = \exp\{g_t + X_t\},
\]

\[
dX_t = \beta (Y_t - X_t) dt + \sigma_X(Z_t) dW_t^{(1)}
\]

\[
dY_t = \alpha (\phi - Y_t) dt + \sigma_Y dW_t^{(2)}
\]

\[
dZ_t = \eta (m - Z_t) dt + \sigma_Z dW_t^{(3)}
\]

\[
d[W^{(1)}, W^{(2)}]_t = \rho_{xy} dt
\]

\[
d[W^{(1)}, W^{(3)}]_t = \rho_{xz} dt
\]

\[
d[W^{(2)}, W^{(3)}]_t = 0
\]
Stochastic Volatility Spot Models

- **Realized volatility**: NYMEX light sweet crude oil
Stochastic Volatility Spot Models

- **Simulated volatility**

\[
\eta = 1, \quad \frac{\sigma Z}{\sqrt{2\eta}} = 0.1
\]

\[
\eta = 10, \quad \frac{\sigma Z}{\sqrt{2\eta}} = 0.1
\]

\[
\eta = 100, \quad \frac{\sigma Z}{\sqrt{2\eta}} = 0.1
\]
Electricity Spot Price Modeling

- For Electricity, typical modeling assumptions assume

\[ d \ln(S_{t-}) = \alpha(\theta - \ln(S_{t-}))\,dt + \sigma\,dW_t + dQ_t. \]

- \( Q_t \) is a Compound Poisson process (or possibly Lévy)

\[ Q_t = \int_0^t \int_{-\infty}^\infty y \, \mu(dy, dt) = \sum_{n=1}^{N(t)} q_i \]

with \( q_i \overset{i.i.d.}{\sim} F_q(u) \)

- When calibrated to real data one finds:
  - Mean-reversion is very high to draw down jumps
  - This pushes diffusive volatility artificially high
Electricity Spot Price Modeling

- We propose a natural extension of the two-factor and three-factor diffusion model:

\[ S_t := \exp \{ g_t + X_t + J_t \}, \]

where the new jump component \( J_t \) is defined via

\[ dJ_t = -\kappa J_{t-} dt + dQ_t, \]

- **Jump** and **diffusion** reversion rates are **decoupled**
- No artificially high diffusive volatilities
Forward Prices
Forward Prices

- The **forward price** for **two-factor model** is affine:

\[ F^{(i)}(t, T) \equiv \mathbb{E}_t^Q \left[ S_T^{(i)} \right] \]

\[ = \exp \left( g_T^{(i)} + G_{t,T}^{(i)} + e^{-\beta_i(T-t)} X_t^{(i)} + M_{t,T}^{(i)} Y_t^{(i)} \right) \]

- For the **three-factor model**, we carry out a **singular perturbation expansion** by assuming:

\[ \bar{\sigma}_Z^2 := \frac{\sigma_Z^2}{2\eta} < +\infty \quad \text{while} \quad \eta \to +\infty \]
Forward Prices

The T-maturity forward price satisfies:

\[
\left( \epsilon^{-1} A_0 + \epsilon^{-\frac{1}{2}} A_1 + A_2 \right) F^\epsilon(t, x, z) = 0,
\]

\[
F_T(x, y, z) = e^{gT + x}
\]

where

\[
A_0 := (m - z) \partial_z + \bar{\sigma}_Z^2 \partial_{zz},
\]

\[
A_1 := \sqrt{2} \rho_{xz} \bar{\sigma}_Z \sigma_X(z) \partial_{xz},
\]

\[
A_2 := \partial_t + \beta(y - x) \partial_x + \alpha(\phi - y) \partial_y + \frac{1}{2} \sigma_x^2(z) \partial_{xx} + \frac{1}{2} \sigma_y^2 \partial_{yy} + \rho_{xy} \sigma_X(z) \sigma_Y \partial_{xy}
\]
Forward Prices

- We solve this PDE, and prove the error bound, to order $\varepsilon^{1/2}$ using **singular perturbation techniques**


**Corollary 0.1** For any fixed $(T, x, y, z) \in \mathbb{R}^+ \times \mathbb{R}^3$ and all $t \in [0, T]$, we have

\[
F_{t,T}^\varepsilon = \left(1 - V_1 h(t, T; 3\beta) \right) \\
- V_2 \frac{\beta}{\alpha_Y - \beta} [h(t, T; 3\beta) - h(t, T; \alpha_Y + 2\beta)] F_{t,T}^{(0)} + O(\varepsilon),
\]

where $F_{t,T}^{(0)}$ is the forward price in the two-factor model with

\[
\sigma_X = \langle \sigma_X(Z) \rangle
\]
Electricity Forward Prices

- The two-factor jump-diffusion model is also affine:

\[
F_{t,T}^{(1)} := \mathbb{E}_t^Q \left[ S_T^{(1)} \right]
\]

\[
= \exp \left\{ A_{t,T} + B_{t,T} X_t^{(1)} + C_{t,T} Y_t^{(1)} + D_{t,T} J_t \right\}
\]

- The infinitesimal generator \( \mathcal{A} \) of the joint processes acts on the forward price process rendering it zero

\[
\mathcal{A} F_{t,T}^{(1)} = 0
\]
Electricity Spot Price Modeling

- The **pricing PDE** then reduces to a system of **coupled Riccati ODEs**

\[
B_t - \bar{\beta}_1 B = 0 ,
\]

\[
C_t + \bar{\beta}_1 B - \bar{\alpha}_1 C = 0 ,
\]

\[
D_t - \kappa D = 0 ,
\]

\[
A_t + \bar{\alpha}_1 \bar{\phi}_1 C + \frac{(\sigma_X^{(1)})^2}{2} B^2 + \frac{(\sigma_Y^{(1)})^2}{2} C^2 + \rho_1 \sigma_X^{(1)} \sigma_Y^{(1)} BC = - \int_{-\infty}^{\infty} \lambda(u) (e^{D\cdot u} - 1) \, dF_l(u). 
\]
Spread Options
Exchange Option Pricing

- The **spread on two forward rates** is a very popular product (different assets):

\[ \text{pay-off} = \max \left( F_{T;T_1}^{(1)} - \alpha F_{T;T_2}^{(2)} - K, \ 0 \right) \]

- Exact solution difficult (impossible? – at least so far!) for \( K \neq 0 \) - Put \( K = 0 \) – **Margrabe** option

- Risk-neutral Pricing implies:

\[ \text{price} = P(t, T) \mathbb{E}^{Q} \left[ \max \left( F_{T;T_1}^{(1)} - \alpha F_{T;T_2}^{(2)}, \ 0 \right) \right] \]

- Use an asset as a **numeraire** to reduce the stochastic dimensionality?
Exchange Option Pricing

- Introduce the measure-$Q^*$ induced by the following Radon-Nikodym derivative process:

$$\left( \frac{dQ^*}{dQ} \right)_t := \mathbb{E}^{Q^*}_t \left[ \frac{S^{(i)}_T}{S^{(i)}_T} \right] = \frac{F^{(2)}_{t,T_2}}{F^{(2)}_{0,T_2}}$$

- The ratio of two forward prices is a $Q^*$-martingale!

$$F_{t;T_1,T_2} := \frac{F^{(1)}_{t,T_1}}{F^{(2)}_{t,T_2}}$$
Exchange Option Pricing

- In particular, for the **two-factor model** we show

\[ F_{T;T_1,T_2} = F_{t;T_1,T_2} \exp\{N\} \quad \text{with} \quad N \overset{\mathcal{Q}}{\sim} \mathcal{N}\left(-\frac{1}{2}(\sigma^*)^2 ; (\sigma^*)^2\right) \]

where \((\sigma^*)^2\) is a deterministic function of the model parameters and times.

**Proposition 0.1**  The risk-neutral value of the \(T\)-maturity forward spread option is

\[ \Pi_{t,T}^F = P(t,T) \left[ F_{t,T_1}^{(1)} \Phi\left(d^* + \sigma_{t,T}^*\right) - \alpha F_{t,T_2}^{(2)} \Phi(d^*) \right] \]

and

\[ d^* := \frac{\ln \left(\frac{F_{t,T_1}}{\alpha F_{t,T_2}}\right) - \frac{1}{2}(\sigma_{t,T}^*)^2}{\sigma_{t,T}^*}. \]
Exchange Option Pricing

For the **three-factor model** we show, using singular perturbation techniques once again that

**Proposition 0.1** The risk-neutral value of the $T$-maturity forward spread option is

\[
\Pi^F_{t,T} = \Pi^{F(0)}_{t,T} + \Pi^{F(1,1)}_{t,T} + \Pi^{F(1,2)}_{t,T} + O(\epsilon)
\]

where $\Pi^{F(0)}_{t,T}$ is the price in the two-factor model, and the corrections depend on the delta’s and the delta-gamma’s of the two-factor model.
Electricity Exchange Option Valuation

- The ratio of forward prices $F_{T,T_1,T_2}$ is still a martingale under this new $Q^*$-measure.
- However, it is no longer Gaussian.
- Use Fourier transform methods to price.
- Introduce the m.g.f. process of $Z_T = \ln F_{T,T_1,T_2}$

$$\Psi_t^{Z_T}(u) := \mathbb{E}^{Q^*}_t \left[ e^{u Z_T} \right]$$

- This too is a martingale under the new measure.
Crack Spread Valuation

- Rewrite the price as follows:

\[ E_{t}^{Q^*} \left[ (e^{Z_{T} - \bar{\alpha}} - 1)_+ \right] := E_{t}^{Q^*} \left[ \eta(Z_{T} - \bar{\alpha}) \right] \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\eta}(-p) \tilde{f}_{Z_{T} - \bar{\alpha}}(p) \, dp \]

The Fourier transform of \( \eta \) is standard:

\[ \tilde{\eta}(p) := \int_{-\infty}^{\infty} e^{ipx} \eta(x) \, dx = \frac{1}{p(i - p)} \]

whenever \( \Re(p) > 1 \).
Crack Spread Valuation

- Rewrite the price as follows:

**Proposition 0.1** The price of the calendar spark spread option is

\[
\Pi_{t,T} = P(t, T) e^{\bar{\alpha}} F_{t,T_2}^{(2)} \int_{-\infty}^{\infty} \frac{e^{-ip\bar{\alpha}} \Psi_t^{Z_T} (ip)}{-p(p + i)} \frac{dp}{2\pi},
\]
Spot Price Dynamics

Calibration Results
Calibration

- The data consists of spot prices and forward curves on corresponding days for crude oil.
Calibration

- We calibrate the model to both data sets simultaneously
- Computing the sum of squared errors of the forward curve given the spot prices

\[ \text{Sum}(t_p, \Theta) := \sum_{q=1}^{n_p} \left[ \log F_{t_p,T_q}^{(i)} - \log F_{t_p,T_q}^{(i)*} \right]^2. \]

Minimizing first w.r.t. to the hidden process \( Y \)

\[ Y_{t_p}^{\#(i)}(\Theta) = \frac{\sum_{q=1}^{n_p} \left[ M_{t_p,T_q}^{(i)} \left( \log F_{t_p,T_q}^{(i)*} - U_{t_p,T_q}^{(i)} \right) \right]}{\sum_{q=1}^{n_p} \left[ M_{t_p,T_q}^{(i)} \right]^2}. \]
Calibration

- Use this estimate in the sum of squared errors for each day separately
- **Minimize** over the remaining parameters

\[
\Theta^* := \text{ArgMin}_{\Theta \in \Omega} \sum_{p=1}^{m} \sum_{q=1}^{n_m} \left[ \left( \frac{\hat{U}_{t_p, T_q}}{M_{t_p, T_q}} + \frac{\hat{M}_{t_p, T_q}}{M_{t_p, T_q}} \cdot Y_{t_p}^{(i)}(\Theta) - \log F_{t_p, T_q}^{(i)*} \right)^2 \right],
\]

- This procedure yields the daily estimates for the hidden process Y and the risk-neutral model parameters
- The **time-series** of X and Y are used to **estimate the real-world** parameters via **regression**
Calibration

- The calibrated *risk-neutral* model parameters are

\[
\begin{array}{cccccc}
\beta & \alpha & \phi & \sigma_X & \sigma_Y & \rho \\
0.31 & 0.15 & 3.3 & 33\% & 63\% & -0.96 \\
\end{array}
\]

- The calibrated *real-world* model parameters are

\[
\begin{array}{cccccc}
\beta & \alpha & \phi & \sigma_X & \sigma_Y & \rho \\
1.06 & 0.73 & 4.2 & 33\% & 63\% & -0.96 \\
\end{array}
\]
Calibration

![Graph showing forward prices over term (years)]
Calibration

![Graph showing relative RMSE over dates from 08/17/03 to 05/13/06. The x-axis represents dates, and the y-axis represents relative RMSE ranging from 0% to 1.8%. The graph shows fluctuations in RMSE over time.]

Energy Spot Price Models and Spread Option Pricing      © S. Jaimungal and S. Hikspoors, 2007       34
Calibration

- We also check the stability of the model parameters through time

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<th># Fwd-Curves</th>
<th>β</th>
<th>α</th>
<th>φ</th>
<th>σ_χ</th>
<th>σ_γ</th>
<th>ρ</th>
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<td>0.38</td>
<td>0.26</td>
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<td>19%</td>
<td>-0.97</td>
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<td>3.27</td>
<td>33%</td>
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<td>-0.96</td>
</tr>
</tbody>
</table>
Concluding Remarks

- We introduced a **two-factor model** containing mean-reversion to a long run mean-reverting level
  - With and without jumps
- We introduced a third fast mean-reverting **stochastic volatility** into the model
- Obtained forward prices & **spreads on forwards** in closed form
- Future work
  - More thorough model calibration – including calibrating to electricity data using particle filter approaches
  - Adding a slow mean-reverting stochastic volatility
Concluding Remarks

Thank You!